

THE LOGISTIC EQUATION AND A LINEAR STOCHASTIC RESONANCE

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We show analytically that a collective action of two correlated Gaussian white noises, a constant external forcing and a periodic signal leads to a linear stochastic resonance (LSR). The resonance persists for long times, survives averaging over the initial phase of the signal and is characterized by a clear maximum of the signal-to-noise ratio (SNR), unlike other cases of the LSR reported previously. We show that the problem of the LSR is closely related to the behavior of a generalized noisy logistic equation.

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1. Introduction

The logistic equation

$$\dot{x} = ax(1 - x), \quad (1)$$

$a > 0$, is one of the most frequently used, and undoubtedly most successful, models of population dynamics. It is now one of the classical examples of self-organization in many natural and artificial systems [1]. It is virtually impossible to list its all applications — for example, the Web search returns more than 60 000 links to pages and on-line publications containing the key words “logistic equation”.

A natural generalization of the model (1) is one in which the deterministic growth rate, a , is perturbed by a stochastic process:

$$\dot{x} = (a + p\xi(t))x(1 - x). \quad (2)$$

Here $\xi(t)$ is the stochastic term. Such equation has been first discussed by Leung [2] and later by many other authors. In particular, authors of Refs. [3,4], by Borel summing a formal series for the expectation value $\langle x(t) \rangle$,

obtained expressions for the nonlinear relaxation time. The noise term $\xi(t)$ was represented by the Gaussian white noise (GWN) in Refs. [2,3] and by a Gaussian colored noise in Ref. [4]. Surprisingly, closed and mathematically exact expressions for the expectation value and the variance of $x(t)$ are still lacking; we have not solved this problem, but we will present a heuristics, based on a simple transformation of Eq. (2) that, as our numerical simulations indicate, can be useful in predicting the behavior of systems described by the noisy logistic equation.

Surprisingly, the problem of the noisy logistic equation is closely related to that of the linear stochastic resonance (LSR). The stochastic resonance (SR) is an example of the constructive role of a noise, where the noise and a dynamical system act together to reinforce a periodic signal [5]; see also Ref. [6] for a review. We should mention that an aperiodic stochastic resonance has also been discovered [7]. The SR now seems to be ubiquitous and has been claimed “an inherent property of rate-modulated series of events” [8]. There is, however, a long-standing debate whether the SR can be at all present in linear systems. This point is important because a linear kinetics (relaxation) is the final stage of most dissipative processes. It would be particularly important to establish the existence of a LSR in systems driven by a GWN because such a noise represents the standard equilibrium fluctuations of macroscopic physical systems. The LSR has been first reported by Fuliński in Ref. [9] and later discussed in Refs. [10–13]. However, it was soon pointed out that the LSR vanished in the GWN limit [10, 12] or after averaging over the initial phase of the signal [11], or was restricted to transient times, even though other constructive effects of the noise persisted in the asymptotic regime as well [13]. Short-lived or phase-dependent phenomena cannot be deemed “unphysical” — on the contrary, they may be very important in many situations, for example in enzymatic reactions in living cells — but there is a general consensus that a fully-fledged stochastic resonance should be characterized by a clear maximum of the signal-to-noise ratio (SNR) which survives the phase averaging and persists for long times, indicating the presence of an optimal noise level.

The first evidence for a LSR that displays a maximum of the SNR was given only recently in Ref. [14]. However, we have shown [15] that this particular case of the SR was of a non-dynamical nature as the output of the dynamical system discussed in Ref. [14] merely reproduced the spectral properties of the input signal, and moreover, the resonant properties of the latter were only an artifact of the chosen parameterization of the noises. In the present paper we show analytically that a simple linear system with correlated multiplicative and additive GWNs and a driving that consists of a constant term and a periodic signal displays a maximum of the SNR that lacks all the above-mentioned deficiencies: The maximum is present in the

asymptotic (long time) regime and not only for transient times, it survives averaging over the initial phase of the signal, and finally, it is a result of a collective action of the system, the noises and the external driving and not merely a reproduction of very particular properties of the input sequence.

2. The logistic equation

Assume that the noise term in the perturbed logistic equation (2) is a GWN with $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. The equation (2) can easily be solved for every specific realization of the noise. The formal solution to Eq. (2) then reads

$$x(t) = \frac{1}{1 + \frac{1-x_0}{x_0} \exp \left[-at - p \int_0^t \xi(t') dt' \right]} . \tag{3}$$

In most practical situations, one is interested not in specific realizations of a stochastic process, but rather in moments of that process, averaged over an ensemble. One is tempted, then, to calculate the moments directly from Eq. (3), expanding it in a power series:

$$\begin{aligned} \langle x(t) \rangle &= \left\langle \sum_{n=0}^{\infty} (-1)^n \left(\frac{1-x_0}{x_0} \exp \left[-at - p \int_0^t \xi(t') dt' \right] \right)^n \right\rangle \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1-x_0}{x_0} \right)^n e^{-nat} \left\langle \exp \left[-np \int_0^t \xi(t') dt' \right] \right\rangle \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1-x_0}{x_0} \right)^n e^{-nat} \exp \left[\frac{1}{2} n^2 p^2 t \right] , \end{aligned} \tag{4}$$

where $\langle \dots \rangle$ stands for an average over the realizations of the noise $\xi(t)$. We have changed the order of the summation and taking the average in Eq. (4) and used the well known fact that for a GWN

$$\left\langle \exp \left[\int_{t_1}^{t_2} f(t') \xi(t') dt' \right] \right\rangle = \exp \left[\frac{1}{2} \int_{t_1}^{t_2} (f(t'))^2 dt' \right] . \tag{5}$$

We can, however, see that the final series in Eq. (4) is clearly divergent and even though a series similar to it has been Borel summed and some useful information has been extracted from it [3, 4], we conclude that the formal solution given by Eq. (3) offers little insight into the properties of Eq. (2).

Observe that a formal substitution

$$y = \frac{1}{x} \quad (6)$$

leads immediately to

$$y(t) = 1 + \frac{1 - x_0}{x_0} \exp \left[-at - p \int_0^t \xi(t') dt' \right] \quad (7)$$

and if the process $y(t)$ is convergent (see below), after a sufficiently long time almost all realization of the process do not differ from unity and the variance of $y(t)$ asymptotically vanishes. This observation does not constitute a *proof* that analogous statements can be made about the logistic process $x(t)$, too, but provides a useful heuristics about the long-time properties of the logistic process. The substitution (6) gives also a possibility of a further generalization of the noisy logistic equation that will be pursued in the next section.

3. The linear transmitter

With the substitution (6) Eq. (2) changes into

$$\dot{y} = -(a + p\xi(t))y + (a + p\xi(t)). \quad (8)$$

Note that this substitution does not affect the stable stationary point $x = 1$ of Eq. (1) and its basin of attraction, but it breaks for the unstable stationary point $x = 0$ of Eq. (1). Therefore, the vicinity of the unstable stationary point may not be represented correctly by the transformation (6).

The equation (8) closely resembles the equation of motion of a linear transmitter with multiplicative and additive noises that has been the subject of our recent research [16, 17]; the principal difference is that the constant additive driving was lacking in models discussed there. The equation (8) can easily be generalized

$$\dot{y} = -(a + p\xi_m(t))y + (b + q\xi_a(t)), \quad (9)$$

where both noises ξ_m, ξ_a are GWNs, but they are correlated: $\langle \xi_m(t)\xi_a(t') \rangle = c\delta(t - t')$. We assume that the noises ξ_m, ξ_a jointly form a two-dimensional Gaussian process (this assumption simplifies the discussion but is not a crucial one [17]):

$$\xi_m(t) = \xi(t), \quad (10a)$$

$$\xi_a(t) = c\xi(t) + \sqrt{1 - c^2}\eta(t), \quad (10b)$$

where ξ, η are uncorrelated (independent) GWNs and $c \in [-1, 1]$. We finally arrive at

$$\dot{y} = -(a + p\xi(t))y + b + qc\xi(t) + q\sqrt{1 - c^2}\eta(t). \tag{11}$$

3.1. Moments of the process $y(t)$

A formal solution to Eq. (11) reads ($y(0) = 0$)

$$y(t) = \int_0^t e^{-a(t-t')} \exp \left[-p \int_{t'}^t \xi(t'') dt'' \right] \left(b + qc\xi(t') + q\sqrt{1 - c^2}\eta(t') \right) dt'. \tag{12}$$

Unlike the solution (3), Eq. (12) can be used directly to calculate the moments:

$$\begin{aligned} \langle y(t) \rangle &= b \int_0^t e^{-a(t-t')} \left\langle \exp \left[-p \int_{t'}^t \xi(t'') dt'' \right] \right\rangle dt' \\ &\quad + q\sqrt{1 - c^2} \int_0^t e^{-a(t-t')} \left\langle \eta(t') \exp \left[-p \int_{t'}^t \xi(t'') dt'' \right] \right\rangle dt' \\ &\quad + qc \int_0^t e^{-a(t-t')} \left\langle \xi(t') \exp \left[-p \int_{t'}^t \xi(t'') dt'' \right] \right\rangle dt'. \end{aligned} \tag{13}$$

The first of the expectation values on the right hand side of Eq. (13) is of the form (5). The second vanishes identically because the noises η and ξ are uncorrelated, and we have calculated the last one in Ref. [17]:

$$\left\langle \xi(t') \exp \left[-p \int_{t'}^t \xi(t'') dt'' \right] \right\rangle = -\frac{1}{2}p \exp \left(\frac{1}{2}p^2(t - t') \right). \tag{14}$$

After collecting all terms we get

$$\langle y(t) \rangle = \frac{b - \frac{1}{2}cpq}{a - \frac{1}{2}p^2} \left(1 - e^{-(a - \frac{1}{2}p^2)t} \right) \xrightarrow{t \rightarrow \infty} y_\infty = \frac{b - \frac{1}{2}cpq}{a - \frac{1}{2}p^2}. \tag{15}$$

The limit in Eq. (15) exists if $a - \frac{1}{2}p^2 > 0$; otherwise the process is divergent. If we want to keep the interpretation that $x = 1/y$ is a measure of a population, the limit in Eq. (15) makes a physical sense if $b - \frac{1}{2}cpq \geq 0$.

We now calculate the second moment of the process $y(t)$:

$$\begin{aligned} \langle y^2(t) \rangle &= \int_0^t dt_1 \int_0^t dt_2 e^{-a(2t-t_1-t_2)} \left\langle \exp \left[-p \int_{t_1}^t \xi(t') dt' - p \int_{t_2}^t \xi(t') dt' \right] \right. \\ &\quad \left. \times \left(b + qc\xi(t_1) + q\sqrt{1-c^2}\eta(t_1) \right) \left(b + qc\xi(t_2) + q\sqrt{1-c^2}\eta(t_2) \right) \right\rangle. \end{aligned} \tag{16}$$

As we can see, one needs to know one more expectation value that we have previously calculated in Ref. [17]:

$$\begin{aligned} &\left\langle \exp \left[\int_0^T f(t')\xi(t') dt' \right] \xi(t_1)\xi(t_2) \right\rangle \\ &= [\delta(t_1 - t_2) + f(t_1)f(t_2)] \exp \left[\frac{1}{2} \int_0^T [f(t')]^2 dt' \right]. \end{aligned} \tag{17}$$

It is now easy to verify that the variance (16) exists if $a - p^2 > 0$. The full expression for the variance is rather long, and therefore we present the final expression in the asymptotic regime only:

$$\begin{aligned} D &= \langle y^2(t) \rangle - \langle y(t) \rangle^2 \xrightarrow{t \rightarrow \infty} \\ &= \frac{4b^2p^2 - 8abcpq + (4a^2 - 4a(1 - c^2)p^2 + (1 - c^2)p^4) q^2}{2(a - p^2)(p^2 - 2a)^2}. \end{aligned} \tag{18}$$

Let us consider the limiting cases. First, for the uncorrelated noises we obtain

$$D = \frac{b^2p^2 + (a - \frac{1}{2}p^2)^2q^2}{2(a - p^2)(a - \frac{1}{2}p^2)^2}, \quad c = 0. \tag{19}$$

As we can see, for the uncorrelated noises D increases monotonically both as a function of the amplitude of the multiplicative noise, p , and as a function of the amplitude of the additive noise, q . Next consider the case of maximally correlated (or maximally anticorrelated) noises:

$$D = \frac{(bp \mp aq)^2}{2(a - p^2)(a - \frac{1}{2}p^2)^2}, \quad c = \pm 1. \tag{20}$$

The variance (20) can *vanish* for certain values of parameters: $D = 0$ if $c = \pm 1$ and $bp \mp aq = 0$. For such parameters, almost all realizations of the

process $y(t)$ asymptotically remain equal their expectation value y_∞ despite the fact that the noises do not cease and constantly act on the system. Note that in this case $y_\infty = \pm q/p$; the negative value corresponds to an “unphysical” realization of the process (see the discussion below Eq. (15)).

In general, the variance D exhibits, as a function of q , a minimum for all $|c| > 0$ and appropriate values of other parameters; some of these sets of parameters may correspond to “unphysical” realizations. D is, by construction, nonnegative and equals zero only at its minimum for $c = \pm 1$.

The fact that the linear transmitter described by Eq. (11) admits, for the maximally (anti)correlated noises, realizations with a vanishing variance can be shown quite elementarily: Let $c = \pm 1$ and $bp \mp aq = 0$. Then Eq. (11) is equivalent to

$$\dot{y} = -(a + p\xi(t))y + (b \pm q\xi(t)) = -(a + p\xi(t)) \left(y \mp \frac{q}{p} \right), \quad (21)$$

$$y(t) = \pm \frac{q}{p} + \left(y(0) \mp \frac{q}{p} \right) \exp \left[-at - p \int_0^t \xi(t') dt' \right]. \quad (22)$$

If the process is convergent, almost all realizations asymptotically approach $y_\infty = \pm q/p$.

3.2. Application to the logistic equation

According to what we have said, the linearized logistic equation (8) clearly leads to a solution with asymptotically vanishing D , or with $\langle y^2(t) \rangle \rightarrow \langle y(t) \rangle^2 \rightarrow 1$. While in general there is no simple relation between the variances $D = \langle y^2(t) \rangle - \langle y(t) \rangle^2$ and $D_x = \langle x^2(t) \rangle - \langle x(t) \rangle^2$, we expect that if $D = 0$ and $y_\infty \neq 0$, D_x vanishes, too. Observe that for sufficiently large times, with $(a - \frac{1}{2}p^2)^{-1}$ being the characteristic time scale, almost all realizations of $y(t)$ reach a constant value, and so does the inverse $1/y(t) = x(t)$. We thus conclude that almost all realizations the process described by the logistic equation with a noisy growth rate (2) reach a constant value, equal to the equilibrium value of the deterministic logistic equation (1). Heuristically this means that once the process reaches a value $x(t) \simeq 1$, the right hand side of Eq. (2) remains nearly equal to zero for all future times even though the growth rate fluctuates. We have confirmed these results by direct numerical simulations.

The equation for the linear transmitter (11) corresponds to a further generalization of the logistic equation

$$\dot{x} = -(a + p\xi(t))x + (b + qc\xi(t) + q\sqrt{1 - c^2}\eta(t))x^2, \quad (23)$$

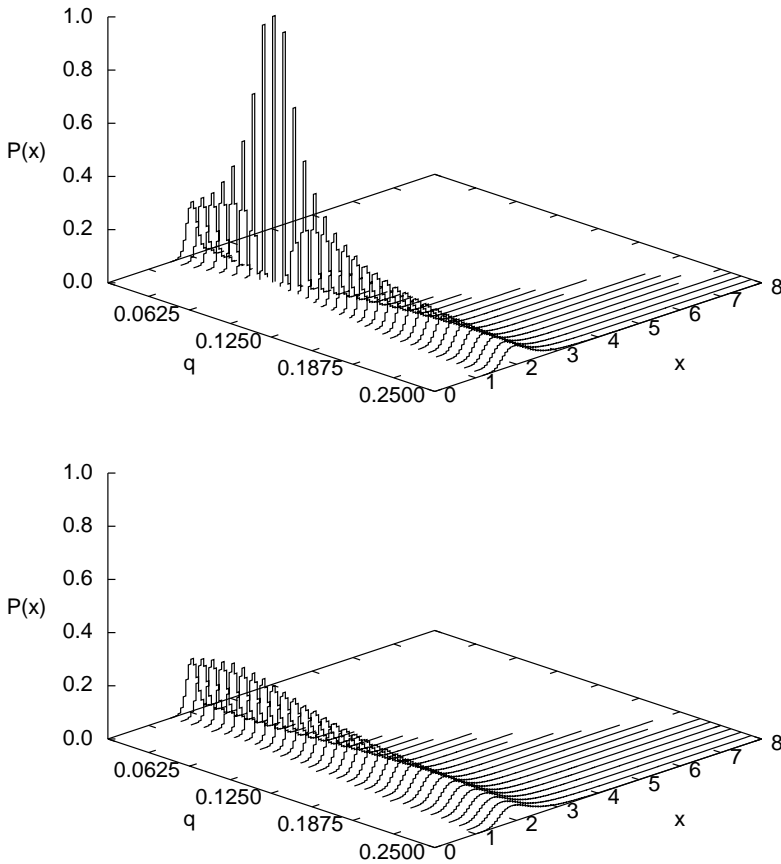


Fig. 1. Average distributions of values of $x(t)$ described by Eq. (23) as a function of the additive noise level, q . Parameters are: $a = 2$, $b = 1$, $p = 0.125$. The correlations are $c = 1$ (upper panel) and $c = 0.5$ (lower panel).

where not only the growth rate, but also the limiting population level fluctuate, and these fluctuations are correlated. The results of the previous subsection suggest that in the presence of correlations, for certain values of the parameters the variance of $x(t)$ should first shrink and then grow as the level of the additive noise increases. We have performed numerical simulations to confirm this. We have solved the nonlinear equation (23) using the Heun scheme [18] with a time step $h = 1/256$. The GWNs have been generated by the Marsaglia algorithm [19] and the famous Mersenne Twister [20] has been used as the underlying uniform generator. We have let the system

to equilibrate for 2^{16} time steps, collected the results for the next 2^{16} steps, calculated the distribution of the values of $x(t)$ that have appeared during the simulation, and averaged the results over 128 realizations of the process.

Selected results are presented in Fig. 1. In case of maximal correlations ($c = 1$, upper panel) the distribution of the values of $x(t)$ first gets narrower as q increases, becomes practically δ -shaped for $q = bp/a$, and then widens as q increases past the resonance. A similar, but much weaker, effect is observed in case of partial correlations ($c = 0.5$, lower panel). Note that for large values of q the distributions become skewed, with a marked preference for values above the expectation value.

4. A linear stochastic resonance

So far we have discussed the systems without any external signal acting on them. Now consider an additively coupled periodic signal:

$$\dot{y} = -(a + p\xi(t))y + b + qc\xi(t) + q\sqrt{1 - c^2}\eta(t) + A\cos(\Omega t + \phi), \quad (24)$$

where ϕ is the initial phase of the signal, A is the amplitude, Ω is the frequency, and all the other parameters are as above. We will, in addition to taking the usual average over realization of the noises, average over the initial phase, as otherwise the correlation function would not correspond to a stationary signal [21]:

$$\langle\langle y(t)y(t + \tau) \rangle\rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle y(t)y(t + \tau) \rangle d\phi. \quad (25)$$

We use the results of Section 3 above and the integrals evaluated in Ref. [17] to calculate the correlation function (25). The calculations are straightforward, but long and tedious and we do not present these details here. All the important technical points are the same as in Section 3. The final result is

$$\begin{aligned} & \langle\langle y(t)y(t + \tau) \rangle\rangle - \langle\langle y(t) \rangle\rangle^2 \xrightarrow{t \rightarrow \infty} C(\tau) \\ & = \frac{A^2 \cos \Omega\tau}{2 [(a - \frac{1}{2}p^2)^2 + \Omega^2]} + \left[\frac{A^2 p^2}{4(a - p^2) [(a - \frac{1}{2}p^2)^2 + \Omega^2]} + D \right] e^{-(a - \frac{1}{2}p^2)\tau}, \end{aligned} \quad (26)$$

where D is given by Eq. (18). Note that this correlation function exists for $a - p^2 > 0$ which does not come as a surprise. Since the correlation function

(26) is stationary, we can use the Wiener–Khinchin theorem to calculate the power spectrum of the process $y(t)$ as the Fourier transform of $C(\tau)$:

$$P(\omega) = 2 \int_0^\infty C(\tau) \cos \omega\tau \, d\tau = P_{\text{signal}} \cdot \delta(\omega - \Omega) + P_{\text{noise}}(\omega), \tag{27}$$

where

$$P_{\text{signal}} = \frac{A^2}{(a - \frac{1}{2}p^2)^2 + \Omega^2}, \tag{28a}$$

$$P_{\text{noise}}(\omega) = \frac{a - \frac{1}{2}p^2}{(a - \frac{1}{2}p^2)^2 + \omega^2} \times \left[\frac{A^2 p^2}{2(a - p^2) [(a - \frac{1}{2}p^2)^2 + \Omega^2]} + \frac{4b^2 p^2 - 8abcpq + (4a^2 - 4a(1 - c^2)p^2 + (1 - c^2)p^4) q^2}{(a - p^2)(p^2 - 2a)^2} \right]. \tag{28b}$$

Using this power spectrum, we can calculate the signal-to-noise ratio (SNR) as the ratio of the power associated with the signal to the power of the noisy background at the frequency of the signal:

$$\text{SNR} = 10 \log_{10} \frac{P_{\text{signal}}}{P_{\text{noise}}(\omega = \Omega)}. \tag{29}$$

By convenience, the SNR is usually measured in dB.

For $c = 1$ this procedure yields

$$\text{SNR} = 10 \log_{10} \frac{2A^2(a - p^2)(a - \frac{1}{2}p^2) [(a - \frac{1}{2}p^2)^2 + \Omega^2]}{A^2 p^2 (a - \frac{1}{2}p^2)^2 + 2 [(a - \frac{1}{2}p^2)^2 + \Omega^2] (bp - aq)^2}. \tag{30}$$

As we can see, the SNR has, as a function of q , a clear maximum for $bp - aq = 0$. The presence of this maximum provides an unequivocal evidence for the stochastic resonance. By differentiating the general expression for the SNR, which is obtained as a ratio of the first expression (28) to the second one and substituting $\omega = \Omega$ in the latter (the logarithm, being a convex function, may be omitted here), it is easy to verify that for all $|c| > 0$ and appropriate values of other parameters, the SNR has a maximum as a function of the additive noise strength, but for small values of $|c|$ this maximum is not very much pronounced, *cf.* Fig. 2. Note that also away from the maximum, larger values of $|c|$ correspond to larger values of the SNR.

Numerical results indicate that there also is a stochastic resonance in the case of a multiplicatively coupled signal, but analytical results are difficult to obtain in this case due to the presence of transcendental function in the expression for the correlation function, *cf.* Refs. [16, 17].

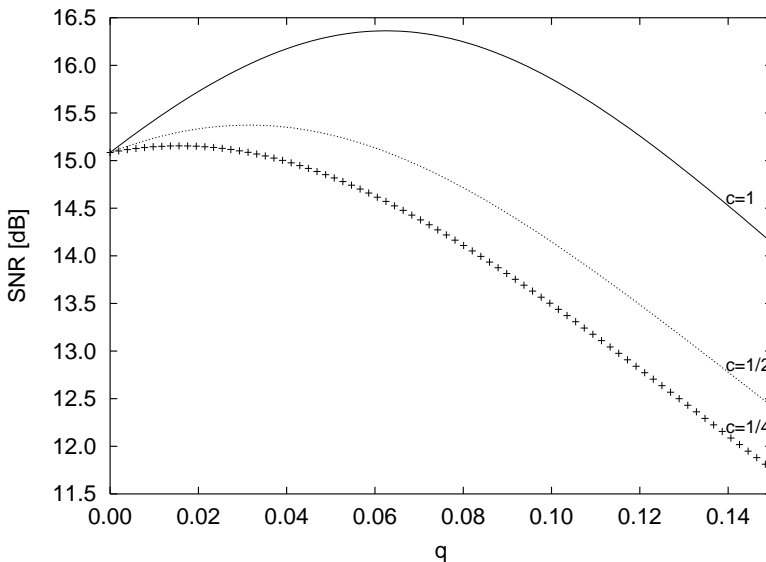


Fig. 2. Signal-to-noise ratio (SNR) for the linear transmitter (24) with various correlations between the noises: $c = 1$ (solid line), $c = 1/2$ (broken line), $c = 1/4$ (crosses). Amplitude of the signal $A = 1$, frequency $\Omega = 2\pi$. Other parameters as on Fig. 1. Curves with $c' = -c$ and $b' = -b$ are identical to those presented above, *cf.* Eq. (28).

5. Discussion

A clear and conclusive evidence that the linear system (24) displays a fully-fledged stochastic resonance when driven by Gaussian white noises is the principal result of this paper. The linear stochastic resonance reported here is characterized by a clear maximum of the SNR, persists for asymptotically long times and survives averaging over the initial phase of the signal. This result closes, we believe, the long debate whether the linear stochastic resonance is at all possible. It is important to realize that there are *two* factors that are needed to produce the SR in the system (24): (*i*) the multiplicative and additive noises have to be correlated, and (*ii*) there should be a constant driving term, $b \neq 0$, present. Berdichevsky and Gitterman in Ref. [12] and the present author in Ref. [17] have also considered correlated multiplicative and additive noises, but without the constant driving, no stochastic resonance has been present, at least for the GWN case. It is worth mentioning that the authors of Ref. [14] also discussed a system with two correlated Gaussian noises, but in that case one of the noises was coupled multiplicatively to the signal.

Note that the system (24) is linear in the sense that its equation of motion is linear with time-dependent coefficients, and thus we call the phenomenon discussed above a “linear stochastic resonance”. There is, however, a point in observation that since noise is meant to represent many unobserved and unaccounted for degrees of freedom — instead of considering impossibly complicated microscopic motions, we mimic their effect by a reasonably simple stochastic process — a multiplicative (nonlinear) coupling between the stochastic process and the observed degrees of freedom means a “hidden” nonlinearity.

We have also shown that the problem of the linear transmitter is closely related to the generalized noisy logistic equation. The analytical results for the transmitter provide some heuristics that might be helpful in predicting the behavior and properties of the noisy logistic process. However, the problem of finding closed and mathematically exact formulas for the moments of the latter remains open.

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