# STOCHASTIC RESONANCE AND RESONANT ENHANCEMENT OF MAGNETIC FLUX IN PERIODICALLY DRIVEN MESOSCOPIC CYLINDERS*

J. Łuczka, J. Dajka, M. Mierzejewski

Institute of Physics, University of Silesia
Uniwersytecka 4, 40-007 Katowice, Poland

P. Hänggi

Institut für Physik, Universität Augsburg
Universitätsstraße 1, 86135 Augsburg, Germany

*(Received February 27, 2006)*

*Dedicated to Professor Peter Talkner on the occasion of his 60th birthday*

Magnetic fluxes in mesoscopic cylinders, driven by an external, time-periodic magnetic field and thermal fluctuations, are investigated. The resonant enhancement of the response of the system is studied and compared with the celebrated phenomenon of Stochastic Resonance phenomenon.

PACS numbers: 64.60.Cn, 05.10.Gg, 73.23.–b

## 1. Introduction

The construction and the function of ultra-small hybrid structures present exciting challenges for future electronics which will build on circuits composed of mesoscopic and nano-elements. Such circuits consisting of molecular wires and loops interrupted by junctions are promising for implementation in a quantum computer [1]. They represent one of the currently studied candidates for solid-state qubits [2]. The geometry of mesoscopic and nano-systems plays a crucial role as it determines also their properties [3,4]. Typical examples that come to mind are the Bohm–Aharonov oscillations in conductance [5], persistent currents [6–8] and the Fano antiresonances [9]. A

* Presented at the XVIII Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 3–6, 2005.

topologically induced persistent current can occur in the presence of a magnetic flux in systems of a cylinder (ring, toroid) geometry. It is a consequence of the phase coherence of electrons and the sensitivity of the wave function to boundary conditions caused by a magnetic flux threading a cylinder or ring. Various aspects of this phenomenon have been theoretically studied in both, single and coupled as well as open and multichannel rings, including the repulsive electron–electron interaction, spin–orbit scattering, equilibrium or nonequilibrium fluctuations and noise [10, 11]. Several experiments confirmed the existence of persistent currents and, as well at least in parts, the theoretical predictions [12]. However, plenty of questions and open problems still remain. One such prominent question is why the experimentally observed currents are much larger than the theoretically predicted values. In this context, it is worth to note that the above mentioned experiments were performed in time-dependent periodic fields. For a non-zero frequency of periodic fields, it is reasonable to expect some extra contribution to the magnetic response of rings.

Motivated by the experimental findings, triggered us to study mesoscopic, topologically induced currents that are driven by external, time-periodic magnetic fields. Our model detailed in [13] reduces to a classical Langevin equation with a quantum-corrected effective potential, which at sufficiently low temperatures can assume a multi-stable shape. In the presence of periodic fields, we then can expect novel phenomena such as the Stochastic Resonance phenomenon [14, 15], an enhancement of escape rates [16,17] or the effect of resonant activation [18]. According to the 'common wisdom', the non-linear behavior of the ensemble-averaged output of the periodically driven system as a function of the temperature in the presence of both multi-stability of the potential and thermal fluctuations is at the heart of the Stochastic Resonance effect [14], possessing many applications also in biological physics [15]. We describe next our model which describes the time-evolution of a magnetic flux threading a cylinder. A Fokker–Planck equation corresponding to the Langevin equation is solved numerically and the results are depicted. We analyze the response of the system to the time-periodic magnetic field. A summary is given at the end of this work.

## 2. Evolution equation for the magnetic flux

Our model system consists of a cylinder which is formed by a collection of rings — constituting the individual current channels. There are $N_z$ channels in the direction of the cylinder axis and $N_r$ in the direction of the cylinder radius. We assume that the thickness of the cylinder wall is small compared with the radius of the cylinder. Because of the mutual inductance between rings, the current in one ring induces a flux in the other rings. In turn, the flux induces a current. We assume that the rings are not contacted. Therefore, there is no tunneling of electrons among the channels and the charge carriers moving in the different rings are independent. It has been shown [19] that the effective interaction between the ring currents, when taken in the self-consistent mean field approximation, results in a magnetic flux $\phi_{\rm ind} = LI_{\rm tot}$ being felt by all electrons. Here, $L$ denotes the cylinder inductance and $I_{\rm tot}$ is the total current in the cylinder. At temperatures $T > 0$, the total magnetic flux $\phi$ consists of a sum of the external flux $\phi_{\rm ext}$ and the flux $\phi_{\rm ind}$ stemming from the total current. The latter is a sum of the dissipative 'normal' Ohmic current $I_{\rm nor}$ and the non-dissipative persistent current $I_{\rm coh}$, resulting from the presence of the 'phase-coherent' electrons in the system [13,20], *i.e.* it assumes the form:

$$\phi = \phi_{\rm ext} + LI_{\rm tot} = \phi_{\rm ext} + L[I_{\rm nor}(\phi,T) + I_{\rm coh}(\phi,T)]\,. \tag{1}$$

Note that $\phi_{\rm ext} \equiv \phi_{\rm ext}(t)$ is induced by an external magnetic field and can either take a fixed value or can be a time-dependent function. Taking into account an explicit form of the "normal" current, as it follows from the Lenz's and Ohm's rules complemented with the Johnson–Nyquist noise term [21], Eq. (1) reads [13,20]

$$\frac{1}{R}\frac{d\phi}{dt} = -\frac{1}{L}[\phi - \phi_{\rm ext}(t)] + I_{\rm coh}(\phi,T) + \sqrt{\frac{2k_{\rm B}T}{R}}\,\Gamma(t)\,, \tag{2}$$

where $R$ is the resistance of the cylinder, $k_{\rm B}$ is the Boltzmann constant and $\Gamma(t)$ denotes a zero-mean Gaussian delta-correlated white noise of unit intensity; *i.e.*, $\langle\Gamma(t)\Gamma(s)\rangle = \delta(t-s)$, modeling the Nyquist equilibrium current noise. This equation takes the form of a classical Langevin equation. The dimensionless form of (2) reads [13]

$$\dot{x} = -V'(x,\tilde{t}) + \sqrt{2D}\,\tilde{\Gamma}(\tilde{t})\,, \tag{3}$$

where the dot denotes a derivative with respect to the re-scaled time $\tilde{t} = t/\tau_0$ with $\tau_0 = L/R$ being the relaxation time of the averaged, normal current. The prime denotes a derivative with respect to the dimensionless flux $x = \phi/\phi_0$, where the flux quantum $\phi_0 = h/e$ is the ratio of the Planck constant $h$ and the elementary charge $e$. The generalized potential reads

$$V(x,\tilde{t}) = \frac{1}{2}x^2 - \lambda(\tilde{t})x - i_0F(x,p,T)\,, \tag{4}$$

where $\lambda(\tilde{t}) = \phi_{\rm ext}(t = \tau_0\tilde{t})/\phi_0$ is the re-scaled flux induced by an external magnetic field. The function $F(x,p,T) = \int f(x,p,T)dx$ characterizes the coherent current and $f(x,p,T) = pf_{\rm e}(x,T) + (1-p)f_{\rm o}(x,T)$, where

$f_{\rm e}(x,T) = \sum_{n=1}^{\infty} A_n(T)\sin(2n\pi x) = f_{\rm o}(x-1/2,T)$. The functions $f_{\rm e}$ and $f_{\rm o}$ describe the coherent current flowing in the channel with an even and odd number of electrons, respectively [8]. The amplitudes $A_n(T)$ are decreasing functions of the temperature. Their explicit forms are presented in [8, 13]. The quantity $p \in [0,1]$ denotes the probability of the occurrence of the single current channel with an even number of electrons. In the following we consider the most natural case $p = 1/2$.

The dimensionless intensity $D$ of re-scaled Gaussian white noise $\widetilde{\Gamma}(\tilde{t}) \equiv \sqrt{\tau_0}\,\Gamma(\tau_0\tilde{t})$ is

$$D = D(T) = \frac{k_{\rm B}T}{2\epsilon_0\delta_0} = \frac{\delta_0 T}{T^*}\,, \tag{5}$$

where the characteristic magnetic energy $\epsilon_0 = \phi_0^2/2L$ and the characteristic noise intensity $\delta_0 = k_{\rm B}T^*/2\epsilon_0$. The characteristic temperature $T^*$ is defined by the relation $k_{\rm B}T^* = \Delta_{\rm F}/2\pi^2$, where $\Delta_{\rm F}$ is the energy gap at the Fermi level. Moreover, the prefactor $i_0$ in (4) depends on the geometry and the material properties of the sample [22]. Although formally the above equations can also be applied to a single mesoscopic ring or toroid, we consider here a cylinder because in this case the prefactor $i_0$ can assume sufficiently large value (because $i_0 \propto N = N_z N_r$). We choose the parameters of the system in such a way that the diffusion coefficient $D \sim \delta_0 T/T^*$ for $\delta_0 \in [10^{-3}, 10^{-2}]$ and $i_0 = 1$. The values of parameters which occur in the amplitudes $A_n(T)$ are the same as in [13, 22]. We note that the resistance $R$ does not enter into the re-scaled equation (3).

As it follows from (1), the total current $I_{\rm tot}$ is linearly related to the magnetic flux $\phi$, *i.e.*, to the dimensionless flux $x$. As a consequence, the properties and behavior of the current follow from the properties and behavior of the magnetic flux. From now on, we will use only the dimensionless variables and omit the 'tilde' for the re-scaled time, $\tilde{t} \equiv t$. The temperature is measured throughout in units of $T^*$.

## 3. Fokker–Planck equation

The Langevin equation (3) defines a Markovian stochastic process $x(t)$. Its probability density $p(x,t)$ obeys the Fokker–Planck equation [23]; *i.e.*,

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}V'(x,t)p(x,t) + D\,\frac{\partial^2}{\partial x^2}p(x,t) \tag{6}$$

with an arbitrary initial condition $p(x,0)$ and the natural boundary conditions, reading $\lim_{|x|\to\infty} p(x,t) = 0$ (this is so because the potential $V(x,t)$ tends to infinity for $|x| \to \infty$). In the case of a time-independent external flux, $\lambda(t) = \lambda$, the corresponding stationary solutions were investigated

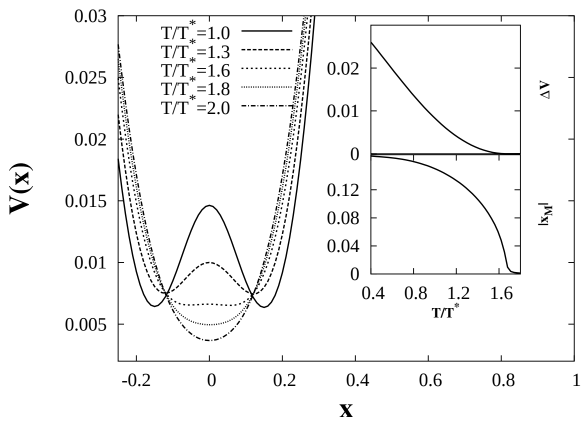


Fig. 1. The generalized potential $V(x,t) \equiv V(x)$ defined by Eq. (4) is depicted in the absence of the external driving, *i.e.* when $\lambda(t) = 0$ for selected values of the scaled temperature $T/T^*$. Upper inset: The corresponding barrier hight $\Delta V$ of the potential as a function of temperature. Lower inset: The position of the minimum $x_{\rm M}$ of the corresponding potential as a function of temperature.

in [13,22]. The dynamics of the system approaching thermal equilibrium has been studied in [24]. In the absence of external driving, *i.e.* when $\lambda(t) = 0$, the potential (4) can assume either a bistable or monostable behavior, see in Fig. 1. In the former case, the corresponding stationary probability density $p(x)$ exhibits two symmetric maxima at $(-x_{\rm M}, x_{\rm M})$ and a minimum at $x_{\rm m} = 0$. In the latter case, it possesses one maximum at $x_{\rm M} = 0$. The extremal values are clearly independent of an initial condition $p(x,0)$. The non-zero maxima of $p(x)$ correspond to the non-zero self-sustaining stationary fluxes and currents.

Let us next study the system dynamics when driven by an external, time-periodic, oscillatory magnetic field of a frequency $\omega$, *i.e.* the case when

$$\lambda(t) = a_0 \sin(\omega t)\,. \tag{7}$$

In this case, the process $x(t)$ becomes a non-stationary stochastic process and the long-time probability density $\lim_{t\to\infty} p(x,t)$ is a periodic function of time [14, 15, 25–28]. As a consequence, the statistical averages are periodic functions of time. In particular, the mean magnetic flux obeys

$$\langle x(t)\rangle = \left\langle x\left(t + 2\frac{\pi}{\omega}\right)\right\rangle \quad \text{for} \quad t\to\infty\,, \tag{8}$$

where

$$\langle x(t)\rangle = \int\limits_{-\infty}^{\infty} xp(x,t)\,dx\,. \tag{9}$$

The probability density $p(x,t)$ has been determined numerically from the Fokker–Planck equation (6). The evolution of $p(x,t)$ has been calculated with the time step $\Delta t = 10^{-4}$ starting out from the Gaussian distribution $p(x,0) = \exp(-x^2/2\sigma)/\sqrt{2\pi\sigma}$, with $\sigma = 0.005$. The results presented below correspond to sufficiently large times, when $p(x,t)$ becomes independent of the initial distribution [25, 29]. In doing so, we have used a grid of 2001 $x$-points uniformly distributed in the interval $[-5,5]$. This interval is much larger than the half-width of both the initial distribution $p(x,0)$ and the stationary distribution $p_{\mathrm{st}}(x)$ in the absence of the external force. The partial derivatives with respect to $x$-variable have been calculated with the help of the 5-point formula.

## 4. Resonant dynamics of the average magnetic flux

In bistable systems driven by a time-periodic forcing signal one can expect a resonant behavior termed the *stochastic resonance* [14, 15]: the response to the external periodic signal can be maximized by increasing the intensity of noise up to an optimal value. The phenomenon is well understood and constitutes a prominent example for the constructive role of noise.

In the following we show that there occurs a resonant enhancement of the output signal in the mesoscopic cylinder to the external, time-varying magnetic field. This effect is a result of an interplay between coherent and dissipative currents flowing in the cylinder as well as the periodic driving and the thermal noise. The phenomenon is somewhat different than the usual Stochastic Resonance occurring in a bistable system for which the potential does not depend on temperature [14]. Here, the generalized potential (4) depends on temperature: it is bistable for low temperatures and successively becomes monostable at higher temperatures. The input signal is characterized by the external periodic magnetic flux (7) of amplitude strength $a_0$. The output signal is characterized by the long-time mean magnetic flux (8), which as a periodic function of time oscillates between maximal and minimal values. The absolute extremal value $b_0 = |\langle x(t)\rangle|_{\max}$ compared to the amplitude $a_0$ denotes the (spectral) amplification factor $A_{\mathrm{f}} = b_0/a_0$ of the output signal [14, 25, 26].

Fig. 2 depicts the dependence of the amplification factor on temperature of the driven system. Different values of the characteristic noise intensity $\delta_0$ in (5) correspond to different characteristics of the cylinder, *i.e.* to different set-ups (*cf.* Eq. (20) in [22]). One can notice that there is an optimal temperature at which the output signal is maximal with the corresponding amplitude $b_0$ of the output being several times larger than the amplitude $a_0$ of the input. This Stochastic Resonance enhancement can be caused either by the occurrence of the bifurcation of the time-dependent solution

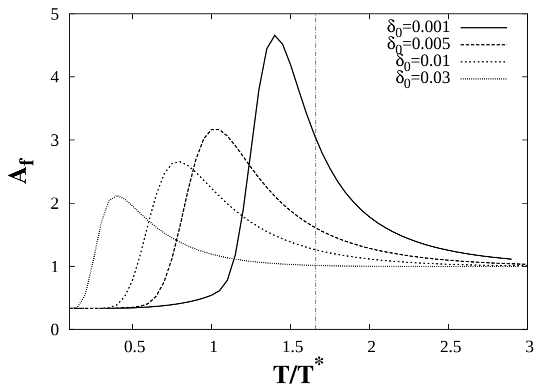


Fig. 2. The amplification factor (see in text and in Ref. [25]) is depicted *vs.* the scaled temperature for several values of the noise intensity $\delta_0$ and a fixed driving strength $a_0 = 0.01$. The vertical line depicts the critical temperature above which the generalized potential $V(x)$ in Fig. 1 becomes monostable.

or by noise. In the former case it is induced by the deterministic crossing of the potential barrier occurring at sufficiently high temperatures or also for a sufficiently large driving amplitude $a_0$ (note the inset in Fig. 1). This effect is detectable for small noise amplitudes, *e.g.* for $\delta_0 = 0.001$ in Fig. 2. The noise-dominated situation corresponds to larger values of $\delta_0$; then the resonant enhancement is due to the standard stochastic resonance mechanism. In the intermediate regime one can not distinguish whether the enhancement of the magnetic flux is caused mainly by 'deterministic' driving or synchronized 'stochastic' escape events.

There is a natural question how the observed resonance-behavior described above is related to the model Stochastic Resonance in continuous or discrete systems in the limit of small (with respect to both noise and the barrier) driving amplitude and small (with respect to the barrier height) noise intensity. The second assumption justifies the approximation of the rate by the so called Kramers rate [16, 17]. The leading linear response approximation for the amplification factor is given then by the relation [14, 25]:

$$A_{\mathrm{f}} = \frac{\langle x^2 \rangle_{|a_0=0}}{D(T)} \frac{2r_{\mathrm{k}}(T)}{\sqrt{4r_{\mathrm{k}}(T)^2 + \omega^2}}\,, \tag{10}$$

where $r_{\mathrm{k}}$ is the Kramers escape rate [16, 17] with temperature $T$ as a control parameter. The comparison between the precise numerical results and the linear response approximation (10) is presented in Fig. 3. Notice that the approximation (10) allows to predict the position of peaks of $A_{\mathrm{f}}$ with surprisingly good accuracy although the used parameters of the system dynamics are beyond of the range of validity of Eq. (10) [14].

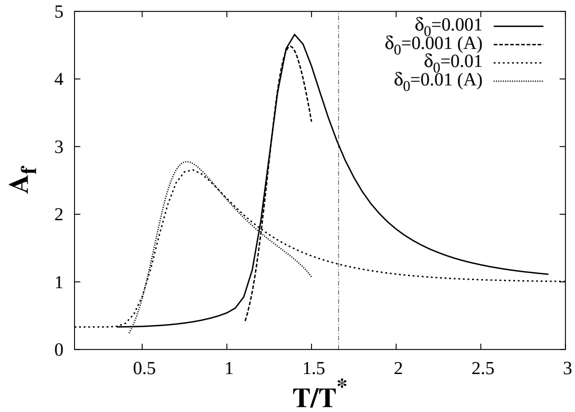


Fig. 3. The comparison of the Stochastic Resonance as quantified by the amplification factor between the numerical data with the analytic linear response formula (10), labeled by A.

The mean energy carried by the magnetic flux is one of the main characteristics of the system. It is proportional to the second statistical moment $\langle \phi^2(t)\rangle \sim \langle x^2(t)\rangle$. In the long time limit, it is also a periodic function of time, *i.e.*, $\langle x^2(t)\rangle = \langle x^2(t+2\pi/\omega)\rangle$. In some range of parameters, we could observe a non-monotonic behavior with a local minimum being exhibited as a function of temperature. Indeed, when the noise parameter $\delta_0$ is sufficiently small, *i.e.* when the coherent current dominates over the dissipative current, there exists an optimal temperature at which the amplitude of the magnetic energy is minimal. The numerical results are plotted in Fig. 4. The non-

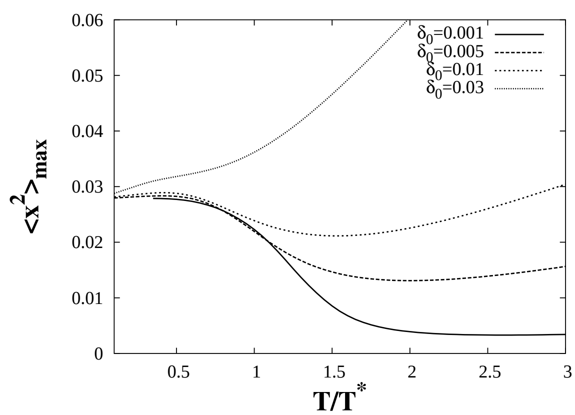


Fig. 4. Amplitude of the re-scaled mean magnetic energy $\langle x^2(t)\rangle$ *vs.* the temperature for several values of the noise strength $\delta_0$.

zero value at $T = 0$ is obvious because then the mean energy of the flux is roughly speaking proportional to the minimum $x_{\mathrm{M}}$ of the generalized potential. In high temperature, however, there are no coherent electrons left and the statistics is governed by the Gaussian distribution for which the second statistical moment becomes a linear function of the scaled temperature.

## 5. Summary

In this work we have investigated some unusual properties of the magnetic flux in mesoscopic cylinders and rings. We showed that due to interplay between coherent and dissipative currents and an external, deterministic periodic driving, one can obtain a substantial gain of the output signal, *i.e.*, the maximum of the amplitude of the mean magnetic flux. The effect is shown to be related both to the deterministic and stochastic properties of the system and, although closely related, is not identical to the stochastic resonance *sensu stricto*. The comparison of the numerical results with the approximate linear response result for the output amplitude of the periodically driven system depicts a satisfactory agreement from the qualitative point of view.

There are at least three reasons for studying persistent currents out of equilibrium. First *(i)*, the non-equilibrium behavior of the system can be radically different from the equilibrium behavior. Our investigations can, in principle, serve as a guideline for experiments on mesoscopic rings subjected to external time-dependent magnetic fields. The second *(ii)* reason is that the presence of peaks observed in the experiment could provide an indirect evidence for the existence of bistability occurring in the system, *i.e.* when the system is able to accommodate, intrinsic, *self-sustaining currents*. Finally *(iii)*, one can then even speculate that such resonant features contribute to the unexpected and yet still unresolved large amplitude of the experimentally measured persistent currents, being several times larger than predicted by present theories.

The work supported by the Foundation for Polish Science (JD, grant SN-302-934/2005), the ESF Program *Stochastic Dynamics: Fundamentals and Applications*, the DAAD-KBN collaboration program (*Stochastic Complexity*) (JŁ and PH) and the Polish State Committee for Scientific Research (KBN) grant PBZ-MIN-008/P03/2003 (JD and JŁ). We would like to acknowledge Peter Talkner for many elucidating, and most inspiring scientific and also non-scientific discussions. Peter Talkner is still young enough to continue doing great science and to identify exciting novel physics phenomena.

## REFERENCES


[1] A. Leggett, B. Ruggiero, P. Silvestrini (eds.), *Quantum Computing and Quantum Bits in Mesoscopic Systems*, Kluwer, New York 2004.

[2] J. Stolze, D. Suter, *Quantum Computing: A Short Course from Theory to Experiment*, Wiley-VCH, Berlin 2004.

[3] Y. Imry, *Introduction to Mesoscopic Physics*, Oxford University Press, New York 1997; E. Akkermans, G. Montambaux, J.-L. Pichard, J. Zinn-Justin (eds.), *Mesoscopic Quantum Physics*, North-Holland, Amsterdam 1995.

[4] S. Kohler, J. Lehmann, P. Hänggi, *Phys. Rep.* **406**, 379 (2005).

[5] T. Ludwig, A.D. Mirlin, Mesoscopic Aharonov-Bohm oscillations in metallic rings, in *Fundamental Problems of Mesoscopic Physics: Interactions and Decoherence*, eds. I.V. Lerner, B.L. Altshuler, Y. Gefen Kluwer, Dordrecht 2004, p.99, [cond-mat/0312577]; T. Dittrich, P. Hänggi, G.-L. Ingold, B. Kramer, G. Schön, W. Zwerger, *Quantum Transport and Dissipation*, Wiley-VCH, Weinheim 1998; B. Spivak *et al.*, *Phys. Rev. Lett.* **95**, 226804 (2005).

[6] F. Hund, *Ann. Phys. (Leipzig)* **32**, 102 (1938).

[7] V. Ambegaokar, U. Eckern, *Phys. Rev. Lett.* **65**, 381 (1990).

[8] H.F. Cheung, Y. Gefen, E.K. Riedel, W.H. Shih *Phys. Rev.* **B37**, 6050 (1989).

[9] L. Jorge *et al.*, *Phys. Rev.* **B39**, 3554 (1989); P.A. Orellana *et al.*, *Phys. Rev.* **B68**, 2195321 (2003).

[10] H.A. Cerdeira, B. Kramer, G. Schön (eds.), *Quantum Dynamics of Submicron Structures*, NATO ASI Series E, Vol. 291, Kluwer, London 1995; T. Ando, Y. Arakawa, K. Furuya, S. Komiyama, H. Nakashima (eds.), *Mesoscopic Physics and Electronics*, Springer, Berlin 1998; U. Eckern, P. Schwab, *J. Low. Temp. Phys.* **126**, 1291 (2002).

[11] K. Czajka, M.M. Maska, M. Mierzejewski, Z. Sledz, *Phys. Rev.* **B72**, 035320 (2005).

[12] L.P. Levy *et al.*, *Phys. Rev. Lett.* **64**, 2074 (1990); V. Chandrasekhar *et al.*, *Phys. Rev. Lett.* **67**, 3578 (1991); D. Mailly *et al.*, *Phys. Rev. Lett.* **70**, 2020 (1993); B. Reulet *et al.*, *Phys. Rev. Lett.* **75**, 124 (1995); E.M.Q. Jariwala *et al.*, *Phys. Rev. Lett.* **86**, 1594 (2001); W. Rabaut *et al.*, *Phys. Rev. Lett.* **86**, 3124 (2001); R. Deblock *et al.*, *Phys. Rev. Lett.* **89**, 206803 (2002).

[13] J. Dajka, J. Łuczka, M. Szopa, E. Zipper, *Phys. Rev.* **B67**, 073305 (2003).

[14] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).

[15] P. Hänggi, *Chem. Phys. Chem.* **3**, 285 (2002).

[16] P. Hänggi, P. Talkner, M. Borkovec, *Rev. Mod. Phys.* **62**, 251 (1990).

[17] P. Hänggi, *J. Stat. Phys.* **42**, 105 (1986); *J. Stat. Phys.* **44**, 1003 (1986); Addendum.

[18] P. Pechukas, P. Hänggi, *Phys. Rev. Lett.* **73**, 2772 (1994); P. Reimann, P. Hänggi, *Lecture Notes Phys.* **484**, 127 (1997).

[19] M. Lisowski, E. Zipper , M. Stebelski, *Phys. Rev.* **B59**, 8305 (1999).

[20] J. Łuczka, J. Dajka, P. Hänggi, *Acta Phys. Pol. B* **36**, 1693 (2005).

[21] J. Dajka, J. Łuczka, M. Szopa, P. Hänggi, *Phys. Stat. Sol. (B)* **242**, 196 (2005).

[22] J. Dajka, M. Kostur, J. Łuczka, M. Szopa, E. Zipper *Acta Phys. Pol. B* **34**, 3793 (2003).

[23] P. Hänggi , H. Thomas, *Phys. Rep.* **88**, 207 (1982).

[24] J. Dajka, J. Łuczka, M. Mierzejewski, P. Hänggi, *J. Phys.: Conf. Series* **30**, 321 (2006).

[25] P. Jung, P. Hänggi, *Phys. Rev.* **A44**, 8032 (1991).

[26] P. Jung, P. Hänggi, *Europhys. Lett.* **8**, 505 (1989).

[27] P. Talkner, *New J. Physics* **1**, 4 (1999).

[28] P. Talkner, J. Łuczka, *Phys. Rev.* **E69**, 046109 (2004).

[29] P. Jung, P. Hänggi, *Phys. Rev.* **A41**, 2977 (1990).