PRINCIPLES OF QUANTUM THEORY OF SPINOR FIELD IN RIEMANNIAN SPACE-TIME

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Basic properties of the quantum spinor field theory in the Riemannian space-time are described. Both real and complex fields are considered.

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The problem of field quantization in Riemannian space-time has been formulated and basically solved in work [1] for the case of the scalar field. In the present paper the problem is defined more exactly and solved for significantly more complicated case of the spinor field. We employ the results of Fock and Ivanenko who first formulated the Dirac equation in the Riemannian space-time [2–3].

1. The Dirac equation in the Cartan form

Following Cartan [4] we shall write the Dirac equation in the form

$$\left(i\hbar \sum_{\nu=0}^{3} H^{\nu} \frac{\partial}{\partial x^{\nu}} + mcH^{4}\right) \psi = 0, \tag{1}$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ are Cartesian coordinates in the Poincaré-Minkowski space-time, m — the electron-positron mass, c — the velocity of light, $\hbar$ — the Planck

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constant. The matrices H are

$$H^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$H^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad H^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H^4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

To pass to the original form of the Dirac equation, it is necessary to perform the following substitution of spinor components:

$$\psi_1 = \frac{\psi_2' - \psi_4'}{\sqrt{2}}, \quad \psi_2 = \frac{\psi_1' - \psi_3'}{\sqrt{2}}, \quad \psi_3 = \frac{\psi_1' + \psi_3'}{\sqrt{2}}, \quad \psi_4 = \frac{-\psi_2' - \psi_4'}{\sqrt{2}}.$$

The matrices H generate the Clifford algebra since

$$H^a H^b + H^b H^a = 2\eta^{ab}, \tag{3}$$

where $\eta^{ab} = 0$ for $a \neq b$, $\eta^{11} = \eta^{22} = \eta^{33} = -\eta^{00} = 1$. By means of the tensor η^{ab} and its inverse $\eta_{ab} = \eta^{ab}$ we shall raise and lower indices. For instance, $H_a = \eta_{ab} H^b$, $H^a = \eta^{ab} H_b$. We shall reserve the Latin letters for indices ranging over the values 0–4 and the Greek letters for those ranging over the values 0–3. We shall omit the summation sign corresponding to any index which occurs twice in a single term. In all subsequent equations a repeated index in any term therefore implies a summation over that index. For example:

$$\sum_{\nu=0}^{3} H^\nu \frac{\partial}{\partial x^\nu} = H^\nu \frac{\partial}{\partial x^\nu}, \quad \sum_{b=0}^{4} \eta_{ab} H^b = \eta_{ab} H^b.$$

The meaning of the tensors η_{ab} and η^{ab} is clear: $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ is the metric form of four-dimensional Poincaré-Minkowski space-time, and $ds^2 = \eta_{ab} dx^a dx^b$ is that of five-dimensional Poincaré-Minkowski space-time. Setting $\Psi = \psi e^{-\frac{imc}{\hbar} x_4}$ we may write Eq. (1) in the form

$$H^a \frac{\partial}{\partial x^a} \Psi = 0. \tag{4}$$

In addition to (3), we have for the matrices H

$$H_0 H_1 H_2 H_3 H_4 = i. \tag{5}$$

Consequently

$$\frac{1}{5!} \varepsilon_{abcpq} H^a H^b H^c H^p H^q = -i, \quad \frac{1}{4!} \varepsilon_{abcpq} H^b H^c H^p H^q = -i H_a. \tag{6}$$

$$\frac{1}{3!}\varepsilon_{abcpq}H^cH^pH^q = i[H_aH_b], \quad \frac{1}{2!}\varepsilon_{abcpq}H^pH^q = i[H_aH_bH_c],$$

$$\varepsilon_{abcpq}H^q = -i[H_aH_bH_cH_p], \quad \varepsilon_{abcpq} = -i[H_aH_bH_cH_pH_q],$$

where brackets denote the alternating product of matrices, ε_{abcpq} is a tensor skew symmetric in all its indices, $\varepsilon_{01234} = 1$. In particular,

$$\frac{1}{4!}\varepsilon_{\alpha\beta\mu\nu}H^\alpha H^\beta H^\mu H^\nu = -iH_4, \tag{7}$$

$$\frac{1}{3!}\varepsilon_{\alpha\beta\mu\nu}H^\beta H^\mu H^\nu = iH_4H_\alpha, \quad \frac{1}{2!}\varepsilon_{\alpha\beta\mu\nu}H^\mu H^\nu = iH_4[H_\alpha H_\beta],$$

$$\varepsilon_{\alpha\beta\mu\nu}H^\nu = -iH_4[H_\alpha H_\beta H_\mu], \quad \varepsilon_{\alpha\beta\mu\nu} = -iH_4[H_\alpha H_\beta H_\mu H_\nu],$$

where $\varepsilon_{\alpha\beta\mu\nu}$ is a skew symmetric tensor in all its indices, $\varepsilon_{0123} = 1$.

2. *Orthogonal basis in a Riemann space-time*

The Riemannian space-time is characterized by the metric form $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, where $g_{\alpha\beta}$ are arbitrary functions of coordinates x^α. By using some linear differential forms

$$f^\alpha = f^\alpha_\beta dx^\beta \tag{8}$$

the metric form can be diagonalized as $ds^2 = \eta_{\alpha\beta}f^\alpha f^\beta$. Solving Eq. (8) with respect to dx^α we have that

$$dx^\alpha = \tilde{f}^\alpha_\beta f^\beta, \tag{9}$$

where $f^\alpha_\gamma \tilde{f}^\gamma_\beta = \delta^\alpha_\beta$ and hence $\tilde{f}^\alpha_\gamma f^\gamma_\beta = \delta^\alpha_\beta$. The basis dual to f^α consists of vector fields

$$e_\alpha = \tilde{f}^\beta_\alpha \frac{\partial}{\partial x^\beta}. \tag{10}$$

We also have that

$$\frac{\partial}{\partial x^\alpha} = f^\beta_\alpha e_\beta. \tag{11}$$

Any vector field can be defined both in the coordinate basis $\frac{\partial}{\partial x}$ and in the basis e: $A^\alpha e_\alpha = a^\alpha \frac{\partial}{\partial x^\alpha}$; therefore $A^\alpha = a^\beta f^\alpha_\beta$, $a^\alpha = A^\alpha \tilde{f}^\alpha_\beta$. The covector field (linear form) can be given analogously in the coordinate basis dx and in the basis f: $A_\alpha f^\alpha = a_\alpha dx^\alpha$, hence $A_\alpha = a_\beta \tilde{f}^\beta_\alpha$, $a_\alpha = A_\beta f^\beta_\alpha$.

The covariant differential of the vector field is

$$\mathscr{D}A^\alpha = dA^\alpha + \omega^\alpha_\mu A^\mu \tag{12}$$

and the covariant differential of the covector field equals

$$\mathcal{D}A_\alpha = dA_\alpha - \omega^\mu_\alpha A_\mu, \tag{13}$$

where ω^α_μ are linear differential forms equal to $\omega^\alpha_\mu = \omega^\alpha_{\beta\mu} f^\beta$. The coefficients $\omega^\alpha_{\beta\mu}$ of these forms are called the components of the connection. Hence one gets the covariant derivatives

$$\mathcal{D}_\beta A^\alpha = e_\beta A^\alpha + \omega^\alpha_{\beta\mu} A^\mu,$$

$$\mathcal{D}_\beta A_\alpha = e_\beta A_\alpha - \omega^\mu_{\beta\alpha} A_\mu, \tag{14}$$

which makes it possible to write the covariant derivative for any tensor. For instance, for tensors of components $A^{\alpha\beta}$, A^α_β and $A_{\alpha\beta}$ we have that

$$\mathcal{D}_\gamma A^{\alpha\beta} = e_\gamma A^{\alpha\beta} + \omega^\alpha_{\gamma\mu} A^{\mu\beta} + \omega^\beta_{\gamma\mu} A^{\alpha\mu}, \tag{15}$$

$$\mathcal{D}_\gamma A^\alpha_\beta = e_\gamma A^\alpha_\beta + \omega^\alpha_{\gamma\mu} A^\mu_\beta - \omega^\mu_{\gamma\beta} A^\alpha_\mu,$$

$$\mathcal{D}_\gamma A_{\alpha\beta} = e_\gamma A_{\alpha\beta} - \omega^\mu_{\gamma\alpha} A_{\mu\beta} - \omega^\mu_{\gamma\beta} A_{\alpha\mu}.$$

The components of the connection are determined from two conditions. The first one is absence of torsion. For the scalar function φ this means $\mathcal{D}_\alpha e_\beta \varphi = \mathcal{D}_\beta e_\alpha \varphi$. Hence there follows the equation for the components of the connection

$$\omega^\gamma_{\alpha\beta} - \omega^\gamma_{\beta\alpha} = c^\gamma_{\alpha\beta}, \tag{16}$$

where the coefficients $c^\gamma_{\alpha\beta}$ are given by the Lie operation

$$e_\alpha e_\beta - e_\beta e_\alpha = c^\gamma_{\alpha\beta} e_\gamma \tag{17}$$

and consequently are equal to

$$c^\gamma_{\alpha\beta} = \tilde{f}^\mu_\alpha \tilde{f}^\nu_\beta \left(\frac{\partial f^\gamma_\mu}{\partial x^\nu} - \frac{\partial f^\gamma_\nu}{\partial x^\mu} \right) = \tilde{f}^\mu_\alpha e_\beta f^\gamma_\mu - \tilde{f}^\mu_\beta e_\alpha f^\gamma_\mu. \tag{18}$$

Without torsion an important corollary follows from (15)

$$\mathcal{D}_\alpha \mathcal{D}_\beta A_\gamma - \mathcal{D}_\beta \mathcal{D}_\alpha A_\gamma = R^\nu_{\gamma,\alpha\beta} A_\nu, \tag{19}$$

where

$$R^\nu_{\gamma,\alpha\beta} = e_\beta \omega^\nu_{\alpha\gamma} - e_\alpha \omega^\nu_{\beta\gamma} + c^\mu_{\alpha\beta} \omega^\nu_{\mu\gamma} + \omega^\mu_{\alpha\gamma} \omega^\nu_{\beta\mu} - \omega^\mu_{\beta\gamma} \omega^\nu_{\alpha\mu} \tag{20}$$

are the components of the Riemann-Christoffel tensor in the basis e, f.

The second condition for finding the components of the connection is the conservation of the metric tensor under the parallel translation. This implies that the covariant derivative $\mathcal{D}_\alpha \eta_{\beta\gamma}$ equals zero. Since $\alpha_\alpha \eta_{\beta\gamma} = 0$ then according to (15) we get one more equation for the components of the connection, viz.

$$\omega^\mu_{\alpha\beta} \eta_{\mu\gamma} + \omega^\mu_{\alpha\gamma} \eta_{\beta\mu} = 0. \tag{21}$$

Denoting

$$\omega_{\gamma\beta\alpha} = \eta_{\nu\mu}\omega^{\mu}_{\alpha\beta}, \qquad c_{\alpha\beta\mu} = c^{\gamma}_{\alpha\beta}\eta_{\gamma\mu}, \tag{22}$$

we obtain from (16) and (21) that

$$\omega_{\nu\beta\alpha} - \omega_{\nu\alpha\beta} = c_{\alpha\beta\nu}, \qquad \omega_{\nu\beta\alpha} + \omega_{\beta\nu\alpha} = 0. \tag{23}$$

Inserting (23) into the identity

$$\begin{aligned}\omega_{\nu\beta\alpha} \equiv {} & \tfrac{1}{2}(\omega_{\nu\beta\alpha}+\omega_{\beta\nu\alpha})+\tfrac{1}{2}(\omega_{\alpha\nu\beta}+\omega_{\nu\alpha\beta})-\tfrac{1}{2}(\omega_{\alpha\beta\nu}+\omega_{\beta\alpha\nu}) \\ & +\tfrac{1}{2}(\omega_{\alpha\beta\nu}-\omega_{\alpha\nu\beta})+\tfrac{1}{2}(\omega_{\nu\beta\alpha}-\omega_{\nu\alpha\beta})-\tfrac{1}{2}(\omega_{\beta\nu\alpha}-\omega_{\beta\alpha\nu})\end{aligned} \tag{24}$$

we arrive at

$$\omega_{\nu\beta\alpha} = \tfrac{1}{2}(c_{\nu\beta\alpha}+c_{\alpha\beta\nu}-c_{\alpha\nu\beta}).$$

Likewise the metric tensor, the matrices H do not alter under the parallel translation. This means that their covariant derivatives are zero. For the matrix $A = A_{\alpha}H^{\alpha}$ we have that

$$\mathscr{D}A = dA+\omega^{\alpha}_{\mu}A^{\mu}H_{\alpha} = dA+\Omega A-A\Omega, \tag{25}$$

where

$$\Omega = \tfrac{1}{4}\,\omega_{\alpha\mu\nu}f^{\nu}H^{\alpha}H^{\mu}. \tag{26}$$

Since for the matrix A (up to higher-order infinitesimals)

$$\mathscr{D}A-dA+A = (1+\Omega)A(1+\Omega)^{-1},$$

then for a spinor the relation should hold

$$\mathscr{D}\psi-d\psi+\psi = (1+\Omega)\psi.$$

Consequently, the covariant differential of a spinor equals

$$\mathscr{D}\psi = d\psi+\tfrac{1}{4}\,\omega_{\alpha\mu\nu}f^{\nu}H^{\alpha}H^{\mu}\psi \tag{27}$$

and the covariant derivative of a spinor is

$$\mathscr{D}_{\nu}\psi = e_{\nu}\psi+\tfrac{1}{4}\,\omega_{\alpha\mu\nu}H^{\alpha}H^{\mu}\psi. \tag{28}$$

The covariant derivative of the conjugate spinor $\overline{\psi} = \psi^{*}H_0$ has the form

$$\mathscr{D}_{\nu}\overline{\psi} = e_{\nu}\overline{\psi}-\overline{\psi}\,\tfrac{1}{4}\,\omega_{\alpha\mu\nu}H^{\alpha}H^{\mu}. \tag{29}$$

There occur objects both of the spinorial and tensorial nature. The rules (14), (28), and (29) allow one to find their covariant derivatives. For example, the covariant derivative of a spinor possesses both the spinorial and vectorial character. The second covariant derivative of a spinor is equal to

$$\mathscr{D}_{\gamma}\mathscr{D}_{\nu}\psi = e_{\gamma}\mathscr{D}_{\nu}\psi+\tfrac{1}{4}\,\omega_{\alpha\mu\gamma}H^{\alpha}H^{\mu}\mathscr{D}_{\nu}\psi-\omega^{\mu}_{\gamma\nu}\mathscr{D}_{\mu}\psi. \tag{30}$$

As for the vector case, the alternating second covariant derivative of a spinor does not contain its derivatives and is expressed by the Riemann-Christoffel tensor

$$\mathscr{D}_\alpha \mathscr{D}_\beta \psi - \mathscr{D}_\beta \mathscr{D}_\alpha \psi = \tfrac{1}{4} H^\mu H^\nu R_{\mu\nu,\alpha\beta} \psi, \tag{31}$$

where

$$R_{\mu\nu,\alpha\beta} = \eta_{\nu\sigma} R^\delta_{\mu,\alpha\beta}. \tag{32}$$

To prove the formula (31), one should use the following relation

$$\tfrac{1}{2}\{[H^\alpha H^\beta][H^\mu H^\nu] - [H^\mu H^\nu][H^\alpha H^\beta]\}$$
$$= \eta^{\mu\beta}[H^\alpha H^\nu] - \eta^{\alpha\mu}[H^\beta H^\nu] + \eta^{\nu\beta}[H^\mu H^\alpha] - \eta^{\nu\alpha}[H^\mu H^\beta]. \tag{33}$$

3. The Dirac equation in the Riemannian space-time

The Dirac equation in the Riemannian space-time is obtained from (1) by replacement of $\frac{\partial \psi}{\partial x^\nu}$ by $\mathscr{D}_\nu \psi$:

$$H^\nu \mathscr{D}_\nu \psi = \frac{imc}{\hbar} H^4 \psi. \tag{34}$$

Just as in the plane case, this equation can be written in the conjugate form

$$\mathscr{D}_\nu \bar{\psi} H^\nu = -\frac{imc}{\hbar} \bar{\psi} H^4. \tag{35}$$

The so-called "square" of the Dirac equation is obtained in the following way. We have

$$\left(H^\alpha \mathscr{D}_\alpha - \frac{imc}{\hbar} H^4\right)\left(H^\beta \mathscr{D}_\beta - \frac{imc}{\hbar} H^4\right) = H^\alpha H^\beta \mathscr{D}_\alpha \mathscr{D}_\beta - \frac{m^2 c^2}{\hbar^2}$$

and, furthermore,

$$H^\alpha H^\beta \mathscr{D}_\alpha \mathscr{D}_\beta = H^\alpha H^\beta \frac{\mathscr{D}_\alpha \mathscr{D}_\beta + \mathscr{D}_\beta \mathscr{D}_\alpha}{2} + H^\alpha H^\beta \frac{\mathscr{D}_\alpha \mathscr{D}_\beta - \mathscr{D}_\beta \mathscr{D}_\alpha}{2}.$$

The first term in the last expression equals

$$H^\alpha H^\beta \frac{\mathscr{D}_\alpha \mathscr{D}_\beta + \mathscr{D}_\beta \mathscr{D}_\alpha}{2} = \frac{H^\alpha H^\beta + H^\beta H^\alpha}{2} \frac{\mathscr{D}_\alpha \mathscr{D}_\beta + \mathscr{D}_\beta \mathscr{D}_\alpha}{2} = \eta^{\alpha\beta} \mathscr{D}_\alpha \mathscr{D}_\beta.$$

The second term according to (31) is

$$H^\alpha H^\beta \frac{\mathscr{D}_\alpha \mathscr{D}_\beta - \mathscr{D}_\beta \mathscr{D}_\alpha}{2} = \tfrac{1}{8} H^\alpha H^\beta H^\mu H^\nu R_{\mu\nu,\alpha\beta}.$$

Since

$$H^{\alpha}H^{\beta}H^{\mu} = [H^{\alpha}H^{\beta}H^{\mu}]+\eta^{\beta\mu}H^{\alpha}+\eta^{\beta\alpha}H^{\mu}-\eta^{\alpha\mu}H^{\beta} \tag{36}$$

and the alternation of the Riemann-Christoffel tensor in three indices gives zero,

$$H^{\alpha}H^{\beta}\frac{\mathscr{D}_{\alpha}\mathscr{D}_{\beta}-\mathscr{D}_{\beta}\mathscr{D}_{\alpha}}{2} = \tfrac{1}{4}\, H^{\alpha}H^{\nu}\eta^{\beta\mu}R_{\mu\nu,\alpha\beta} = \tfrac{1}{4}\, H^{\alpha}H^{\nu}R_{\nu\alpha} = \tfrac{1}{4}\, R.$$

Thereby we obtain the "squared" Dirac equation

$$\left(\eta^{\alpha\beta}\mathscr{D}_{\alpha}\mathscr{D}_{\beta}+\tfrac{1}{4}\, R-\frac{m^2c^2}{\hbar^2}\right)\psi = 0. \tag{37}$$

In the general case this system of equations of second order by no means breaks down into four individual equations for each component of a spinor.

The Dirac equation is simplified significantly for the case of the orthogonal coordinates (if those are available), when it is possible to construct the Lamé basis, i.e. to put $f_{\beta}^{\alpha} = h^{\alpha}\delta_{\beta}^{\alpha}$. By using the formula (36) we obtain that

$$\tfrac{1}{4}\,\omega_{\alpha\mu\nu}H^{\nu}H^{\alpha}H^{\mu} = \tfrac{1}{4}\,\omega_{[\alpha\mu\nu]}H^{\nu}H^{\alpha}H^{\mu}+\tfrac{1}{2}\,\eta^{\alpha\nu}\omega_{\alpha\mu}H^{\mu}.$$

Since

$$\omega_{[\alpha\mu\nu]} = \tfrac{1}{2}\, c_{[\alpha\mu\nu]}, \qquad \eta^{\alpha\nu}\omega_{\alpha\mu\nu} = c_{\alpha\mu}^{\alpha},$$

and, in the Lamé basis

$$c_{\alpha\beta}^{\gamma} = \frac{1}{h^{\alpha}h^{\beta}}\left[\delta_{\alpha}^{\gamma}\frac{\partial h^{\gamma}}{\partial x^{\beta}}-\delta_{\beta}^{\gamma}\frac{\partial h^{\gamma}}{\partial x^{\alpha}}\right]$$

and, consequently,

$$c_{[\alpha\mu\nu]} = 0, \quad \tfrac{1}{2}\, c_{\alpha\mu}^{\alpha} = \frac{1}{\sqrt{hh^{\mu}}}\frac{\partial}{\partial x^{\mu}}\sqrt{\frac{h}{h^{\mu}}},$$

where $h = h^0h^1h^2h^3$, then in the Lamé basis

$$\tfrac{1}{4}\,\omega_{\alpha\mu\nu}H^{\nu}H^{\alpha}H^{\nu} = \sum_{\nu=0}^{3}\frac{H^{\mu}}{\sqrt{hh^{\mu}}}\frac{\partial}{\partial x^{\mu}}\sqrt{\frac{h}{h^{\mu}}}.$$

In the Lamé basis Eq. (34) has thus the following form

$$\sum_{\mu=0}^{3}\frac{H^{\mu}}{\sqrt{hh^{\mu}}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{\frac{h}{h^{\mu}}}\,\psi\right) = \frac{imc}{\hbar}\, H^{4}\psi \tag{38}$$

and Eq. (35) is written as

$$\sum_{\mu=0}^{3}\frac{1}{\sqrt{hh^{\mu}}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{\frac{h}{h^{\mu}}}\,\bar{\psi}\right)H^{\mu} = -\frac{imc}{\hbar}\,\bar{\psi}H^{4}. \tag{39}$$

4. *Anticommutator of two spinor fields*

In order to preserve the fundamental concepts of quantum field theory, we will consider only those Riemannian space-times which admit the space-like hypersurfaces dividing the space-time into two parts. One of these separated parts may be regarded as the Past, another — the Future, and the dividing hypersurface itself — the Present. Such hypersurfaces will be called complete. We shall suppose that a solution of the Dirac equation throughout the whole space-time is given uniquely by values of a spinor field on a complete hypersurface. On the hypersurface itself, however, a spinor field may be given in an arbitrary way.

Consider now the system which consists of the Dirac equation and its conjugate

$$H^{\nu}\mathscr{D}_{\nu}u = \frac{imc}{\hbar} H^{4}u, \qquad \mathscr{D}_{\nu}\bar{v}H^{\nu} = -\frac{imc}{\hbar}\bar{v}H^{4}. \tag{40}$$

Let u, $\bar{v}$ be the solution of the system. The divergence of the vector $S^{\nu} = -\bar{v}H^{\nu}u$, which is equal to $\mathscr{D}_{\nu}S^{\nu} = -(\mathscr{D}_{\nu}\bar{v})H^{\nu}u - \bar{v}H^{\nu}(\mathscr{D}_{\nu}u)$, vanishes because of (40). Hence, it follows that all the integrals

$$\int_{\Sigma} S_{\mu}d\sigma^{\mu} = \int_{\Sigma} \begin{vmatrix} q_1^0 & q_1^1 & q_1^2 & q_1^3 \\ q_2^0 & q_2^1 & q_2^2 & q_2^3 \\ q_3^0 & q_3^1 & q_3^2 & q_3^3 \\ S^0 & S^1 & S^2 & S^3 \end{vmatrix} \tag{41}$$

over complete hypersurfaces are equal to each other. In the integral (41) q_1^{α}, q_2^{α}, q_3^{α} are vectors of elementary displacements along the hypersurface Σ. If e.g. the hypersurface is defined by the equations $x^{\alpha} = T^{\alpha}(q^1, q^2, q^3)$ then $q_1^{\alpha} = f_{\mu}^{\alpha}\frac{\partial T^{\mu}}{\partial q^1}dq^1, \dots$. In other words, the integral (41) equals the integral of the exterior form [5]

$$\int_{\Sigma} S_{\mu}d\sigma^{\mu} = \int_{\Sigma} \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\mu}S^{\mu}f^{\alpha}\wedge f^{\beta}\wedge f^{\gamma}. \tag{42}$$

According to (7) the form integrated here is as follows

$$\frac{1}{3!}\varepsilon_{\alpha\beta\gamma\mu}S^{\mu}f^{\alpha}\wedge f^{\beta}\wedge f^{\gamma} = \frac{i}{3!}\bar{v}H_4H_{\alpha}H_{\beta}H_{\gamma}uf^{\alpha}\wedge f^{\beta}\wedge f^{\gamma}$$

$$= \frac{i}{3!}\bar{v}H_4F\wedge F\wedge Fu, \tag{43}$$

where $F = H_{\alpha}f^{\alpha}$. Thus, the integral (41) can be written in the form

$$\int_{\Sigma} S_{\mu}d\sigma^{\mu} = i\int_{\Sigma} \bar{v}H_4[Q_1Q_1Q_3]u, \tag{44}$$

where

$$Q_1 = H_\alpha q_1^\alpha, \quad Q_2 = H_\alpha q_2^\alpha, \quad Q_3 = H_\alpha q_3^\alpha.$$

This integral defines the scalar product in the space of solutions of the system (40).

The spinor field is quantized using the Fermi statistics. That is to say, a pair of spinor fields ψ, $\overline{\psi}$ are required to generate the (infinite-dimensional) Clifford algebra. The algebra generators ψ, $\overline{\psi}$ along the complete hypersurface are linearly independent. Now consider the following vectors from the linear envelope of algebra generators

$$U = i\int_\Sigma \overline{\psi} H_4[Q_1Q_2Q_3]u, \qquad V^* = i\int_\Sigma \bar{v} H_4[Q_1Q_2Q_3]\psi. \tag{45}$$

The integrals (45) as well as the integral (44) do not depend on the complete hypersurface chosen, because $u, \overline{\psi}$ and $\psi, \bar{v}$ obey the equations (40). The sum $U+V^*$ represents the common element of the envelope. It should be stressed here that the spinor fields $u, \bar{v}$ keep on being nonquantized. Setting

$$(U+V^*)^2 = i\int_\Sigma \bar{v} H_4[Q_1Q_2Q_3]u, \tag{46}$$

we introduce the symmetric scalar product in the generator envelope which exactly expresses the quantization principle by the Fermi statistics.

Since the pairs $u, 0$ and $0, \bar{v}$ satisfy Eqs (40) then from (46), it follows that

$$U^2 = 0, \qquad V^{*2} = 0 \tag{47}$$

and, consequently, the anticommutator $\{UV^*\}$ is

$$\{UV^*\} = UV^* + V^*U = i\int_\Sigma \bar{v} H_4[Q_1Q_2Q_3]u. \tag{48}$$

Substituting (45) for V^* into (48), we obtain (because of the arbitrariness of $\bar{v}$ on Σ) that on Σ

$$\{\psi(x)U\} = u(x). \tag{49}$$

Inserting (45) for U into (48) we get that

$$\{V^*\overline{\psi}(x)\} = \bar{v}(x). \tag{50}$$

Since the pair $\psi, \overline{\psi}$ obeys the system (40), then the pair $\{\psi U\}, \{V^*\overline{\psi})$ obeys the same system (40), as well. However, this system is also satisfied by the pair $u, \bar{v}$. So far as the two latter pairs coincide on Σ, then due to the supposed uniqueness of the Cauchy problem solution for (40), the equalities (49) and (50) prove to be correct not only on Σ but also throughout the whole space-time. Similarly, one can deduce from (47), that for any space-time point the following equalities hold:

$$\{\psi(x)V^*\} = 0, \qquad \{U\overline{\psi}(x)\} = 0. \tag{51}$$

Because of the arbitrariness of u, $\bar{v}$ on Σ, we obtain from (51) for any point y from Σ.

$$\{\psi_p(x)\psi_q(y)\} = 0, \quad \{\bar{\psi}_q(y), \bar{\psi}_p(x)\} = 0, \tag{52}$$

where p and q numerate components of the spinors ψ and $\bar{\psi}$. In virtue of the uniqueness of the solution for the Cauchy problem for (40), the equalities (52) prove to be valid for any world point y too.

Let us denote through $\{\psi(x)\bar{\psi}(y)\}$ the matrix with components $\{\psi_p(x)\bar{\psi}_q(y)\}$. This matrix satisfies the following algebraic condition

$$\{\psi(x)\bar{\psi}(y)\}H^0 = H_0\{\psi(y)\bar{\psi}(x)\}^\dagger, \tag{53}$$

where "†" means the hermitian conjugation of the matrix. From (45) and (48) it follows, that $\{\psi(x)\bar{\psi}(y)\}$ provided it is possible to draw a complete hypersurface through two (different) world points x and y. On writting (49) and (50) in the developed form

$$u(x) = i\int_\Sigma \{\psi(x)\bar{\psi}(y)\}H_4[Q_1Q_2Q_3]u(y),$$

$$\bar{v}(x) = i\int_\Sigma \bar{v}(y)H_4[Q_1Q_2Q_3]\,\{\psi(y)\bar{\psi}(x)\}, \tag{54}$$

we see that the anticommutator $\{\psi(x)\bar{\psi}(y)\}$ gives a solution of the Cauchy problem for the system (40). Since the anticommutator $\{\psi(x)\bar{\psi}(y)\}$ satisfies the same system itself, then in accordance with (54) we arrive at

$$\{\psi(x)\bar{\psi}(y)\} = i\int_\Sigma \{\psi(x)\bar{\psi}(z)\}H_4[Q_1Q_2Q_3]\,\{\psi(z)\bar{\psi}(y)\}. \tag{55}$$

Now, let for any complete hypersurface Σ and the system (40) the Cauchy problem be solved by some method

$$u(x) = i\int_\Sigma \bar{S}(x, y)H_4[Q_1Q_2Q_3]u(y), \tag{56}$$

$$\bar{v}(x) = i\int_\Sigma \bar{v}(y)H_4[Q_1Q_2Q_3]S(y, x).$$

Comparing (56) with (54), we find $\{\psi(x)\bar{\psi}(y)\} = \bar{S}(x, y)$, $\{\psi(y)\bar{\psi}(x)\} = S(y, x)$ on the direct product $M\times\Sigma$, where M is the whole space-time. In virtue of (53) the functions $\bar{S}$ and S are related via the condition $\bar{S}(x, y)H^0 = H_0S^\dagger(y, x)$. Following (55) we find that

$$\{\psi(x)\bar{\psi}(y)\} = i\int_\Sigma \bar{S}(x, z)H_4[Q_1Q_2Q_3]S(z, y) \tag{57}$$

on $M\times M$.

It should be noted that if the pair $(u, \bar{v})$ is a solution of (40), then the pair $(u, \bar{v})^* = (v, \bar{u})$ is a solution of (40) as well. The solution is called real, if $(u, \bar{v})^* = (u, \bar{v})$ i.e. $u = v$. The element $U+U^*$ of the generator envelope which corresponds to the real solution, is called real or hermitian.

5. *Commutator of two scalar fields*

It is of interest to compare the spinor case with the scalar one. Probably, the requirement of the uniqueness of a solution of the Cauchy problem for the system (40) coincides with that for the scalar equation

$$\eta^{\alpha\beta}\mathscr{D}_\alpha\mathscr{D}_\beta\Phi+\frac{R}{6}\,\Phi=\left(\frac{mc}{\hbar}\right)^2\Phi, \tag{58}$$

considered in the work [1]. Let u, v be two solutions of this equation. Then the divergence of the vector $S_\mu = (u\mathscr{D}_\mu v - v\mathscr{D}_\mu u)$ is equal to zero, and hence it follows that all the integrals (42) over complete hypersurfaces are mutually equal. Their common value defines antisymmetric scalar product in a space of solutions of Eq. (58). A scalar field Φ is quantized using the Bose statistics. This means that the values of Φ on the complete hypersurface Σ and the values of its normal derivative are both regarded as generators of an algebra which is an infinite-dimensional analogue of the algebra of quantum mechanics. The common element of the linear envelope of generators has the following form

$$U=\int\limits_\Sigma (u\Phi_\mu-\Phi u_\mu)d\sigma^\mu, \tag{59}$$

where $\Phi_\mu = \mathscr{D}_\mu\Phi$, $u_\mu = \mathscr{D}_\mu u$. The integral (59) does not depend on the choice of the complete hypersurface, since u and Φ both satisfy Eq. (58). The scalar field u keeps on to be regarded as nonquantized. Putting

$$\langle UV\rangle = UV-VU = i\hbar\int\limits_\Sigma (uv_\mu - vu_\mu)d\sigma^\mu \tag{60}$$

for any two elements U, V of the type (59), we introduce the antisymmetric scalar product in the envelope of generators which is the exact expression of the quantization principle for the Bose statistics.

If one inserts into (60) an expression of the type (59) for V, then one can notice without difficulty the following. Since the functions v and $v_{(n)} = v_\alpha n^\alpha$, where n^α is a normal to Σ, may take arbitrary values on Σ, we get that on Σ

$$\langle\Phi(x)U\rangle = i\hbar u(x),\qquad \langle\Phi_{(n)}(x)U\rangle = i\hbar u_{(n)}(x).$$

However, the commutator $\langle\Phi(x)U\rangle$ obeys Eq. (58) together with $\Phi(x)$. From the uniqueness of the solution of the Cauchy problem for Eq. (58) it follows that

$$\langle\Phi(x)U\rangle = i\hbar u(x) \tag{61}$$

for any world point x. On writing this equation in the developed form

$$u(x)=\int\limits_\Sigma [\Delta(x,y)u_\mu(y)-\Delta_\mu(x,y)u(y)]d\sigma^\mu \tag{62}$$

we see that the commutator

$$\Delta(x,y)=\frac{i}{\hbar}\langle\Phi(x)\Phi(y)\rangle = -\Delta(y,x) \tag{63}$$

provides a solution of the Cauchy problem for Eq. (58). $\Delta_\mu(x, y)$ means the covariant derivative of $\Delta(x, y)$ with respect to the second argument. Because the commutator $\Delta(x, y)$ itself satisfies Eq. (58), we have by (62) that

$$\Delta(x, y) = \int_\Sigma [\Delta_\mu(y, z)\Delta(z, x) - \Delta_\mu(x, z)\Delta(z, y)]d\sigma^\mu. \tag{64}$$

Now let for any complete hypersurface Σ the Cauchy problem for Eq. (58) be solved by some method:

$$u(x) = \int_\Sigma [T(x, y)u_\mu(y) - T_\mu(x, y)u(y)]d\sigma^\mu. \tag{65}$$

Then comparing (65) with (62), we notice that $\Delta(x, y) = T(x, y)$, $\Delta_\mu(x, y)\, n^\mu = T_\mu(x, y)n^\mu$ on $M \times \Sigma$. Following (64) and (63) we obtain that

$$\frac{i}{\hbar}\langle \Phi(x)\Phi(y)\rangle = \int_\Sigma [T_\mu(x, z)T(y, z) - T_\mu(y, z)T(x, z)]d\sigma^\mu \tag{66}$$

on $M \times M$. The hermiticity condition for the case of the real field Φ under consideration is formulated very simply: an element of the type (59) is called real or hermitian if u is a real solution of Eq. (58).

In the case of the complex scalar field φ, we should consider the system of equations

$$\eta^{\alpha\beta}\mathscr{D}_\alpha\mathscr{D}_\beta u + \frac{R}{6}u = \left(\frac{mc}{\hbar}\right)^2 u,$$

$$\eta^{\alpha\beta}\mathscr{D}_\alpha\mathscr{D}_\beta v^* + \frac{R}{6}v^* = \left(\frac{mc}{\hbar}\right)^2 v^* \tag{67}$$

and the pair (φ, φ^*) is to obey this system. The common element of the envelope of generators has the form $U + V^*$ where

$$U = \int_\Sigma (u\varphi_\mu^* - \varphi^* u_\mu)d\sigma^\mu, \tag{68}$$

$$V^* = \int_\Sigma (v^*\varphi_\mu - \varphi v_\mu^*)d\sigma^\mu.$$

The antisymmetric scalar product in the envelope is introduced by the condition

$$\langle U_1 + V_1^*, U_2 + V_2^*\rangle = i\hbar \int_\Sigma (u_1 v_{2\mu}^* - v_2^* u_{1\mu} - u_2 v_{1\mu}^* + v_1^* u_{2\mu})d\sigma^\mu. \tag{69}$$

Hence it follows that

$$\langle U_1 U_2\rangle = 0, \quad \langle V_1^* V_2^*\rangle = 0 \tag{70}$$

and

$$\langle U V^*\rangle = i\hbar \int_\Sigma (u v_\mu^* - v^* u_\mu)d\sigma^\mu. \tag{71}$$

From (70) we obtain

$$\langle\varphi(x)\varphi(y)\rangle = 0, \quad \langle\varphi^*(x)\varphi^*(y)\rangle = 0. \tag{72}$$

From (71) we obtain

$$\langle\varphi(x)U\rangle = i\hbar u(x), \quad \langle\varphi^*(x)V^*\rangle = i\hbar v^*(x). \tag{73}$$

By writing the two latter equalities in the developed form, we derive the solution of the Cauchy problem for the system (67). But as this system is twice repeated Eq. (58), we infer, by comparing with (62), that

$$\frac{i}{\hbar}\langle\varphi(x)\varphi^*(y)\rangle = \frac{i}{\hbar}\langle\varphi^*(x)\varphi(y)\rangle = \frac{i}{\hbar}\langle\phi(x)\phi(y)\rangle = \Delta(x, y). \tag{74}$$

If the pair (u, v^*) satisfies the system (67), then the pair $(u, v^*)^* = (v, u^*)$, obviously, obeys the same system. A solution of the system (67) is called real if $(u, v^*)^* = (u, v^*)$ i.e., $u = v$. The element $U+U^*$ of the generator envelope corresponding to the real solution, is called real or hermitian.

6. Current vector and charge operator

The complex field is well adjusted to describe charged particles. In the scalar case, the current vector is defined by (71) and equals

$$J_\mu = \frac{ie}{\hbar}(\varphi^*\varphi_\mu - \varphi_\mu^*\varphi). \tag{75}$$

For the real field $\phi = \phi^*$ and the current vector vanishes. In the spinor case, the current vector is given by (46) and is equal to

$$J_\mu = e\bar{\psi}H_\mu\psi. \tag{76}$$

In both cases, the charge operator is defined via the integral

$$\hat{e} = \int_\Sigma J_\mu d\sigma^\mu \tag{77}$$

over the complete hypersurface Σ and does not depend upon the choice of Σ, since $\mathscr{D}_\mu J^\mu = 0$. In the spinor case, the integral (77) can be written as

$$\hat{e} = -ie\int_\Sigma \bar{\psi}H_4[Q_1Q_2Q_3]\psi. \tag{78}$$

In the formulae (76) and (78) $-e$ is the electric charge of an electron, in (75) e is the electric charge of a meson.

7. Angular momentum operators

The Killing vector field K^α obeys the equation

$$\mathscr{D}_\mu K_\nu + \mathscr{D}_\nu K_\mu = 0. \tag{79}$$

Because of the identity (19), $\mathscr{D}_\alpha\mathscr{D}_\mu K_\nu - \mathscr{D}_\mu\mathscr{D}_\alpha K_\nu = R^\beta_{\nu,\alpha\mu}K_\beta$ and by differentiating the Killing equation we obtain that $\mathscr{D}_\alpha\mathscr{D}_\mu K_\nu + \mathscr{D}_\alpha\mathscr{D}_\nu K_\mu = 0$. Hence, as a result of the trick applied to both Eqs (23), it follows that

$$\mathscr{D}_\alpha\mathscr{D}_\mu K_\nu = \tfrac{1}{2}(R^\beta_{\alpha,\nu\mu} + R^\beta_{\nu,\alpha\mu} - R^\beta_{\mu,\alpha\nu})K_\beta = R^\beta_{\alpha,\nu\mu}K_\beta. \tag{80}$$

Eq. (79) possesses a nontrivial solution if and only if there is an isometry group in the space--time. Therefore the equality $f^{\alpha'}_{\beta'}(x') = f^\alpha_\beta(x)$ holds. In this equality, prime marks the result of an operation from the group. For the infinitesimal transformation $x^{\alpha'} = x^\alpha + K^\mu f^\alpha_\mu t$ this implies that $K^\mu e_\mu f^\alpha_\beta = 0$. Consider the corresponding transformation of the basis

$$f^{\alpha'} = f^{\alpha'}_{\beta'}(x')dx^{\beta'} = f^\alpha_\beta[dx^\beta + dK^\mu \tilde{f}^\beta_\mu t + K^\mu df^\beta_\mu t].$$

Since $c^\alpha_{\mu\nu} = \tilde{f}^\beta_\mu e_\nu f^\alpha_\beta - \tilde{f}^\beta_\nu e_\mu f^\alpha_\beta$, $K^\mu c^\alpha_{\mu\nu} = K^\mu \tilde{f}^\beta_\mu e_\nu f^\alpha_\beta$. However, $f^\alpha_\beta d\tilde{f}^\beta_\mu = -\tilde{f}^\beta_\mu df^\alpha_\beta$. Consequently $f^{\alpha'} = f^\alpha + dK^\alpha t - K^\mu c^\alpha_{\mu\nu} f^\nu = f^\alpha + (\mathscr{D}_\nu K^\alpha - K^\mu\omega^\alpha_{\mu\nu})f^\nu t$. Thus the basis undergoes an infinitesimal rotation given by the antisymmetric matrix $\mathscr{D}_\nu K_\alpha + K^\mu\omega_{\nu\alpha\mu}$. Hence it follows, that the increment of spinor field under the infinitesimal transformation considered is equal to

$$\psi'(x') - \psi(x) = t[K^\mu e_\mu + \tfrac{1}{4}K^\mu\omega_{\nu\alpha\mu}H^\nu H^\alpha$$
$$+ \tfrac{1}{4}(\mathscr{D}_\nu K_\alpha)H^\nu H^\alpha]\psi = t[K^\nu\mathscr{D}_\nu + \tfrac{1}{4}(\mathscr{D}_\alpha K_\beta)H^\alpha H^\beta]\psi.$$

To each Killing field K^α there corresponds the angular momentum operator of the spinor field, which equals

$$\hat{K} = -i\hbar[K^\mu\mathscr{D}_\mu + \tfrac{1}{4}(\mathscr{D}_\alpha K_\beta)H^\alpha H^\beta]. \tag{81}$$

Let us now prove that it commutes with the operator $H^\nu\mathscr{D}_\nu - \dfrac{imc}{\hbar}H_4$. It is evident that $\hat{K}H^4 = H^4\hat{K}$ and we have to prove only the equality $\hat{K}H^\nu\mathscr{D}_\nu = H^\nu\mathscr{D}_\nu\hat{K}$. We have

$$\frac{i}{\hbar}[H^\nu\mathscr{D}_\nu\hat{K} - \hat{K}H^\nu\mathscr{D}_\nu]$$
$$= K^\mu H^\nu(\mathscr{D}_\nu\mathscr{D}_\mu - \mathscr{D}_\mu\mathscr{D}_\nu) + \tfrac{1}{4}(\mathscr{D}_\nu\mathscr{D}_\alpha K_\beta)H^\nu H^\alpha H^\beta$$
$$+ (\mathscr{D}_\nu K^\mu)H^\nu\mathscr{D}_\mu + \tfrac{1}{4}(\mathscr{D}_\alpha K_\beta)(H^\nu H^\alpha H^\beta - H^\alpha H^\beta H^\nu)\mathscr{D}_\nu.$$

Since

$$H^\nu H^\alpha H^\beta - H^\alpha H^\beta H^\nu = 2\eta^{\alpha\nu}H^\beta - 2\eta^{\beta\nu}H^\alpha,$$

then

$$(\mathscr{D}_\nu K^\mu)H^\nu\mathscr{D}_\mu + \tfrac{1}{4}(\mathscr{D}_\alpha K_\beta)(H^\nu H^\alpha H^\beta - H^\alpha H^\beta H^\nu)\mathscr{D}_\nu = 0.$$

Next, taking into account Eq. (31), we find that

$$\frac{i}{\hbar}[H^{\nu}\mathscr{D}_{\nu}\hat{K}-\hat{K}H^{\nu}\mathscr{D}_{\nu}] = \tfrac{1}{4}\{K^{\mu}R_{\alpha\beta,\mu\nu}+(\mathscr{D}_{\nu}\mathscr{D}_{\alpha}K_{\beta})\}H^{\nu}H^{\alpha}H^{\beta}.$$

Because of (80), the sum in braces equals zero. Our assertion is thus proven. One has also the important corollary: If ψ obeys the Dirac equation (34), then $\hat{K}\psi$ also satisfies the same equation. To any operator $\hat{K}$ with this property, there corresponds secondary quantized operator

$$\hat{\mathfrak{h}} = i\int_{\Sigma}\bar{\psi}H_4[Q_1Q_2Q_3]\hat{K}\psi \tag{82}$$

which does not depend on Σ.

8. *Energy-momentum tensor*

Components of the energy-momentum tensor in an arbitrary orthogonal basis are of the same form as in the plane space-time in Cartesian coordinates:

$$T_{\mu\nu} = \frac{i\hbar}{4}[\bar{\psi}H_{\mu}\psi_{\nu}-\bar{\psi}_{\nu}H_{\mu}\psi+\bar{\psi}H_{\nu}\psi_{\mu}-\bar{\psi}_{\mu}H_{\nu}\psi], \tag{83}$$

where $\psi_{\mu} = \mathscr{D}_{\mu}\psi, \quad \bar{\psi}_{\mu} = \mathscr{D}_{\mu}\bar{\psi}$.

Let us prove that both divergences of the tensor $\bar{\psi}H_{\mu}\psi_{\nu}-\bar{\psi}_{\nu}H_{\nu}\psi$ are zero. We have

$$\mathscr{D}^{\nu}\bar{\psi}H_{\mu}\psi_{\nu} = \bar{\psi}^{\nu}H_{\mu}\psi_{\nu}+\bar{\psi}H_{\mu}\mathscr{D}^{\nu}\psi_{\nu}.$$

By (37) we obtain that

$$\mathscr{D}^{\nu}\bar{\psi}H_{\mu}\psi_{\nu} = \bar{\psi}^{\nu}H_{\mu}\psi_{\nu}+\left(\frac{m^2c^2}{\hbar^2}-\tfrac{1}{4}R\right)\bar{\psi}H_{\mu}\psi.$$

Consequently

$$\mathscr{D}^{\nu}(\bar{\psi}H_{\mu}\psi_{\nu}-\bar{\psi}_{\nu}H_{\mu}\psi) = 0. \tag{84}$$

Next, we have that

$$\mathscr{D}^{\mu}\bar{\psi}H_{\mu}\psi_{\nu} = \bar{\psi}_{\mu}H^{\mu}\psi_{\nu}+\bar{\psi}H^{\mu}\mathscr{D}_{\mu}\mathscr{D}_{\nu}\psi$$

$$= \bar{\psi}_{\mu}H^{\mu}\psi_{\nu}+\bar{\psi}\mathscr{D}_{\nu}H^{\mu}\psi_{\mu}+\bar{\psi}H^{\mu}(\mathscr{D}_{\mu}\mathscr{D}_{\nu}-\mathscr{D}_{\nu}\mathscr{D}_{\mu})\psi.$$

By (34) and (35) we get that

$$\mathscr{D}^{\mu}\bar{\psi}H_{\mu}\psi_{\nu} = \bar{\psi}H^{\mu}(\mathscr{D}_{\mu}\mathscr{D}_{\nu}-\mathscr{D}_{\nu}\mathscr{D}_{\mu})\psi.$$

From (31) and (36) we find that

$$\mathscr{D}^{\mu}\bar{\psi}H_{\mu}\psi_{\nu} = -\tfrac{1}{2}\bar{\psi}H^{\beta}\psi R_{\beta\nu}.$$

Consequently

$$\mathscr{D}^{\mu}(\bar{\psi}H_{\mu}\psi_{\nu}-\bar{\psi}_{\nu}H_{\mu}\psi) = 0. \tag{85}$$

From (84) and (85), it follows that the divergence of the energy-momentum tensor is zero

$$\mathscr{D}^{\mu}T_{\mu\nu} = 0. \tag{86}$$

For the Killing vector field, because of Eq. (79), we obtain that $\mathscr{D}_{\mu}(T^{\mu\nu}K_{\nu}) = 0$. Therefore the integral

$$\hat{\mathfrak{h}} = \int_{\Sigma} T_{\mu\nu}K^{\nu}d\sigma^{\mu} \tag{87}$$

does not depend on a choice of the complete hypersurface Σ. This integral will be called secondary quantized operator of the angular momentum. Now let us prove that this operator can be represented in the form (82), where $\hat{K}$ is the operator (81).

For the proof, the equality

$$\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}H_{\nu}H^{\alpha}]\psi = \bar{\psi}H_{\nu}\psi_{\mu}-\bar{\psi}_{\mu}H_{\nu}\psi-\bar{\psi}H_{\mu}\psi_{\nu}+\bar{\psi}_{\nu}H_{\mu}\psi \tag{88}$$

should be noted. This equality follows immediately from the identity (36) and Eqs (34) and (35). Owing to (88), the energy-momentum tensor may be written in the form

$$T_{\mu\nu} = \frac{i\hbar}{2}[\bar{\psi}H_{\mu}\psi_{\nu}-\bar{\psi}_{\nu}H_{\mu}\psi]+\frac{i\hbar}{4}\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}H_{\nu}H^{\alpha}]\psi. \tag{89}$$

Hence,

$$T_{\mu\nu}K^{\nu}+\bar{\psi}H_{\mu}\hat{K}\psi = \frac{i\hbar}{4}K^{\nu}\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}H_{\nu}H^{\alpha}]\psi$$

$$-\frac{i\hbar}{4}\bar{\psi}H_{\mu}H^{\alpha}H^{\nu}\psi(\mathscr{D}_{\alpha}K_{\nu})-\frac{i\hbar}{2}(\bar{\psi}_{\nu}H_{\mu}\psi+\bar{\psi}H_{\mu}\psi_{\nu})K^{\nu}.$$

Next, we have that

$$K^{\nu}\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}H_{\nu}H^{\alpha}]\psi = \mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}KH^{\alpha}]\psi-\bar{\psi}[H_{\mu}H^{\nu}H^{\alpha}]\psi(\mathscr{D}_{\alpha}K_{\nu}),$$

where $K = K^{\nu}H_{\nu}$, and in virtue of (36) and the Killing equation (79), we obtain that

$$T_{\mu\nu}K^{\nu}+\bar{\psi}H_{\mu}\hat{K}\psi = \frac{i\hbar}{4}\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}KH^{\alpha}]\psi$$

$$+\frac{i\hbar}{2}\{\bar{\psi}H^{\alpha}\psi(\mathscr{D}_{\alpha}K_{\mu})-K^{\alpha}\bar{\psi}_{\alpha}H_{\mu}\psi-K^{\alpha}\bar{\psi}H_{\mu}\psi_{\alpha}\}.$$

Finally, from Eqs. (34), (35) and (79) we find

$$\mathscr{D}_{\alpha}\bar{\psi}(K_{\mu}H^{\alpha}-K^{\alpha}H_{\mu})\psi = \bar{\psi}H^{\alpha}\psi\mathscr{D}_{\alpha}K_{\mu}-K^{\alpha}\bar{\psi}_{\alpha}H_{\mu}\psi-K^{\alpha}\bar{\psi}H_{\mu}\psi_{\alpha}$$

and then

$$T_{\mu\nu}K^{\nu}+\bar{\psi}H_{\mu}\hat{K}\psi = \frac{i\hbar}{4}\,\mathscr{D}_{\alpha}\bar{\psi}[H_{\mu}KH^{\alpha}]\psi+\frac{i\hbar}{2}\,\mathscr{D}_{\alpha}\bar{\psi}(K_{\mu}H^{\alpha}-K^{\alpha}H_{\mu})\psi. \tag{90}$$

Since the right of this equality is a divergence of a skew-symmetric tensor, the integral of this divergence over the complete hypersurface is zero. Thus,

$$\begin{aligned}\int_{\Sigma} T_{\mu\nu}K^{\nu}d\sigma^{\mu} &= -\int_{\Sigma}\bar{\psi}H_{\mu}\hat{K}\varphi d\sigma^{\mu}\\ &= i\int_{\Sigma}\bar{\psi}H_{4}[Q_{1}Q_{2}Q_{3}]\hat{K}\psi,\end{aligned} \tag{91}$$

q.e.d.

9. Transformations of orthogonal basis

The metric form ds^2 defines an orthogonal basis with an accuracy up to an orthogonal transformation only. Let $ds^2 = \eta_{\alpha\beta}f^{\alpha}f^{\beta} = \eta_{\alpha\beta}\eta'^{\alpha}\eta'^{\beta}$, $f^{\alpha} = L^{\alpha}_{\beta}f'^{\beta}$ and, inversely, $f'^{\alpha} = \tilde{L}^{\alpha}_{\beta}f^{\beta}$, where L and $\tilde{L}$ are some matrices depending on coordinates x. Then $\eta_{\alpha\sigma}\tilde{L}^{\sigma}_{\beta} = \eta_{\beta\sigma}L^{\sigma}_{\alpha}$. We also have that

$$e'_{\alpha} = L^{\beta}_{\alpha}e_{\beta},\quad e_{\alpha} = \hat{L}^{\beta}_{\alpha}e'_{\beta},\quad \tilde{f}^{\alpha}_{\beta} = \tilde{f}'^{\alpha}_{\gamma}\tilde{L}^{\gamma}_{\beta},\quad f^{\beta}_{\alpha} = f'^{\gamma}_{\alpha}L^{\beta}_{\gamma}.$$

Substituting the two latter formulae into (20), we find that

$$c^{\gamma}_{\alpha\beta} = c'^{\sigma}_{\mu\nu}\tilde{L}^{\mu}_{\alpha}\tilde{L}^{\nu}_{\beta}L^{\gamma}_{\sigma}+\tilde{L}^{\sigma}_{\alpha}e_{\beta}L^{\gamma}_{\sigma}-\tilde{L}^{\sigma}_{\beta}e_{\alpha}L^{\gamma}_{\sigma}. \tag{92}$$

Consequently,

$$\omega_{\alpha\beta\gamma} = \omega'_{\mu\nu\sigma}\tilde{L}^{\mu}_{\alpha}\tilde{L}^{\nu}_{\beta}\tilde{L}^{\sigma}_{\gamma}+\eta_{\mu\nu}\tilde{L}^{\mu}_{\alpha}e_{\gamma}\tilde{L}^{\nu}_{\beta}. \tag{93}$$

From these, there follow directly the formulae $\mathscr{D}_{\beta}A^{\alpha} = \tilde{L}^{\mu}_{\beta}L^{\alpha}_{\nu}\mathscr{D}'_{\mu}A'^{\nu}$, $\mathscr{D}_{\beta}A_{\alpha} = \tilde{L}^{\mu}_{\beta}\tilde{L}^{\nu}_{\alpha}\mathscr{D}'_{\mu}A'_{\nu}$ and analogous formulae for other tensors.

Consider now in what way the covariant differential of a spinor transforms. Any Lorentz transformation $f'^{\alpha} = \tilde{L}^{\alpha}_{\beta}f^{\beta}$ can be expanded in a product of some number p of symmetries[1]. And the symmetry with respect to the plane orthogonal to the unit vector a^{α} is expressed via the formula $f'^{\alpha} = f^{\alpha}-2a^{\alpha}a_{\beta}f^{\beta}$. Since $-AH^{\alpha}A = H^{\alpha}-2a^{\alpha}A$, where $A = a_{\alpha}H^{\alpha}$, then under any Lorentz transformation

$$\begin{aligned}&(-1)^{p}S^{-1}H^{\alpha}S = \tilde{L}^{\alpha}_{\beta}H^{\beta},\quad (-1)^{p}SH^{\alpha}S^{-1} = L^{\alpha}_{\beta}H^{\beta},\\ &(-1)^{p}SH_{\alpha}S^{-1} = \tilde{L}^{\beta}_{\alpha}H_{\beta},\quad (-1)^{p}S^{-1}H_{\alpha}S = L^{\beta}_{\alpha}H_{\beta},\end{aligned} \tag{94}$$

where $S = A_{p}\ldots A_{1}$, $S^{-1} = A_{1}\ldots A_{p}$.

Therefore, because of (93), the matrix (26) transforms in the following way

$$\Omega = S^{-1}\Omega'S+\tfrac{1}{4}\,S^{-1}H_{\mu}Sd(S^{-1}H^{\mu}S).$$

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[1] The number p is even, if $\det|L^{\alpha}_{\beta}| = 1$, and is odd if $\det|L^{\alpha}_{\beta}| = -1$ i.e. $(-1)^{p} = \det|L^{\alpha}_{\beta}|$.

Now let us prove that

$$\tfrac{1}{4} S^{-1}H_\mu Sd(S^{-1}H^\mu S) = S^{-1}dS. \tag{95}$$

For $p = 1$ this equality can be easily verified. Indeed,

$$\tfrac{1}{4} AH_\mu Ad(AH^\mu A) = \tfrac{1}{4}(2a_\mu A-H_\mu)d(2a^\mu A-H^\mu)$$

$$= \tfrac{1}{2}(2a_\mu A-H_\mu)(Ada^\mu+a^\mu dA) = AdA. \tag{96}$$

Here we take into account that $2a_\mu da^\mu = d(a_\mu a^\mu) = 0$ and $A \cdot dA+dA \cdot A = dA^2 = 0$ for $A^2 = a_\mu a^\mu = 1$. We next prove that if Eq. (95) holds for some p, then it holds, as well, for $p+1$. Thus, we must prove that from (95), there follows the formula

$$\tfrac{1}{4} S^{-1}AH_\mu ASd(S^{-1}AH^\mu AS) = S^{-1}Ad(AS). \tag{97}$$

For brevity the index $p+1$ of the matrix A is omitted. We have

$$\tfrac{1}{4} S^{-1}AH_\mu ASd(S^{-1}AH^\mu AS) = \tfrac{1}{4} S^{-1}AH_\mu AS(dS^{-1})AH^\mu AS$$

$$+\tfrac{1}{4} S^{-1}AH_\mu A(d(AH^\mu A))S+\tfrac{1}{4} S^{-1}AH_\mu H^\mu AdS.$$

Because $H_\mu H^\mu$ is equal to the number which represents a dimension of the space-time, the third term is

$$\tfrac{1}{4} S^{-1}AH_\mu H^\mu AdS = \tfrac{1}{4} S^{-1}H_\mu H^\mu dS.$$

The second term, by (96) is equal to

$$\tfrac{1}{4} S^{-1}AH_\mu A(d(AH^\mu A))S = S^{-1}A(dA)S.$$

And the first term is

$$\tfrac{1}{4} S^{-1}(2a_\mu A-H_\mu)S(dS^{-1})(2a^\mu A-H^\mu)S = \tfrac{1}{4} S^{-1}H_\mu S(dS^{-1})H^\mu S.$$

Consequently,

$$\tfrac{1}{4} S^{-1}AH_\mu ASd(S^{-1}AH^\mu AS)$$

$$= \tfrac{1}{4} S^{-1}H_\mu Sd(S^{-1}H_\mu S)+S^{-1}A(dA)S.$$

So, from (95), the formula (97) follows. By induction the equality (95) has been proved and then

$$\Omega = S^{-1}\Omega' S+S^{-1}dS. \tag{98}$$

Hence we find that

$$\mathscr{D}\psi = d\psi+\Omega\psi = S^{-1}(d\psi'+\Omega'\psi') = S^{-1}\mathscr{D}'\psi' \tag{99}$$

where

$$\psi' = S\psi, \tag{100}$$

i.e., ψ and $\mathcal{D}\psi$ transform by the same law. Since

$$S^{+}H_0 = (-1)^p H_0 S^{-1}, \tag{101}$$

the conjugate spinor transforms by the law

$$\bar{\psi}' = (-1)^p \bar{\psi} S^{-1}. \tag{102}$$

In the same way its covariant differential transforms:

$$\begin{aligned}\mathcal{D}\bar{\psi} = d\bar{\psi} - \bar{\psi}\Omega &= (-1)^p (d\psi' - \bar{\psi}'\Omega')S \\ &= (-1)^p (\mathcal{D}'\psi')S.\end{aligned} \tag{103}$$

Now it is not difficult to prove that Eqs. (34) and (35) are covariant under transformations of orthogonal basis. Indeed, the transformation rules for the covariant derivative of a spinor follow from (39) and (103)

$$\begin{aligned}\mathcal{D}_\nu \psi &= \tilde{L}^\mu_\nu S^{-1} \mathcal{D}'_\mu \psi', \\ \mathcal{D}_\nu \bar{\psi}' &= (-1)^p \tilde{L}^\mu_\nu (\mathcal{D}'_\mu \psi')S.\end{aligned} \tag{104}$$

By (94) we obtain that

$$H^\nu \mathcal{D}_\nu \psi = (-1)^p S^{-1} H^\mu \mathcal{D}'_\mu \psi', \qquad \mathcal{D}_\nu \bar{\psi} H^\nu = \mathcal{D}'_\mu \bar{\psi}' H^\mu S.$$

Besides, we get that

$$(-1)^p S H^4 S^{-1} = H^4. \tag{105}$$

Consequently,

$$H^\nu \mathcal{D}_\nu \psi - \frac{imc}{\hbar} H^4 \psi = (-1)^p S^{-1} \left(H^\nu \mathcal{D}'_\nu \psi' - \frac{imc}{\hbar} H^4 \psi' \right),$$

$$\mathcal{D}_\nu \bar{\psi} H^\nu + \frac{imc}{\hbar} \bar{\psi} H^4 = \left(\mathcal{D}'_\nu \bar{\psi}' H^\nu + \frac{imc}{\hbar} \bar{\psi}' H^4 \right) S,$$

and Eqs. (34) and (35) are seen to be covariant.

Finally one can easily show that the current vector and the energy-momentum tensor transform in the expected way:

$$J^\alpha = L^\alpha_\mu J'^\mu, \qquad T_{\mu\nu} = \tilde{L}^\alpha_\mu \tilde{L}^\beta_\nu T'_{\alpha\beta}.$$

Editorial note. This article was proofread by the editors only, not by the authors.

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