DILATIONS AND LIGHT–HEAVY NEUTRINO MIXINGS*

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A dilation procedure is presented for the interval neutrino mixing matrix in order to explore possible unitary extensions of the three-dimensional neutrino mixings. Limits on light–heavy neutrino mixings are considered.

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1. Introduction

Neutrino physics is a fascinating area of present experimental and theoretical research. Currently, we know that neutrinos oscillate between flavour states, what proves that they are not massless particles [1, 2]. This discovery was a milestone in particle physics. However, there are still many unsolved problems connected to neutrinos. Among them is a problem of the mass hierarchy in the neutrino sector [3, 4]. Moreover, we still do not know why neutrinos are much lighter than charged fermions and we also do not know whether they are Dirac or Majorana particles [5–7]. Finally, we are not sure if there exist only three types of neutrinos, *i.e.* electron, muon and tau, as described by the Standard Model (SM) of elementary particles [6, 7]. Nevertheless, many important physical phenomena can be explained as long as new additional massive neutrino states exist [6–8]. Hints of such new physics in the neutrino sector can be hidden, *e.g.* in the neutrino mixing matrix as a deviation from unitarity of the PMNS mixing matrix. This issue was considered in [9] where a novel approach to study the neutrino mixing matrix was introduced based on advanced methods from matrix theory and convex analysis [10–12]. The main object of interest in this approach is an interval matrix obtained from global fits to current experimental data [13–16]

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$$\mathrm{V}_{\mathrm{osc}} = \left(\begin{array}{ccc} 0.799 \div 0.845 & 0.514 \div 0.582 & 0.139 \div 0.155 \\ -0.538 \div -0.408 & 0.414 \div 0.624 & 0.615 \div 0.791 \\ 0.22 \div 0.402 & -0.73 \div -0.567 & 0.595 \div 0.776 \end{array} \right) . \quad (1)$$

An interval matrix (1) is to be understood as a family of matrices with elements contained in the specified intervals. Further, in [9], the notion of a contraction in the context of neutrino physics was introduced which characterizes physically meaningful mixing matrices. It was shown that the set of all physically allowed mixing matrices could be represented as a convex hull of contractions, spanned by unitary matrices with parameters restricted by experimental data. Finally, taking matrices from this convex set, possible beyond the Standard Model (BSM) scenarios triggered by a theory of unitary dilations was considered. In the present work, we focus on the last issue and show how the procedure of a unitary dilation of contractions from $\mathrm{V}_{\mathrm{osc}}$ to the higher-dimensional unitary BSM mixing matrix can be used. In this way, present PMNS-based global mixing data can be confronted with additional limits on mixings between SM and BSM neutrino fields.

2. Unitary dilation

To motivate the construction, let us start with a simple remark. In the neutrino mixing analysis, we would like to study possible extensions of the classical three neutrinos scenario, hence we consider an initial mixing matrix V as a principal submatrix of some larger one

$$V \to \left(\begin{array}{cc} V & V_{lh} \\ V_{hl} & V_{hh} \end{array} \right) . \quad (2)$$

In the matrix theory, this is called a matrix dilation. Since in the case of neutrino mixing (up to some possible exotic subtleties), the complete matrix has to be unitary, we are interested only in unitary dilations. However, not all matrices can be dilated to a unitary matrix. In fact, such a possibility is restricted to contractions, *i.e.* matrices with a spectral (operator) norm less or equal to one. Although the formal definition of the spectral norm $\|V\| = \max_{\|x\|=1} \|Vx\|$ is not suitable for our analysis, it can be characterized by the largest singular value of a matrix. Then, it provides also a simple condition for V being a contraction

$$\|V\| = \max_{\|x\|=1} \|Vx\| = \sigma_{\max}(V) \leq 1 \, . \quad (3)$$

Suppose (3) holds; then V can be dilated to a unitary matrix with the use of the Cosine–Sine (CS) decomposition [10]

$$U \equiv \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{m-n} \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix} . \tag{4}$$

One would like to know what are the possible dimensions of U since, from the point of view of particle physics, they are connected to the number of neutrinos in the BSM scenario. To show this, we will follow an example given in [9], although implementing also additional restrictions on light–heavy neutrino mixings.

3. Constraints for the light–heavy mixing

Due to the attractiveness of the idea of additional massive neutrinos in the BSM scenario, the possibility of their detection is intensively studied. There is a variety of processes from which a signature of sterile neutrinos can be visible. These include both production and decay of these particles. Among them, we can mention the following processes: $pp \to lljj$, $e^-e^- \to W^-W^-$, neutrinoless double beta decay, invisible Z-boson decays, neutrino oscillations and cosmological effects [17–23]. These processes depend on values of the elements of the light–heavy part of the mixing matrix $|V_{lh}|$, where $l = e, \mu, \tau$ and $h = 1, 2, \ldots, N$. Here, we adopt heavy neutrino mass-independent constraints as discussed in [24]

$$\sum_{h=1}^{N} |V_{eh}|^2 \leq 0.0030 \,, \qquad \sum_{h=1}^{N} |V_{\mu h}|^2 \leq 0.0032 \,, \qquad \sum_{h=1}^{N} |V_{\tau h}|^2 \leq 0.0062 \,. \tag{5}$$

In [20], a scenario with one additional sterile neutrino is considered and upper bounds for individual entries of the light–heavy part of the mixing matrix are given as a function of additional neutrino mass in wide range of its masses of 10 eV $\lesssim m_N \lesssim$ 1 TeV. As our method of analysis of the mixing matrix depends only on experimental results for three SM neutrinos, we do not want to restrict ourselves neither to a specific model nor to the particular number of additional neutrinos. Thus, we accommodate in discussion general constraints (5) in the case of heavy sterile neutrinos.

4. Analysis

Now, we will use theory of the unitary dilations to three-dimensional mixing matrices and then compare light–heavy mixings obtained in this way with constraints given by (5). First, we have to construct physically allowed mixing matrices, *i.e.* contractions within $\mathrm{V}_{\rm osc}$. The set of such matrices can be defined as a convex hull Ω of unitary matrices with parameters restricted by experiments [9]. Here, we will create a physical contraction as a simple convex combination of two 3×3 unitary matrices and then compare its unitary dilation (4) with limits (5). We start from two unitary matrices U_1 and U_2 with the following set of parameters:

$$\begin{aligned} U_1 &\rightarrow \theta_{12} = 31.38^\circ\,, \quad \theta_{23} = 38.4^\circ\,, \quad \theta_{13} = 7.99^\circ\,, \\ U_2 &\rightarrow \theta_{12} = 35.99^\circ\,, \quad \theta_{23} = 52.8^\circ\,, \quad \theta_{13} = 8.90^\circ\,. \end{aligned} \tag{6}$$

Now, we take their convex combination $V' = \sum_{i=1}^{n} \alpha_i U_i$ with $\sum_{i=1}^{n} \alpha_i = 1$, *e.g.*

$$V' = \tfrac{1}{2}U_1 + \tfrac{1}{2}U_2 = \begin{pmatrix} 0.822 & 0.549 & 0.147 \\ -0.469 & 0.521 & 0.701 \\ 0.311 & -0.643 & 0.687 \end{pmatrix}. \tag{7}$$

It turns out that for this matrix, a dilation as defined by (4) gives a unitary matrix which does not fulfill (5), for details, see formula (44) in [9]. In general, contraction condition (3) and limits (5) give very strong constraints on possible dilation matrices. This can be seen by imposing (5) directly to randomly generated matrices within $\mathrm{V}_{\mathrm{osc}}$. We have found that among 10^9 randomly generated matrices that lie inside $\mathrm{V}_{\mathrm{osc}}$, only one matrix satisfies (5) and $\|V\| \leq 1$, namely

$$\begin{pmatrix} 0.816 & 0.557 & 0.148 \\ -0.453 & 0.459 & 0.764 \\ 0.357 & -0.692 & 0.624 \end{pmatrix}. \tag{8}$$

The above matrix has the following set of singular values $\{1, 0.999, 0.997\}$. An error estimated, based on the precision of the initial interval matrix (1), is at the level of 0.003 [9], so nothing decisive can be said about deviations of singular values from unity. It should be stressed that the number of singular values strictly less than one controls a dimension of the smallest possible unitary dilation in a way that this number stands for a dimension enlargement of the initial matrix. Thus, in this particular case, taking seriously the possibility that two singular values are smaller than one, just for illustration, we obtain the following 5×5 unitary dilation matrix:

$$\left(\begin{array}{ccc|cc} 0.816 & 0.557 & 0.148 & 0.0444 & -0.00429 \\ -0.453 & 0.459 & 0.764 & 0.0228 & 0.0247 \\ 0.357 & -0.692 & 0.624 & 0.0449 & 0.0497 \\ \hline 0.0244 & 0.0222 & -0.00477 & -0.637 & 0.769 \\ 0.0344 & -0.0131 & 0.0717 & -0.768 & -0.636 \end{array}\right). \tag{9}$$

(Observe that a dilation matrix of dimension four is also possible in this case within given error estimation by assuming that two singular values of (8) are equal to one and the third is less than one.)

To improve the strategy of searching for possible BSM unitary matrices based on the CS decomposition (4) with restrictions (5), instead of taking as a point of departure $\mathrm{V_{osc}}$ and unitary dilation of matrices within it, we can transform the block matrix (2) corresponding to the complete unitary mixing in the following way:

$$\begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} V_{lh} & V \\ V_{hh} & V_{hl} \end{pmatrix} . \tag{10}$$

This allows us to consider unitary dilations of matrices V_{lh} on which we impose directly limits like (5) and, moreover, we fix a three-dimensional submatrix V for which elements lie within $\mathrm{V_{osc}}$. After this operation, we can reverse this transformation to get the complete unitary mixing matrix of form of (2). This procedure will need separate careful studies.

5. Summary

In neutrino oscillation experiments, usually the three-dimensional unitary matrix U_{PMNS} is considered. If there are additional neutrinos which mix with the observed three neutrino species, the three-dimensional neutrino mixing matrix $\mathrm{V_{osc}}$ should keep marks of these additional neutrino states, and cannot be unitary. We argue that there is a more natural approach to data analysis in a quest for extra neutrino flavours. Namely, we use matrix theory concepts of contraction and dilation. There is a theorem in the matrix analysis [11] that any submatrix V of the unitary matrix U is a contraction, *i.e.* $\|V\| \leq 1$. This purely mathematical fact makes it possible to extract physically allowed mixing matrices from the experimentally derived interval matrix $\mathrm{V_{osc}}$ by taking $\|\mathrm{V_{osc}}\| \leq 1$. Consequences of this approach have been discussed in detail in [9]. Here, we focused on the dilation procedure, which allows to extend experimentally determined neutrino mixings based on the interval matrix (1) to the complete unitary matrix of dimension higher than three. We showed that limits on light–heavy neutrino mixings (5) together with a contraction constraint (3) restrict strongly possible mixing structures and outline how effectively accommodate these limits in searching for extra neutrino species by constructing higher than three-dimensional unitary matrices.

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