MATHISSON'S NEW MECHANICS: ITS AIMS AND REALISATION*

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In 1937, Myron Mathisson published in this journal a paper entitled *A New Mechanics of Material Systems*. This showed for the first time how an extended body in general relativity could be described by an infinite set of multipole moments and how approximate equations of motion could be obtained by retaining only a finite number of these moments. He obtained such equations of motion when only the monopole and dipole moments are retained and also partial results when the quadrupole moment is also retained. This was the start of a programme of work that he continued until his death in 1940. This paper identifies the aims of this work and the obstacles that still needed to be overcome. It outlines subsequent developments by the present author and others that have continued this programme and brought it to fulfillment.

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1. Introduction

It gives me great pleasure to present a paper at this meeting on the life and work of Myron Mathisson. Although he died in the year that I was born, through his papers he has probably had a greater influence on my work than any living scientist.

I first came upon his work as a graduate student working on equations of motion in general relativity. I had come to this topic through the papers of Papapetrou [1] and Corinaldesi and Papapetrou [2] published together in 1951 under the title *Spinning Test-Particles in General Relativity*, a test particle being a body whose mass is, in some appropriate sense, negligible. Papapetrou considered a test particle sufficiently small that only the

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monopole and dipole moments of its energy-momentum tensor $T^{\alpha\beta}$ need be considered, a so-called pole–dipole particle. He showed that for the purpose of studying its motion, the monopole moment of such a particle could be described by a scalar M and the dipole moment by an antisymmetric tensor $S^{\alpha\beta}$, these representing its mass and spin, *i.e.* angular momentum. These were shown to satisfy the equations

$$\frac{\delta}{ds}\left(Mv^\alpha + v_\beta \frac{\delta S^{\alpha\beta}}{ds}\right) = \tfrac{1}{2} v^\beta S^{\gamma\delta} R^\alpha_{\cdot\beta\gamma\delta} \tag{1}$$

and

$$\frac{\delta S^{\alpha\beta}}{ds} + v^\alpha v_\gamma \frac{\delta S^{\beta\gamma}}{ds} - v^\beta v_\gamma \frac{\delta S^{\alpha\gamma}}{ds} = 0\,. \tag{2}$$

Here s is the proper time along the world line of the body, $v^\alpha \equiv dx^\alpha/ds$ is its four-velocity, δ/ds denotes the absolute derivative with respect to s and $R^\alpha_{\cdot\beta\gamma\delta}$ is the curvature tensor of the spacetime. The signature of the spacetime metric is taken as -2, so that $v_\alpha v^\alpha = 1$.

The treatment by Papapetrou was extremely non-covariant. The quantities M and $S^{\alpha\beta}$ were constructed from non-covariant parts and had to be proved to be a scalar and tensor respectively. Similarly the equations as originally derived had to be manipulated into the above covariant forms. To my mind these aspects were unsatisfactory. Indeed it was this unsatisfactory nature of the derivation that led me to study the problem further. But the final equations were covariant and, so it seemed, new. Papapetrou pointed out that in a flat spacetime they agreed with the results of a study of the pole–dipole particle in special relativity by Mathisson [3] in 1937.

In due course I decided, for completeness, to look up the 1937 paper of Mathisson. I discovered to my astonishment that not only was the treatment actually in general relativity but it was also covariant. Mathisson derived essentially the same equations of motion as Papapetrou, including the spin-curvature interaction term that is arguably the most important discovery of the work. Indeed, he went further by considering the quadrupole terms. He showed that the equation of motion for the spin then gains additional terms representing a torque exerted by the gravitational field, expressed in terms of the quadrupole moment and the curvature tensor. On top of all this, the title of the paper was, in translation from its original German, *A New Mechanics of Material Systems*, a far more all-embracing and inspiring title than that of Papapetrou.

The year was 1964. I felt compelled to find out what further work Mathisson had done on this "new mechanics". I discovered, to my great sadness, that he had died in 1940 while still working on this topic. I decided to continue Mathisson's programme of work as I understood it to be. This came to dominate the next ten years of my life.

2. Mathisson's multipole moments

To see the aims of Mathisson's work we need to look at his characterisation of the multipole moments of a body. In contrast to Papapetrou, Mathisson treated an extended body and defined an infinite set of covariant multipole moments for it. We consider a body occupying a world tube of finite spatial extent and choose a timelike world line L within this world tube. For present purposes L is otherwise arbitrary, but at a later stage we will wish to impose conditions that restrict it to be a suitably defined mass centre. Precisely what conditions are best suited for this is in fact one of the big questions of this subject.

Mathisson showed that there exists an infinite set of multipole moments $m^{\alpha\beta}$, $m^{\alpha\beta\gamma}$, $m^{\alpha\beta\gamma\delta}$,... such that

$$\int T^{\alpha\beta}\varphi_{\alpha\beta} Dx = \int_L \Big(m^{\alpha\beta}\varphi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\varphi_{\alpha\beta} + \frac{1}{2!}\, m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta} + \cdots \Big) ds \quad (3)$$

for all symmetric tensor fields $\varphi_{\alpha\beta}$ of compact support. Here $T^{\alpha\beta}$ is the energy-momentum tensor of the body, which is taken to be symmetric. The integral on the left extends over all space, with $Dx \equiv \sqrt{-g}\, d^4x$ being the spacetime volume element. That on the right is over the world line L on which s is proper time, ∇_α denotes covariant differentiation and $\nabla_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta$. The moments are not uniquely determined by this but they become so if we require in addition that

(a) they are symmetric on their last two indices, *i.e.* those contracted with $\varphi_{\alpha\beta}$, and separately are symmetric on all their other indices, *i.e.* those contracted with the derivative operators, and

(b) they are orthogonal to the four-velocity $v^\alpha \equiv dx^\alpha/ds$ of L on all indices except the last two.

The infinite set of these moments was called by Mathisson the *gravitational skeleton* of the body, or in a later paper its *dynamical skeleton.*

3. The variational equation of dynamics

If we take

$$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}\,, \quad (4)$$

where ω_α is an arbitrary vector field of compact support and round brackets around indices denote symmetrisation then the left side of the defining equation (3) vanishes as a consequence of the conservation equation

$$\nabla_\beta T^{\alpha\beta} = 0\,. \quad (5)$$

The moments therefore satisfy

$$\int\limits_L \left(m^{\alpha\beta}\varphi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\varphi_{\alpha\beta} + \frac{1}{2!}\, m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta} + \cdots \right) ds = 0 \qquad (6)$$

for all such $\varphi_{\alpha\beta}$. Mathisson called this the *variational equation of dynamics*, the variation in question being the ability to vary ω_α arbitrarily. It is a constraint on the gravitational skeleton and is the central equation of his programme of work.

Mathisson's "new mechanics" consists of determining the consequences of this variational equation, so providing a description of an extended body and its motion in terms of parameters similar to those of Newtonian rigid body mechanics, rather than the description provided by the energy-momentum tensor which corresponds more to that of Newtonian continuum mechanics. However, he was only able to work with this equation by truncating the series after the first few terms, the justification being that successively higher moments should have less and less effect on the motion if the gravitational field varies only slowly across the body. The truncated equation then leads both to restrictions on the form of the moment tensors and to differential equations of motion that they must satisfy.

When only the first two moments are retained, these restrictions imply that these moments are determined by a scalar M and an antisymmetric tensor $S^{\alpha\beta}$ such that

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} + \frac{\delta}{ds}\left(q^{(\alpha}v^{\beta)}\right) \qquad (7)$$

and

$$m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)} + v^\gamma q^{(\alpha}v^{\beta)}\,, \qquad (8)$$

where

$$p^\alpha = Mv^\alpha + v_\beta\frac{\delta S^{\alpha\beta}}{ds} \qquad (9)$$

and

$$q^\alpha = S^{\alpha\beta}v_\beta\,. \qquad (10)$$

These satisfy the same equations of motion (1) and (2) as were later derived by Papapetrou. For reasons described below, Mathisson derived these results only in the special case $q^\alpha = 0$ but the general case given here gives greater insight into the structure of the results.

With use of the auxiliary vector p^α the equations (1) and (2) can be written in the form

$$\frac{\delta p^\alpha}{ds} = \frac{1}{2}v^\beta S^{\gamma\delta}R^{\alpha}_{\cdot\beta\gamma\delta}\,, \qquad \frac{\delta S^{\alpha\beta}}{ds} = 2p^{[\alpha}v^{\beta]}\,, \qquad (11)$$

where square brackets around indices denote antisymmetrisation. Indeed the equations in this form imply that p^α has the form given in (9), where M is now defined by $M = p^\alpha v_\alpha$, so that alternatively p^α and $S^{\alpha\beta}$ can be regarded as giving the primary description with M being the auxiliary variable. This is perhaps a more elegant view of the results.

If the expressions (7) and (8) are put back into the defining relation (3) for the moments, the terms involving q^α combine into a single derivative that integrates to zero. That defining equation is therefore unaffected if expressions (7) and (8) are replaced by the simpler ones

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} \qquad \text{and} \qquad m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)}\,. \tag{12}$$

These are the forms taken by (7) and (8) when $q^\alpha = 0$ but in the case of general q^α they no longer satisfy the orthogonality condition (b). This is a first indication that (b) is not the most appropriate addition to condition (a) to ensure uniqueness of the moments. We shall see below that modification of this orthogonality condition is one of the key steps towards solving the variational equation exactly.

4. Centre of mass

In their component form, there are ten independent variables in equations (1) and (2), namely M, three independent components of the unit vector v^α and six independent components of the antisymmetric tensor $S^{\alpha\beta}$. There are, however, only seven independent equations since the inner product of (2) with v_β is satisfied identically. These equations are therefore not determinate, which is simply a reflection of the fact that for a given body, the world line L can be chosen arbitrarily.

Mathisson wished to impose further conditions to select a specific L that could be considered as the world line of the centre of mass of the body. By examining the form taken by the moments in special relativity, Mathisson showed that the q^α defined by (10) could be interpreted as the static mass dipole moment of the body and that L could be chosen so that $q^\alpha = 0$ along it. In Newtonian mechanics the static mass dipole moment vanishes when taken about the centre of mass. He, therefore, took

$$v_\beta S^{\alpha\beta} = 0 \tag{13}$$

as the condition to characterise L as the world line of the centre of mass in general relativity.

With this condition the equations (1) and (2) become determinate. However, they also become third order differential equations for $x^\alpha(s)$, so that initial values of position, velocity and acceleration are all needed in order to determine a unique solution.

This raises the question of whether this is a real physical phenomenon or an artefact of the condition (13). Unfortunately this question cannot be answered within the framework of the pole–dipole approximation, as there can only be one choice of L for which the neglect of quadrupole and higher moments can be valid. Indeed, this neglect can only be truly justified in the limiting case of a point test particle, when L is taken as the world line of the particle itself. Whether or not (13) holds is then without doubt a question about the physical properties of the particle. In this context (13), or any alternative, is generally called a *supplementary condition* rather than a mass centre definition.

As mentioned above, Mathisson imposed condition (13) *before* he exploited the variational equation, so that his equations of motion were only derived for this special case. This is why Mathisson's equations were said above to be only *essentially* the same as those of Papapetrou. Papapetrou did not impose this condition. He was aware of its difficulties but he offered no alternative of general applicability.

A more detailed understanding came first from the situation in special relativity, in which the alternative form (11) for the equations of motion simplifies to

$$\frac{dp^\alpha}{ds} = 0\,, \qquad \frac{dS^{\alpha\beta}}{ds} = 2p^{[\alpha}v^{\beta]}\,. \tag{14}$$

These equations were studied by Weyssenhoff and Raabe [4] in 1947, who deduced them on very different grounds from those of Mathisson. They considered a point particle with four-velocity v^α that was endowed with an internal angular momentum (spin) $S^{\alpha\beta}$ satisfying (13). They accepted the possibility that its four-momentum p^α need not be parallel to v^α. Then the equations (14) represent conservation of momentum and of total (orbital plus spin) angular momentum. They defined its mass by $M = p^\alpha v_\alpha$ and deduced (9). Since p^α is now constant in time, a Lorentz frame can be chosen in which the spatial components of p^α are zero. They showed that in this frame the general solution of the equations is uniform motion in a circle of arbitrary radius r at an angular velocity determined by r and the spin/mass ratio S/M where $S^2 = \frac{1}{2}S^{\alpha\beta}S_{\alpha\beta}$. These motions were interpreted as real physical motions for such a particle.

The same equations were studied in the context of an extended body in special relativity by Møller [5] in 1949. He showed that for an arbitrary world line L the equations (14) are satisfied exactly by the momentum p^α, and angular momentum $S^{\alpha\beta}$ about a point $z(s)$ of L, defined as integrals of $T^{\alpha\beta}$ by

$$p^\alpha = \int_\Sigma T^{\alpha\beta}\, d\Sigma_\beta \tag{15}$$

and

$$S^{\alpha\beta} = 2\int_{\Sigma} X^{[\alpha}T^{\beta]\gamma}\, d\Sigma_{\gamma}\,, \tag{16}$$

where $X^{\alpha} = x^{\alpha} - z^{\alpha}(s)$. The integration is over any cross-section of the body by a hypersurface Σ, since the symmetry of $T^{\alpha\beta}$ together with the conservation law (5) ensures that for fixed s these integrals are independent of the choice of Σ. In this situation, any condition that restricts L is merely one of convention. If (13) is adopted then the various circular motions permitted by the equations are merely different choices of L. Møller called the points on these lines *pseudocentres of gravity*, so that what one is doing in prescribing an initial acceleration for a solution of the equations is simply picking out which pseudocentre of gravity is to be described. The condition (13) represents the vanishing of the mixed components $S^{\alpha 4}$ for $\alpha = 1, 2, 3$ in the frame in which the 3-velocity is zero. Møller pointed out that a unique L can be determined by requiring these components to vanish instead in the zero 3-momentum frame, which is uniquely determined since p^{α} is constant. He called this the *true centre of gravity.* It is characterised in a general frame by

$$p_{\beta}S^{\alpha\beta} = 0\,. \tag{17}$$

In 1959 Tulczyjew [6] revisited the pole–dipole equations in general relativity, considered as approximations for an extended body. Following the results of Weyssenhoff and Raabe, he identified the p^{α} of (9) as the momentum of the body in this approximation. He rejected (13) as a mass centre definition due to its non-uniqueness and observed that in special relativity (17) does determine a unique world line. With this as motivation, he then adopted (17) as the condition to determine the centre of mass in general relativity "owing to the lack of another definition". The combination of (11) with (17) provides equations for $x^{\alpha}(s) \in L$ that are only of second order, so it is no longer possible to prescribe an initial acceleration.

This was the first definition of the mass centre of an extended body in general relativity that avoided the problem of non-uniqueness. In contrast to the situation in special relativity, however, where (15) and (16) provide well defined expressions for p^{α} and $S^{\alpha\beta}$ in terms of $T^{\alpha\beta}$, the momentum and spin that occur in the Tulczyjew condition are defined only within the pole–dipole approximation. This was overcome by the present author [7] in 1964, where explicit expressions were given for p^{α} and $S^{\alpha\beta}$ as integrals of $T^{\alpha\beta}$ that were generalisations of (15) and (16) and were shown to satisfy equations (11) in the pole–dipole approximation. Taken with the Tulczyjew condition (17) this provided the first definition of the mass centre in general relativity that was not dependent on an approximation procedure.

Even this is still unsatisfactory, however. Any change to the p^α and $S^{\alpha\beta}$ defined there that leaves them unaltered in the pole–dipole approximation would affect the definition of the mass centre but would not affect the equations of motion to that approximation. It is therefore only one of an infinite set of possible definitions that lead to distinct world lines but which agree with one another in the limiting case of flat spacetime. A satisfactory choice from this set can only be made on physical grounds when equations of motion are available to an arbitrarily high order of multipole moments.

The centre of mass problem is therefore intimately bound up with Mathisson's programme of work on equations of motion.

5. The problems ahead

To see the problems facing the Mathisson programme, it is necessary to form an overall view. Mathisson's approach to equations of motion falls into two distinct parts. The first part is to show that moments exist that satisfy the defining equation and any supplementary conditions such as (a) and (b) above. The second part is to exploit the consequences of the variational equation. In his 1937 paper his treatment of the variational equation was approximate, in that it required a truncation. In contrast his existence proof for the moments dealt with the infinite set of moments without approximation. However, it left the uniqueness question unanswered, as the orthogonality condition (b) was imposed only on the moments retained in the truncated variational equation.

An improved and more detailed proof was given by Bielecki, Mathisson and Weyssenhoff [8] in 1939. This imposed both (a) and (b) from the outset and proved both the existence and uniqueness of the moments. Or at least it did so subject to one explicit proviso, namely that infinite series converge and functions are analytic to whatever extent is required.

This proviso is important, as the proof requires the moments at any point $z \in L$ to be determined by the value of $T^{\alpha\beta}$ on a hypersurface through z that has some freedom in its specification. This implies some form of analyticity of $T^{\alpha\beta}$ that is not physically reasonable. It is permissible to require $\varphi_{\alpha\beta}$ to be analytic as this is just an auxiliary field introduced for convenience, but the analyticity must not apply also to $T^{\alpha\beta}$. The problem originates not in the proof but in the moment defining equation itself. Mathisson [9] studied this issue again in 1940, this time in the simpler case of special relativity, in a paper simply titled *The Variational Equation of Relativistic Dynamics*. He was there able to give expressions for the moments as explicit integrals of $T^{\alpha\beta}$ over hyperplanes orthogonal to the chosen world line L. These were shown to satisfy the defining equations but they were not deduced from it. Indeed, the flawed nature of the defining equations makes this task impossible.

These problems suggest that it may be preferable to abandon the Mathisson approach and instead seek a covariant version of the Papapetrou method. Seek a set of multipole moments defined from the outset as explicit tensor-valued integrals over cross-sections of the body and study their properties directly. This would by-pass the difficulties associated with Mathisson's moment defining equation and leave only a study of the consequences for these moments of the energy-momentum conservation equation. This study would correspond to the solving of Mathisson's variational equation. One would expect to have to retain only the first few moments in this study, but that would correspond to Mathisson's need to truncate the variational equation. This approach was adopted in 1962 by Tulczyjew and Tulczyjew [10] and in 1964, with different definitions, by the present author in [7]. But once one sees how to construct such tensor-valued integrals it becomes clear that there is an infinite number of possibilities to choose from, no one choice being more natural, in some sense, than the rest.

Perhaps it does not matter which we choose, as the quantities such as p^α and $S^{\alpha\beta}$ that appear in the final equations of motion are constructs from these moments rather than being moment tensors themselves. But the goal must be to avoid the need to neglect the higher moments, as this truncation itself is fundamentally flawed. We have only to look at Papapetrou's original forms (1) and (2) for the equations of motion of a pole–dipole particle to see that even in a flat spacetime the dipole construct enters the equation (1) governing the monopole construct M and the velocity v^α. If higher moments were retained we would expect further contributions to this equation. We have seen that p^α provides a preferable description of the monopole structure in that the spin $S^{\alpha\beta}$ then only occurs in the monopole equation of motion in combination with the curvature tensor. But what happens to this if we retain higher moments? Equations of motion that include quadrupole terms were obtained by Taub [11] in 1965 and Madore [12] in 1969. Both of these resulted in an equation of motion for the momentum in which the quadrupole moment, like the dipole moment $S^{\alpha\beta}$, interacts with an undifferentiated curvature tensor. This conflicts with the justification offered for the neglect of the higher moments, namely that they interact with successively higher derivatives of the curvature tensor.

The quadrupole extensions of Taub and Madore were unsatisfactory also for another reason. In addition to the neglect of octopole and higher moments that is in the nature of this approximation, they both had to neglect certain of the components of the quadrupole moment on the grounds of smallness. So even a complete treatment of the quadrupole case was proving elusive.

So how might we continue to higher orders and avoid truncation. We can hope that by a judicious choice of our original moment definitions the consequences of the energy-momentum conservation equation might be manageable, and indeed even sufficiently systematic that we can handle them to all orders. But we need a guide to help towards this judicious choice. There really is only one guide available. It is to return to Mathisson's approach, with its implicit definition of the moments through a defining equation, and to use the variational equation as our guide. That equation has a deceptive simplicity. Our goal is to find our way through that deception.

So in the 1960's I was led back to Mathisson's new mechanics of 1937 to provide the way forward. The aims were two-fold.

1. To remove all assumptions of convergence and analyticity from the statement of the moment defining equation and its associated existence and uniqueness proof.

2. To solve the variational equation exactly, and only then to truncate the result to provide equations of motion to any desired order of approximation.

I regard my achievement of these two aims in [13–15] as the realisation of Mathisson's new mechanics.

Realisation of 1. led to explicit and unique expressions for the moment tensors as integrals of $T^{\alpha\beta}$. Realisation of 2. led to unambiguous analogues of p^α and $S^{\alpha\beta}$ that generalise (15) and (16) and have a natural identification as the momentum and spin of the body. This enables a third aim to be added.

3. To show that the Tulczyjew condition (17) can be used to determine L uniquely and that this L has properties which enable it to be identified naturally as the centre of mass line in general relativity.

Through the work of Ehlers and Rudolph [16] and of Schattner [17, 18], this too has been achieved.

The following sections outline the means by which these aims have been achieved.

6. Fourier transformation: special relativity

The first clue towards realising the aims of Mathisson's new mechanics can be found by considering the simpler environment of special relativity. Consider a rectangular coordinate system so that the components $g_{\alpha\beta}$ of the metric tensor are constant, but not necessarily diagonal. The coordinates x^α can then be treated as components of a position vector.

Mathisson's defining equation (3) for the moments can be rewritten in full as

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\, Dx = \int\limits_L \sum \frac{1}{n!} m^{\delta\cdots\gamma\alpha\beta}(s)\nabla_{\delta\cdots\gamma}\varphi_{\alpha\beta}(z(s))\, ds\,. \tag{18}$$

Here L is parameterised as $z^\alpha(s)$, where now for generality s is not necessarily proper time. Here and throughout, n is the number of indices in the set marked with dots, in this case $\delta\dots\gamma$, and we allow the possibilities $n=0$ and $n=1$. We seek to avoid the requirement for $\varphi_{\alpha\beta}$ to be analytic. To do so we introduce the Fourier transform $\widetilde{\varphi}_{\alpha\beta}$ defined by

$$\widetilde{\varphi}_{\alpha\beta}(k) = \int \varphi_{\alpha\beta}(x)\exp(\mathrm{i}\,k\cdot x)\,Dx\,, \tag{19}$$

where $k\cdot x \equiv k_\alpha x^\alpha$. For more complicated cases we shall also write the Fourier transform $\widetilde{\varphi}_{\alpha\beta}$ as $F[\varphi_{\alpha\beta}]$.

If we express $\varphi_{\alpha\beta}$ in terms of $\widetilde{\varphi}_{\alpha\beta}$ in (18), we get

$$\begin{aligned}\int T^{\alpha\beta}\varphi_{\alpha\beta}\,Dx \;=\; & \frac{1}{(2\pi)^4}\int_L ds\sum\int Dk\,\frac{(-\mathrm{i})^n}{n!}\,k_\delta\cdots k_\gamma \\ & \times m^{\delta\cdots\gamma\alpha\beta}(s)\widetilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}\,k\cdot z)\,.\end{aligned} \tag{20}$$

If we now exchange the order of the summation and the k-space integration we get

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\,Dx = M^{\alpha\beta}[\Phi_{\alpha\beta}]\,, \tag{21}$$

where

$$M^{\alpha\beta}[\Phi_{\alpha\beta}] = \frac{1}{(2\pi)^4}\int_L ds\int Dk\,\widetilde{m}^{\alpha\beta}(s,k)\widetilde{\Phi}_{\alpha\beta}(z(s),k) \tag{22}$$

with

$$\widetilde{m}^{\alpha\beta}(s,k) = \sum\frac{(-\mathrm{i})^n}{n!}\,k_\delta\cdots k_\gamma m^{\delta\cdots\gamma\alpha\beta}(s) \tag{23}$$

and

$$\widetilde{\Phi}_{\alpha\beta}(z,k) = \widetilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}\,k\cdot z). \tag{24}$$

Note that $\widetilde{\Phi}_{\alpha\beta}(z,k)$ is simply the Fourier transform of $\varphi_{\alpha\beta}$ about z as origin.

In this form the moment defining equation no longer requires $\varphi_{\alpha\beta}$ to be analytic. We have, of course, achieved this by exchanging a summation and an integration that cannot in general be validly exchanged. We adopt this new form as an improved defining equation in special relativity.

7. Fourier transformation: general relativity

To extend this definition to a curved spacetime, note that since the moments are tensors at $z(s)\in L$, k_α must be considered as a vector at the same point. This makes $\widetilde{m}^{\alpha\beta}$ be a tensor field on the tangent space $T_z(M)$ to the spacetime manifold M at $z(s)$, so $\widetilde{\Phi}_{\alpha\beta}$ must be likewise.

With these interpretations, equations (21), (22) and (23) can be taken over unchanged into a curved spacetime. We are left only with needing a generalisation of (24). There is no problem in defining Fourier transforms on $T_z(M)$ since it is always a flat manifold. We can therefore take $\widetilde{\Phi}_{\alpha\beta}$ to be the Fourier transform of a $\Phi_{\alpha\beta}$ that is also a tensor field on $T_z(M)$. This leaves us simply to decide how $\Phi_{\alpha\beta}$ is to be defined in terms of $\varphi_{\alpha\beta}$.

When M is flat we have a natural identification of the manifold $T_z(M)$ with M, which is the underlying reason why (19) is meaningful in special relativity. In a curved spacetime we relate the two manifolds by means of the exponential map $\mathrm{Exp}_z : T_z(M) \to M$. If $X \in T_z(M)$ and $x = \mathrm{Exp}_z X$ then the derivative map $((\mathrm{Exp}_z)_*)_X$ of Exp_z at X maps $T_X(T_z(M))$ isomorphically onto $T_z(M)$. This mapping between tangent spaces has a unique extension to a mapping of the corresponding tensor algebras. We denote this by replacing the $*$ by A (for Algebra) in the notation. By letting X vary we get a map $(\mathrm{Exp}_z)_\mathrm{A}$ from tensor fields on $T_z(M)$ to tensor fields on M. If we have such a tensor field for each z, we can apply the corresponding map to each of them to obtain a family of tensor fields on M parameterised by z, *i.e.* a two-point function with scalar character at z and tensor character at the second point x. We let Exp_A denote this overall map and Exp^A denote its inverse. These can both be formalised as maps between appropriate vector bundles.

We see that Exp^A acts on two-point tensors with scalar character at one point, say z. A special case of this is an ordinary tensor field treated as such a two-point tensor that is independent of z. We can therefore take

$$\Phi_{\alpha\beta} = \mathrm{Exp}^\mathrm{A}\varphi_{\alpha\beta}\,. \tag{25}$$

For each z, this defines $\Phi_{\alpha\beta}$ as a tensor field on $T_z(M)$.

If M is flat and we identify each of its tangent spaces with M itself then this gives in any rectangular coordinate system on M that

$$\Phi_{\alpha\beta}(z, X) = \varphi_{\alpha\beta}(x)\,, \qquad \text{where} \quad X \equiv x - z\,. \tag{26}$$

In the expression for X the points $x, z \in M$ are being identified with their position vectors relative to the coordinate origin, their difference being their relative position vector considered as an element of $T_z(M)$. If we take the Fourier transform of this on $T_z(M)$ we recover (24), which was our starting point in special relativity.

Equations (21), (22) and (23), when taken with (25) instead of (24), serve to define the infinite set of moments $m^{\alpha\beta}$, $m^{\alpha\beta\gamma}$, $m^{\alpha\beta\gamma\delta}, \ldots$ equally well in both special and general relativity in a manner that avoids the requirement for $\varphi_{\alpha\beta}$ to be analytic. We now adopt this as a new definition of the moments in general relativity. We have seen that in special relativity it is effectively a transformation of Mathisson's original definition (3). In general relativity, however, there is no longer any sense in which it is a transformation of Mathisson's original equation. We are making a real change in the definition of the moments in order to eliminate the analyticity requirement.

It should be noted that although there always exists a neighbourhood N_z of the origin in $T_z(M)$ that is mapped diffeomorphically by Exp_z onto its image in M, in general N_z cannot be extended to the whole of $T_z(M)$. For simplicity the presentation in this paper will be written as if N_z can be so extended, but the full derivations in [15] do not make this unrealistic simplification.

8. The hypersurfaces of integration

This change to the moment definitions enables us to deduce that the moments are expressible as integrals of $T^{\alpha\beta}$ over a uniquely determined set of hypersurfaces. We shall adopt additional conditions to ensure uniqueness of the moments, as in section 2. For later convenience, however, we shall make a slight generalisation of its condition (b) by replacing the four-velocity v^α by an arbitrary timelike field of unit vectors n^α along L, so that

$$n_\delta m^{\delta\cdots\gamma\beta\alpha} = 0 , \qquad \text{for } n \geq 1 . \tag{27}$$

Recall that n is the number of indices in the set marked with dots, so that this applies to moments with three or more indices.

Choose a Minkowskian coordinate system on $T_{z(s)}(M)$ such that $n_\alpha \neq 0$ only for $\alpha = 4$. Then (23) and (27) show that $\widetilde{m}^{\alpha\beta}(s,k)$ is independent of k_4. Recall now the result that for the Fourier transform $\widetilde{f}(k)$ of a function $f(k)$ of a single variable, we have

$$\int \widetilde{f}(k)\, dk = 2\pi f(0) . \tag{28}$$

This enables us to perform the k_4 integration in the s-integrand on the right of equation (22) to show that its value for fixed s depends on $\varPhi_{\alpha\beta}(z(s), X)$ only through its value on the hyperplane $X^4 = 0$, *i.e.* $n_\alpha X^\alpha = 0$. This hyperplane is mapped by Exp_z into the hypersurface $\varSigma(s)$ formed by all geodesics through $z(s)$ orthogonal to n_α. It follows that for each s, the s-integrand on the right of equation (22) depends on $\varphi_{\alpha\beta}$ only through its values on $\varSigma(s)$.

Now define a scalar function $\tau(x)$ by $\tau(x) = s$ if $x \in \varSigma(s)$ and let w^α be any vector field such that $w^\alpha \nabla_\alpha \tau = 1$. Then we have a corresponding decomposition

$$\int T^{\alpha\beta} \varphi_{\alpha\beta}\, Dx = \int ds \int\limits_{\varSigma(s)} T^{\alpha\beta} \varphi_{\alpha\beta} w^\gamma \, d\varSigma_\gamma , \tag{29}$$

of the left of equation (21), where $d\Sigma_\alpha$ is the vector-valued surface element on $\Sigma(s)$. Since both these s-integrals are expressions for $M^{\alpha\beta}[\Phi_{\alpha\beta}]$, it follows that the two s-integrands are equal, so that

$$\int_{\Sigma(s)} T^{\alpha\beta}\varphi_{\alpha\beta}w^\gamma\, d\Sigma_\gamma = \frac{1}{(2\pi)^4}\int Dk\, \widetilde{m}^{\alpha\beta}(s,k)\widetilde{\Phi}_{\alpha\beta}(z(s),k)\,. \tag{30}$$

It is now straightforward to express $\varphi_{\alpha\beta}$ in terms of $\widetilde{\Phi}_{\alpha\beta}$ and so to identify $\widetilde{m}^{\alpha\beta}(s,k)$ as an integral over $\Sigma(s)$. By expanding the resulting integrand as a series in k_α we may obtain explicit expressions for the moments $m^{\delta\cdots\gamma\alpha\beta}$ as integrals of $T^{\alpha\beta}$ over $\Sigma(s)$.

The moment defining equations of Sections 6 and 7, together with the symmetry and orthogonality conditions, provide an implicit definition of the moments. The above result shows that this implicit definition determines unique explicit expressions for the moments as integrals of $T^{\alpha\beta}$, a result that eluded Mathisson as we saw in Section 5. We are, therefore, already part way toward our goal.

9. Further revision of the moment definitions

We shall not give further detail here of the explicit expressions for the moments of the previous section as there are further modifications needed to the moment definitions before we can reach the goal of an exact solution to the variational equation. This time the variational equation itself is to be our guide. This equation is obtained by taking $\varphi_{\alpha\beta}$ to have the form

$$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)} \tag{31}$$

for an arbitrary vector field ω_α of compact support. We begin by investigating to what extent ω_α is determined if we only know $\varphi_{\alpha\beta}$. Observe first that if $\varphi_{\alpha\beta} = 0$ then ω_α is a Killing vector field. Moreover, any Killing vector field satisfies the equation of geodesic deviation along any geodesic. It follows that in the case of a nonzero $\varphi_{\alpha\beta}$, ω_α should be expected to satisfy an inhomogeneous form of the equation of geodesic deviation with the source term determined by $\varphi_{\alpha\beta}$.

This is indeed so. It is easily shown that ω_α satisfies

$$\frac{\delta^2}{du^2}\omega_\alpha + \omega_\beta\dot{x}^\gamma\dot{x}^\delta R^{\beta}_{\cdot\gamma\delta\alpha} = \dot{x}^\beta\dot{x}^\gamma\nabla_{\{\beta}\varphi_{\alpha\gamma\}}\,, \tag{32}$$

along all affinely parameterised geodesics $x^\alpha(u)$, where $\dot{x}^\alpha = dx^\alpha/du$ and curly brackets around three indices are defined by

$$A_{\{\alpha\beta\gamma\}} = A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta}\,. \tag{33}$$

The sign convention for the curvature tensor is such that

$$\nabla_{[\beta\gamma]}\omega_\alpha = \tfrac{1}{2}R^{\delta}_{\cdot\alpha\beta\gamma}\omega_\delta\,. \tag{34}$$

By integrating the equation for ω_α along all geodesics through a fixed point z, we can, therefore, find ω_α everywhere if we only know ω_α and $\nabla_\beta\omega_\alpha$ at z, as these values determine the required initial conditions for any geodesic. But $\nabla_{(\beta}\omega_{\alpha)} = \varphi_{\beta\alpha}$, so in fact we only need

$$A_\alpha(z) \equiv \omega_\alpha(z) \qquad \text{and} \qquad B_{\alpha\beta}(z) \equiv \nabla_{[\alpha}\omega_{\beta]}(z)\,. \tag{35}$$

Given a base point z, this construction will determine a vector field ω_α from an arbitrary tensor field $\varphi_{\alpha\beta}$, for any values of A_α and antisymmetric $B_{\alpha\beta}$ at z. We shall let $\lambda_\alpha(z,x)$ be the vector field so obtained when we take $A_\alpha = 0$ and $B_{\alpha\beta} = 0$. It is, of course, also a functional of the field $\varphi_{\alpha\beta}$, but we shall leave this dependence implicit. Similarly we let $\xi_\alpha(z,x)$ be the vector field so obtained when we take $\varphi_{\alpha\beta} = 0$ but leave A_α and $B_{\alpha\beta}$ arbitrary, its dependence on these two tensors at z again being left implicit. If $\varphi_{\alpha\beta}$, A_α and $B_{\alpha\beta}$ are constructed as above from a given ω_α then we have

$$\omega_\alpha(x) = \lambda_\alpha(z,x) + \xi_\alpha(z,x) \tag{36}$$

for any z. Note that ξ_α satisfies the true equation of geodesic deviation, *i.e.* with no source term.

To continue, we need to be able to differentiate fields such as $\Phi_{\alpha\beta}(z,X)$ in a covariant manner, with respect to both $z \in M$ and $X \in T_z(M)$. We give the required formulae for a field $\Psi^{\alpha}_{\cdot\beta}$ so as to illustrate the terms that arise from both contravariant and covariant indices. We define

$$\nabla_{\alpha*}\Psi^{\beta}_{\cdot\gamma} = \frac{\partial}{\partial z^\alpha}\Psi^{\beta}_{\cdot\gamma} - \Gamma^{\varepsilon}_{\alpha\delta}X^\delta\frac{\partial}{\partial X^\varepsilon}\Psi^{\beta}_{\cdot\gamma} + \Gamma^{\beta}_{\alpha\delta}\Psi^{\delta}_{\cdot\gamma} - \Gamma^{\delta}_{\alpha\gamma}\Psi^{\beta}_{\cdot\delta} \tag{37}$$

and

$$\nabla_{*\alpha}\Psi^{\beta}_{\cdot\gamma} = \frac{\partial}{\partial X^\alpha}\Psi^{\beta}_{\cdot\gamma}\,, \tag{38}$$

where the Levi–Civita connection $\Gamma^{\alpha}_{\beta\gamma}$ is evaluated at z. The first of these differs from the usual covariant derivative only through the addition of one term involving a derivative with respect to X^α. The second needs no connection terms since the components of X^α form a rectangular coordinate system on the flat tangent space. As shown in [15], these operations have a natural expression within the theory of connections on vector bundles. It can be shown that $\nabla_{\alpha*}$ commutes with Fourier transformation and that $\nabla_{\alpha*}X^\beta = 0$.

With this notation it can be shown that

$$\mathrm{Exp}^{\mathrm{A}}\nabla_{(\alpha}\lambda_{\beta)} = \nabla_{*(\alpha}\mathrm{M}_{\beta)} - \tfrac{1}{2}\Lambda^{\gamma}\nabla_{*\{\alpha}G_{\gamma\beta\}} \tag{39}$$

and

$$\begin{aligned} \Xi_{\alpha\beta} &\equiv \mathrm{Exp}^{\mathrm{A}}\nabla_{(\alpha}\xi_{\beta)} \\ &= \tfrac{1}{2}A^{\gamma}\nabla_{\gamma*}G_{\alpha\beta} + B^{\gamma\delta}X_{\delta}G_{\alpha\beta\gamma} - B^{\gamma\delta}\nabla_{*(\alpha}(G_{\beta)\gamma}X_{\delta})\,, \end{aligned} \tag{40}$$

where

$$G_{\alpha\beta} = \mathrm{Exp}^{\mathrm{A}}g_{\alpha\beta}\,, \qquad G_{\alpha\beta\gamma} = \tfrac{1}{2}\nabla_{*\{\alpha}G_{\gamma\beta\}} \tag{41}$$

and

$$\Lambda^{\alpha} = \mathrm{Exp}^{\mathrm{A}}\lambda^{\alpha}\,, \qquad \mathrm{M}_{\alpha} = \mathrm{Exp}^{\mathrm{A}}\lambda_{\alpha} = G_{\alpha\beta}\Lambda^{\beta}\,. \tag{42}$$

The A^{α} and $B^{\alpha\beta}$ are evaluated at z and are the values used in the construction of ξ^{α}. Indices of tensors on $T_z(M)$ are raised and lowered with $g_{\alpha\beta}(z)$, which is the flat metric on this tangent space, and is why we need different symbols for the lifts of the contravariant and covariant forms of the vector field $\lambda_{\alpha}(z,x)$. The field $\Xi_{\alpha\beta}(z,X)$ is defined by equation (40).

If $\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}$ as is used in the variational equation then these results give

$$\Phi_{\alpha\beta} + \tfrac{1}{2}\Lambda^{\gamma}\nabla_{*\{\alpha}G_{\gamma\beta\}} = \nabla_{*(\alpha}\mathrm{M}_{\beta)} + \Xi_{\alpha\beta}\,. \tag{43}$$

The right hand side is particularly simple as its first term only involves partial differentiation and its second term is completely determined by the parameters A^{α} and $B^{\alpha\beta}$ at z. The left hand side is well defined for a general field $\varphi_{\alpha\beta}(x)$ as this completely determines $\lambda_{\alpha}(z,x)$ and hence also $\Lambda^{\alpha}(z,X)$.

We capitalise on this simplicity by modifying the definition of the moments to take advantage of it. We replace equation (21) by

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\,Dx = M^{\alpha\beta}\Big[\Phi_{\alpha\beta} + \tfrac{1}{2}\Lambda^{\gamma}\nabla_{*\{\alpha}G_{\gamma\beta\}}\Big]\,, \tag{44}$$

where the functional $M^{\alpha\beta}$ remains defined by (22) and (23). The additional term vanishes in a flat spacetime since $G_{\alpha\beta}$ is then constant, so we are only modifying the gravitational contribution to the moments.

10. The orthogonality conditions revisited

The variational equation now becomes

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)} + \Xi_{\alpha\beta}] = 0\,. \tag{45}$$

Since Fourier transformation gives

$$F[\nabla_{*(\alpha}\mathrm{M}_{\beta)}] = -\mathrm{i}k_{(\alpha}\widetilde{\mathrm{M}}_{\beta)}\,, \tag{46}$$

we have from (22) and (23) that

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)}] = \frac{1}{(2\pi)^4}\int ds \int Dk\, \widetilde{t}^{\alpha}\widetilde{\mathrm{M}}_{\alpha}\,, \tag{47}$$

where

$$\widetilde{t}^{\alpha}(s,k) \equiv -\mathrm{i}\, k_\beta \widetilde{m}^{\alpha\beta}(s,k) = \sum \frac{(-\mathrm{i})^n}{n!}\, k_\gamma \cdots k_\beta t^{\gamma\cdots\beta\alpha}\,, \tag{48}$$

with

$$t^{\alpha} = 0\,, \qquad \text{and} \qquad t^{\delta\cdots\gamma\beta\alpha} = (n+1)m^{(\delta\cdots\gamma\beta)\alpha} \qquad \text{for } n \geq 0\,. \tag{49}$$

For $n = 0$ and $n = 1$ the tensors $m^{\delta\cdots\gamma\beta\alpha}$ and $t^{\delta\cdots\gamma\beta\alpha}$ are equivalent in that each can be expressed in terms of the other. For $n \geq 2$, *i.e.* when these tensors have four or more indices, we can decompose $m^{\delta\cdots\gamma\beta\alpha}$ through the use of symmetry operations into two tensors, one being $t^{\delta\cdots\gamma\beta\alpha}$ and the other being the tensor with the same number of indices from the set

$$J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha} \equiv m^{\zeta\cdots\varepsilon[\delta[\beta\gamma]\alpha]} \qquad \text{for } n \geq 0\,. \tag{50}$$

The nested square brackets denote antisymmetrisation independently over pairs of indices, the opening and closing brackets being paired in order from left to right. In (50) this means that antisymmetrisation is applied independently to the index pairs $\delta\gamma$ and $\beta\alpha$. It follows from these definitions that the J's have the symmetry properties

$$J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha} = J^{(\zeta\cdots\varepsilon)[\delta\gamma][\beta\alpha]} \quad \text{and} \qquad J^{\zeta\cdots\varepsilon\delta[\gamma\beta\alpha]} = 0 \qquad \text{for } n \geq 0\,, \tag{51}$$

$$J^{\zeta\cdots[\varepsilon\delta\gamma]\beta\alpha} = 0 \qquad \text{for } n \geq 1\,, \tag{52}$$

which for $n = 0$ and $n = 1$ are the symmetries of the curvature tensor $R_{\delta\gamma\beta\alpha}$ and its first derivative $\nabla_\varepsilon R_{\delta\gamma\beta\alpha}$, in the latter case when the Bianchi identities are taken into account.

A similar decomposition by symmetry operations can be made of the set (27) of orthogonality conditions. It follows from (50) that (27) implies

$$n_\zeta J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha} = 0 \qquad \text{for } n \geq 1\,. \tag{53}$$

The remaining restrictions in the set (27) are

$$n_\delta m^{\delta(\gamma\cdots\beta)\alpha} = 0 \qquad \text{for } n \geq 1\,. \tag{54}$$

So although the orthogonality conditions can be decomposed and one set (53) involves only the J's, the other set (54) does not involve only the t's.

We can largely achieve such a separation if we replace the members of set (54) with $n \geq 2$ by the set

$$n_\delta t^{\delta\gamma\cdots\beta\alpha} \equiv (n+1)\, n_\delta m^{(\delta\gamma\cdots\beta)\alpha} = 0 \qquad \text{for } n \geq 2\,. \tag{55}$$

These have the same symmetry properties as those of set (54) and hence they impose the same number of constraints. They have the advantage, however, of constraining precisely those parts of the m's that occur in the variational equation. We, therefore, make this change, which is of course a further change to the definition of the moments.

At the same time we omit the remaining condition $n = 1$ of the set (54). We cannot extend (55) to include $n = 1$ since for this case (54) is symmetric but (55) is not, so this would increase the number of constraints and we have already shown that the original set (27) suffices to ensure uniqueness of the moments. We shall see in due course precisely what freedom is left in the moments as a result of this omission.

11. Solution of the variational equation

We now show that as a consequence of the constraints (55), the variational equation (45) can be solved exactly. Substitute from (48) into (47) and from the result into (45). Then separate out the two terms in the summation whose t tensors are not constrained by (55). Integrate these two terms over k and then express the result in terms of the m's and ω_α instead of the t's and λ_α. With use of the expression (40) for $\Xi_{\alpha\beta}$, the variational equation can then be put in the form

$$\begin{aligned}\int ds\Big(& m^{\beta\alpha}\nabla_\beta\omega_\alpha(z) + m^{\gamma\beta\alpha}\nabla_{\gamma\beta}\omega_\alpha(z) + A^\alpha F_\alpha + \tfrac{1}{2}B^{\alpha\beta}(K_{\alpha\beta} + L_{\alpha\beta})\\ &+ \frac{1}{(2\pi)^4}\int Dk\,\widetilde{\mathrm{M}}_\alpha \sum_{n\geq 3}\frac{(-\mathrm{i})^n}{n!}\,k_\gamma\cdots k_\beta t^{\gamma\cdots\beta\alpha}\Big) = 0\,,\end{aligned} \tag{56}$$

where

$$F_\alpha = \frac{1}{2(2\pi)^4}\int \widetilde{m}^{\beta\gamma}\nabla_{\alpha *}\widetilde{G}_{\beta\gamma}\,Dk\,, \tag{57}$$

$$K_{\alpha\beta} = \frac{(-2\mathrm{i})}{(2\pi)^4}\int \widetilde{t}^{\gamma}\nabla_{*[\alpha}\widetilde{G}_{\beta]\gamma}\,Dk \tag{58}$$

and

$$L_{\alpha\beta} = \frac{2\mathrm{i}}{(2\pi)^4}\int \widetilde{G}_{\gamma\delta[\alpha}\nabla_{*\beta]}\widetilde{m}^{\gamma\delta}\,Dk\,. \tag{59}$$

Let, as before, $\Sigma(s)$ be the hypersurface formed by all geodesics through $z(s)$ that are orthogonal at z to $n_\alpha(s)$. The method of Section 8 then shows that as a consequence of (55) the k-space integral in (56) depends on $\lambda_\alpha(z(s), x)$ only through its values on $\Sigma(s)$. The construction of λ_α in Section 9 shows in turn that its values on $\Sigma(s)$ are determined by the values of $\omega_\alpha(x)$ on $\Sigma(s)$.

We will show that all the other terms in the s-integrand of (56) can be reduced to a form with this restricted dependence and, moreover, also dependent on ω_α only through the value of it and its first two derivatives at z. Once this is done, since the vector field ω_α is arbitrary it follows that the s-integrand must vanish separately for each value of s. But this integrand can be regarded as a generalised function, *i.e.* a continuous linear functional, on M itself rather than on $\Sigma(s)$. Its vanishing, therefore, implies that the infinite series in the k-space integral must be identically zero, as must be the coefficients of ω_α and its first two derivatives in the other terms of this integrand. Note that these latter terms correspond to terms in the infinite series that are constant, linear and quadratic in k_α and it is because such terms are absent that the series and the remainder of the integrand must vanish separately.

We now follow this through. At each point of L, define the projection operators

$$P_\alpha^{\cdot\beta} \equiv \chi n_\alpha v^\beta \qquad \text{and} \qquad Q_\alpha^{\cdot\beta} \equiv A_\alpha^\beta - P_\alpha^{\cdot\beta}\,, \tag{60}$$

where $\chi \equiv 1/(n_\alpha v^\alpha)$ and A_α^β is the unit tensor. Then the values at $z(s)$ of

$$\omega_\alpha\,, \quad Q_\beta^{\cdot\gamma}\nabla_\gamma\omega_\alpha \qquad \text{and} \qquad Q_\gamma^{\cdot\varepsilon}Q_\beta^{\cdot\delta}(\nabla_{(\varepsilon\delta)}\omega_\alpha + h_{\varepsilon\delta}n^\zeta\nabla_\zeta\omega_\alpha)\,, \tag{61}$$

can all be evaluated from knowledge of $\omega_\alpha(x)$ on $\Sigma(s)$. Here $h_{\beta\alpha}$ is symmetric and is the extrinsic curvature tensor (or second fundamental form) of $\Sigma(s)$, defined by

$$h_{\beta\alpha} = n_\beta n^\gamma \nabla_\gamma n_\alpha - \nabla_\beta n_\alpha\,, \tag{62}$$

where n_α is the field of unit normals to $\Sigma(s)$, which of course agrees with $n_\alpha(s)$ on L. In a flat spacetime $\Sigma(s)$ is a hyperplane so $h_{\beta\alpha} = 0$ and in a Minkowskian coordinate system the quantities on which the projection operators act reduce to the first and second partial derivatives of ω_α.

Decompose the derivatives of ω_α in (56) into the projections listed in (61) together with similar projections in which one or more of the Q operators is replaced by the corresponding P. Wherever a P operator occurs, use the identity

$$P_\beta^{\cdot\gamma}\nabla_\gamma \equiv \chi\, n_\beta \frac{\delta}{ds} \tag{63}$$

and perform an integration by parts with respect to s. In this way all P operators can be eliminated to leave, as the s-integrand, a reduced expression involving only the projections listed in (61) and so, as required, dependent on ω_α only through its values on $\Sigma(s)$.

We perform this reduction process in stages, drawing conclusions as we go along. We have already seen that the mere existence of this reduction requires the infinite series in the k-space integral of (56) to vanish, so that

$$t^{\gamma\cdots\beta\alpha} = 0 \quad \text{for } n \geq 3\,. \tag{64}$$

Taken with (58), this is sufficient to give $K_{\alpha\beta} = 0$ as the terms that remain evaluate to zero identically.

It is easily seen that only one term in the reduced expression will involve $\nabla_{(\gamma\beta)}\omega_\alpha$ and that vanishing of its coefficient gives

$$Q_\varepsilon^{\cdot\gamma} Q_\delta^{\cdot\beta} m^{(\epsilon\delta)\alpha} = 0\,. \tag{65}$$

From this $m^{(\gamma\beta)\alpha}$ must have the form

$$m^{(\gamma\beta)\alpha} = \frac{1}{2} v^{(\gamma} S^{\beta)\alpha} \tag{66}$$

for some tensor $S^{\alpha\beta}$ that at this stage is not necessarily antisymmetric. This gives

$$m^{\gamma\beta\alpha} = \frac{1}{2}\Big(S^{[\gamma\beta]} v^\alpha + S^{[\gamma\alpha]} v^\beta + S^{(\beta\alpha)} v^\gamma \Big) \tag{67}$$

and hence

$$m^{\gamma\beta\alpha}\nabla_{\gamma\beta}\omega_\alpha = \frac{1}{2} S^{\beta\alpha} \frac{\delta}{ds} \nabla_\beta \omega_\alpha + \frac{1}{2}\omega_\alpha v^\beta S^{[\gamma\delta]} R^\alpha{}_{\cdot\beta\gamma\delta}\,. \tag{68}$$

The first term on the right of (68) requires an integration by parts during the reduction process, following which the coefficient of $\nabla_\beta\omega_\alpha$ in the reduced expression can be read off and its vanishing seen to give

$$Q_\gamma^{\cdot\beta}\left(m^{\gamma\alpha} - \frac{1}{2}\frac{\delta}{ds} S^{\gamma\alpha} + \frac{1}{2} L^{\gamma\alpha} \right) = 0\,. \tag{69}$$

The bracketed expression, therefore, has the form

$$m^{\beta\alpha} - \frac{1}{2}\frac{\delta}{ds} S^{\beta\alpha} + \frac{1}{2} L^{\beta\alpha} = v^\beta p^\alpha \tag{70}$$

for some vector p^α. The symmetric and antisymmetric parts of this give

$$m^{\alpha\beta} = p^{(\alpha} v^{\beta)} + \frac{1}{2}\frac{\delta}{ds} S^{(\alpha\beta)} \tag{71}$$

and

$$\frac{\delta}{ds} S^{[\alpha\beta]} = 2p^{[\alpha} v^{\beta]} + L^{\alpha\beta}\,. \tag{72}$$

It also shows that

$$\left(m^{\beta\alpha} - \frac{1}{2}\frac{\delta}{ds}S^{\beta\alpha} + \frac{1}{2}L^{\beta\alpha}\right)\nabla_\beta\omega_\alpha = p^\alpha\frac{\delta}{ds}\omega_\alpha\,. \tag{73}$$

This requires another integration by parts during the reduction process before the coefficient of ω_α can be read off and its vanishing seen to give

$$\frac{\delta}{ds}p^\alpha = \frac{1}{2}v^\beta S^{[\gamma\delta]}R^{\alpha}_{\cdot\beta\gamma\delta} + F^\alpha\,. \tag{74}$$

This completes the solution of the variational equation but it has left us with some freedom in the definitions of $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$. This was anticipated in Section 10 when we omitted to make a replacement for the case $n = 1$ of the set (54) in our revision of the orthogonality conditions. It is easily shown that the change from Mathisson's original moment defining equation (3) to the final form (44) affects only the quadrupole and higher moment terms. The contributions to this equation from the monopole and dipole moments $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$ can, therefore, be found by substituting from (71) and (67) into (3). The two terms involving $S^{(\alpha\beta)}$ then combine into a total s-derivative that integrates to zero, leaving only contributions from p^α and $S^{[\alpha\beta]}$. It follows that $S^{(\alpha\beta)}$ can be chosen arbitrarily without affecting the values of any other moments.

One possibility would be to retain the condition $n_\gamma m^{\gamma\beta\alpha} = 0$ omitted from the set (54). This would determine $S^{(\alpha\beta)}$ in terms of $S^{[\alpha\beta]}$. In the case $n^\alpha = v^\alpha$ this recovers the forms (7) and (8) for $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$ seen in Section 3 in connection with the pole–dipole approximation. We shall make the simpler choice

$$S^{(\alpha\beta)} = 0\,, \tag{75}$$

which, in the context of Section 3, recovers the forms given in (12).

This completes the set of conditions required to ensure uniqueness of the moments.

12. Description of an extended body

In the light of these results, an extended body in general relativity can be described by an infinite set of tensors defined along an arbitrarily chosen world line L described by a parameter s which is not necessarily proper time. The set consists of a momentum vector p^α, an antisymmetric spin tensor $S^{\beta\alpha}$ and a set $J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha}$, for $n \geq 0$, of quadrupole and higher moments that satisfy the symmetry conditions (51) and (52). Here, as throughout, n is the number of indices in the set indicated by dots, including the two delimiting indices in the count.

These moments are not uniquely determined, but they become so if we choose an arbitrary field n^α of timelike unit vectors along L and require the J's to satisfy the orthogonality conditions (53). Note that even though the lowest order moment affected by the orthogonality conditions is the octopole moment, the values of all moments, including the momentum and spin, are affected by this choice.

The original set $m^{\beta\cdots\alpha}$ of moments that formed the starting point of the development is now given by

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} \quad \text{and} \quad m^{\alpha\beta\gamma} = S^{\alpha(\beta}v^{\gamma)}\,, \tag{76}$$

which follow from (71), (67) and (75), and

$$m^{\alpha\cdots\beta\gamma\delta\varepsilon} = \frac{4n}{n+2}\,J^{(\alpha\cdots\beta|\delta|\gamma)\varepsilon} \qquad \text{for } n \geq 1\,, \tag{77}$$

which follows from (50), (49) and (64). The vertical bars '|' enclose indices that are to be excluded from the symmetrisation. The moments are defined implicitly by their relationship with the energy-momentum tensor, given in terms of the m's by (44) and its related equations.

As a consequence of the energy-momentum conservation law (5) the moments satisfy the two equations of motion

$$\frac{\delta}{ds}p^\alpha = \frac{1}{2}v^\beta S^{\gamma\delta}R^{\alpha}_{\cdot\beta\gamma\delta} + F^\alpha \tag{78}$$

and

$$\frac{\delta}{ds}S^{\alpha\beta} = 2p^{[\alpha}v^{\beta]} + L^{\alpha\beta}\,, \tag{79}$$

where F^α and $L^{\alpha\beta}$ are the gravitational force and torque due to the quadrupole and higher moments, given by (57) and (59). Note that although (57) and (59) apparently also involve the monopole and dipole moments, those contributions vanish identically. These equations of motion reduce to the pole–dipole equations (11) when F^α and $L^{\alpha\beta}$ are neglected.

It is important to note that equations (78) and (79) are the *only* restrictions placed on the moments by (5). Suppose that tensor fields p^α, $S^{\beta\alpha}$ and $J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha}$ are given along L subject only to the equations of motion (78) and (79) and the above symmetry and orthogonality conditions. Then (44) determines a $T^{\alpha\beta}$, which in general will be a distribution (generalised function) rather than an ordinary tensor field, that satisfies (5) identically. Use of the variational equation has extracted *all* the consequences of energy-momentum conservation.

To complete the proof of the above results we still need to show that moments do exist that satisfy our new moment defining equation and with the symmetry and orthogonality conditions we have required. This has been done in [13–15] and it leads to explicit and uniquely determined expressions for p^α, $S^{\alpha\beta}$ and the J's as integrals of the energy-momentum tensor over the hypersurfaces $\Sigma(s)$ formed by the geodesics through $z(s)$ orthogonal to n^α. It is through the dependence of the hypersurface of integration on n^α that *all* the moments are affected by the orthogonality conditions even though those conditions apply only to the octopole and higher moments.

The explicit expressions for p^α and $S^{\alpha\beta}$ are particularly simple. Treat $z(s) \in L$ as a fixed point and let A_α, $B_{\alpha\beta}$ be an arbitrary vector and anti-symmetric tensor at z. As in Section 9 let $\xi_\alpha(z,x)$, considered as a vector function of x, be the solution of the equation of geodesic deviation that satisfies

$$\xi_\alpha = A_\alpha\,, \qquad \nabla_\alpha \xi_\beta = B_{\alpha\beta}\,, \qquad \text{at} \;\; x = z\,. \tag{80}$$

Then

$$A_\alpha p^\alpha + \frac{1}{2} B_{\alpha\beta} S^{\alpha\beta} = \int\limits_{\Sigma(s)} \xi_\alpha T^{\alpha\beta}\, d\Sigma_\beta\,. \tag{81}$$

It is possible to give ξ_α explicitly in terms of A_α and $B_{\alpha\beta}$ by the use of two-point tensors and so to resolve (81) into explicit integrals for p^α and $S^{\alpha\beta}$, as shown in [13], but this adds little for present purposes so it will not be given here. The special case of a flat spacetime is of interest, however. In a Minkowskian coordinate system

$$\xi_\alpha(z,x) = A_\alpha - B_{\alpha\beta} X^\beta \qquad \text{where} \quad X^\alpha = x^\alpha - z^\alpha\,, \tag{82}$$

so that (81) gives

$$p^\alpha \; = \; \int\limits_{\Sigma(s)} T^{\alpha\beta}\, d\Sigma_\beta\,, \tag{83}$$

$$S^{\alpha\beta} \; = \; 2 \int\limits_{\Sigma(s)} X^{[\alpha} T^{\beta]\gamma}\, d\Sigma_\beta\,. \tag{84}$$

The hypersurface $\Sigma(s)$ is specified in these expressions as the hyperplane through $z(s)$ orthogonal to n^α. However, the values of these integrals are independent of the choice of the hypersurface of integration. We have in fact recovered the definitions (15) and (16) of momentum and angular momentum in special relativity given in Section 4, but now they appear as part of a systematic treatment of multipole moments.

13. Equations of motion in the multipole approximation

If we substitute for $\widetilde{m}^{\alpha\beta}$ from (23) into the expressions (57) and (59) for F^α and $L^{\alpha\beta}$, we may exchange the order of the summation and the k-space integration. The integration may then be performed term by term to obtain a formal multipole expansion of the force and torque which can be truncated to give a multipole approximation to any desired order. The multipole expansion is most easily expressed in terms of the *extensions* $g_{\alpha\beta,\gamma\cdots\delta}$ of the metric tensor $g_{\alpha\beta}$ as defined by Veblen and Thomas [19]. These are the tensors that reduce at the pole of a normal coordinate system to the partial derivatives of the metric tensor. This is a natural construction in our context as the image under the exponential map of a rectangular coordinate system in the tangent space $T_z(M)$ is a normal coordinate system on M with z as its pole. We obtain

$$F_\alpha = \frac{1}{2}\sum_{n\geq 2}\frac{1}{n!}m^{\varepsilon\cdots\delta\gamma\beta}\nabla_\alpha g_{\gamma\beta,\varepsilon\cdots\delta} \tag{85}$$

and

$$L^{\alpha\beta} = \sum_{n\geq 1}\frac{1}{n!}g^{\gamma[\alpha}m^{\beta]\varepsilon\cdots\delta\zeta\eta}g_{\{\gamma\eta,\zeta\}\varepsilon\cdots\delta}\,. \tag{86}$$

The extensions are easily evaluated in terms of the curvature tensor and the lowest two are

$$g_{\alpha\beta,\gamma\delta} = -\tfrac{2}{3}R_{\alpha(\gamma\delta)\beta}\,, \qquad g_{\alpha\beta,\gamma\delta\varepsilon} = -\nabla_{(\gamma}R_{|\alpha|\delta\varepsilon)\beta}\,. \tag{87}$$

With these results we have completed the realisation of the first two aims laid down in Section 5.

14. Centre of mass revisited

We return now to a further study of the centre of mass. The expressions for p^α and $S^{\alpha\beta}$ given by (81) show that their values at z depend on L and the vector field n^α along it only through the point z and the value of n^α at this point. Although this may seem natural, the expressions for the J's are significantly more complicated. They depend not only on z and n^α but also on their first derivatives along L at z. This dependence arises through the vector field w^α introduced in Section 8 in the splitting of an integral over M into one over $\Sigma(s)$ followed by one over L. Although it is perhaps surprising, the expressions for p^α and $S^{\alpha\beta}$ do not involve w^α and so are free of this derivative dependence.

The expressions for p^α and $S^{\alpha\beta}$ can, therefore, be considered as functions of a general point $z \in M$ and timelike unit vector n^α at z. As seen in Section 4, in a flat spacetime the function $p^\alpha(z,n)$ is independent of n^α so at any point z we can choose n^α to be parallel to $p^\alpha(z)$. This suggests that in a curved spacetime we can expect to find an n^α that is parallel to $p^\alpha(z,n)$ even though this is now an implicit equation, *i.e.* we can choose

$$p^\alpha(z,n) = M(z)n^\alpha\,, \tag{88}$$

which also serves to define the mass $M(z)$. With this choice of n^α we can consider p^α and $S^{\alpha\beta}$ to be functions only of z. The Tulczyjew condition

$$p_\beta S^{\alpha\beta} = 0\,, \tag{89}$$

then becomes an equation for z and the locus of its solutions can be expected to be a uniquely determined world line that we may take as our base line L. It has been proved by Schattner [17, 18] that subject to mild restrictions on the strength of the gravitational field, these expectations are fulfilled. These conditions do determine the field n^α and the line L uniquely.

It is clear that this choice of L agrees in special relativity with the centre of mass line of Møller [5] described in Section 4, but this is not a very stringent test. That section also mentions an infinite set of possible definitions for centre of mass in general relativity, of which a proposal by the present author in [7] is just one, all of which would pass this same test. They all use the Tulczyjew condition (89) but differ in their definitions of p^α and $S^{\alpha\beta}$. The present proposal, based on the p^α and $S^{\alpha\beta}$ of Section 12, is at first sight simply another of this set. A more stringent test is, therefore, needed to show why it is a preferred choice.

Such a test is provided by motion in a spacetime of constant curvature. The homogeneity and isotropy of such a spacetime should imply the absence of any gravitational force or torque on any extended test body and the world line of its centre of mass would be expected to be a geodesic. We shall show that the present proposal passes this test.

In a spacetime of constant curvature the vector field ξ_α of Section 9 is a Killing vector field, *i.e.* satisfying $\nabla_{(\alpha}\xi_{\beta)} = 0$, for all choices of the initial values A_α and antisymmetric $B_{\alpha\beta}$. Then (40) gives $\Xi_{\alpha\beta} = 0$. It is $\Xi_{\alpha\beta}$ that gives rise to the terms in (56) containing A_α and $B_{\alpha\beta}$, so the vanishing of $\Xi_{\alpha\beta}$ implies $F_\alpha = 0$ and $K_{\alpha\beta} + L_{\alpha\beta} = 0$. Since we saw in Section 11 that $K_{\alpha\beta} = 0$ in all cases, we therefore have

$$F_\alpha = 0 \qquad \text{and} \qquad L_{\alpha\beta} = 0\,, \tag{90}$$

so that the gravitational force and torque vanish. But in a space of constant curvature

$$R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\,, \tag{91}$$

where k is its scalar curvature. Eqs. (78) and (79), therefore, simplify to

$$\frac{\delta p^\alpha}{ds} = kS^{\alpha\beta}v_\beta\,, \qquad \frac{\delta S^{\alpha\beta}}{ds} = 2p^{[\alpha}v^{\beta]}\,. \tag{92}$$

Application of δ/ds to (89) and use of (88) and (92) then gives

$$M^2(v^\alpha - n_\beta v^\beta n^\alpha) = kS^{\alpha\beta}S_{\beta\gamma}v^\gamma\,. \tag{93}$$

Now (89) implies that, considered as a matrix, $S^{\alpha\beta}$ is singular. Its rank is even, since it is antisymmetric, so its rank must be 2. Hence $S^{[\alpha\beta}S^{\gamma]\delta} = 0$, from which it follows that

$$S^{\alpha\gamma}S_{\gamma\delta}S^{\delta\beta} + S^2S^{\alpha\beta} = 0\,, \tag{94}$$

where $S^2 \equiv \frac{1}{2}S^{\alpha\beta}S_{\alpha\beta}$. Multiplication of (93) by $S_{\delta\alpha}$ with further use of (89) then gives

$$(M^2 + kS^2)S_{\alpha\beta}v^\beta = 0\,. \tag{95}$$

It follows that apart from one exceptional case $S_{\alpha\beta}v^\beta = 0$, which with (93) shows v^α to be parallel to p^α. If we take s to be proper time along L, which we have not yet required, then $p^\alpha = Mv^\alpha$. The equations of motion (92) then reduce to

$$\frac{dM}{ds} = 0\,, \qquad \frac{\delta v^\alpha}{ds} = 0\,, \qquad \frac{\delta S^{\alpha\beta}}{ds} = 0\,, \tag{96}$$

so that the mass M is constant, L is a geodesic and $S^{\alpha\beta}$ is covariant constant along it, as was claimed.

This calculation is the first place that the choice of parameter s on L has had any significance. It is easily seen from (44) and (22) that the effect of a change of parameter on the moments is to scale the m's uniformly for each s but that in general the scale factor will be s-dependent. This corresponds to scaling the J's but leaving p^α and $S^{\alpha\beta}$ unchanged. This is the only remaining freedom in the definition of the moments and it is a trivial one. Although the natural choice at first seems to be to take s as proper time along L, *i.e.* $v^\alpha v_\alpha = 1$, many expressions are simplified if we instead choose s so that $v^\alpha n_\alpha = 1$. Two examples are (60) and (93) but there are many others. This is not a compelling reason, however, so it is perhaps best to leave the choice open.

The exceptional case mentioned above is when $M^2 + kS^2 = 0$. When this holds, (89) no longer determines L uniquely. This is not physically relevant as it corresponds to impossibly extreme circumstances, but it is of theoretical interest as it shows that physical conditions do have to be imposed on the body to ensure this uniqueness. A set of sufficient conditions has been given by Schattner [17, 18]. A momentum-velocity relationship has been deduced in the general case from (78), (79) and (89) by Ehlers and Rudolph [16]. This too has an exceptional case. It determines v^α uniquely only if $M^2 + \frac{1}{4}R_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta} \neq 0$, of which the condition for a spacetime of constant curvature is a special case.

These results are sufficient to distinguish the world line characterised by (88) and (89), with the p^α and $S^{\alpha\beta}$ of (81), as a preferred choice for the definition of the centre of mass in general relativity. This construction for p^α and $S^{\alpha\beta}$ arises naturally from the completion of the Mathisson programme presented here, which ties the solution of the centre of mass problem firmly into this theory of multipole moments. With this work we have realised the third aim laid down in Section 5.

15. Conclusions

In 1937 Mathisson published a paper that introduced a new approach to the problem of motion in general relativity, in which an extended body was described by an infinite set of multipole moments. A central feature of his approach was what he termed the *variational equation of dynamics*. Other authors subsequently developed the multipole approach by means other than through the variational equation. They were able to re-derive Mathisson's results but not to extend them much further.

This paper has shown how the multipole programme initiated by Mathisson can be taken to completion, so obtaining equations of motion to an arbitrarily high multipole order, by bringing the variational equation back and using it as a key to further development. This extension of Mathisson's work leads to unambiguous expressions for the momentum and spin of an extended body as integrals of the energy-momentum tensor. It has also been shown how these expressions form the basis for a definition of the centre of mass of such a body that has features which distinguish it among a large class of similar definitions.

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