STATISTICAL INFERENCE OF MULTIVARIATE DISTRIBUTION PARAMETERS FOR NON-GAUSSIAN DISTRIBUTED TIME SERIES*

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We consider a portfolio of stocks whose returns conform to a stationary, multivariate distribution whose all integer moments are finite. For this portfolio we derive the distribution of eigenvalues of various sample covariance matrices and the moments of the eigenvalue distribution, for a particular type of distribution, in terms of the parameters of the portfolio distribution.

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1. Introduction

There are some ideas from nuclear and statistical physics that appear to be helpful in multivariate analysis. Here we mention the computation of a partition sum of a one-dimensional Coulomb gas [1] or planar diagram expansions of theories for strong interactions [2]. The modeling of a Hamiltonian of a nucleus as a random matrix, a random matrix whose elements are chosen such that the spectrum coincides with the measured spectra of nuclei [3] has stimulated investigations of Random Matrix Ensembles (to be termed the Random Matrix Theory). The matrices in the ensembles are invariant under particular similarity transformations what implies that certain properties of the ensemble, like probability densities of eigenvalues, are universal. The idea appeared to use these universal properties for stochastic inference in finance. One [4–8] derived a qualitative way of filtering noise from financial time series. Interesting applications for market factor models like the Capital Asset Pricing Model [9] appeared. One also speculated if it was possible to infer multivariate distribution of stocks from the distribution of eigenvalues of a sample covariance matrix.

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Since, however, the filtering noise method is only based on a result valid for the number of stocks $N \to \infty$ and the length of time series $T \to \infty$ and since it is based on the assumption of Gaussian fluctuations and it also assumes a particular form of the population covariance matrix caution has to be taken in the blind use of that result. In our opinion progress in statistical inference will only be possible if new analytical results are obtained. Some articles [10, 11] extended the theory to correlated Gaussian matrices (Wishart ensembles). The work [10] is a considerable step forward but the theory still assumes a particular form of the population covariance matrix (translational-invariance) the corrections for non-stationarity are made *ad hoc* and no closed system of equations was found for the estimators of the spectral moments of the population covariance matrix as a function of the spectral moments of the sample covariance matrix. It was also not clear how to compute confidence intervals for the estimators mentioned above. In this work we formulate a generic theory of the distribution of eigenvalues of a sample covariance matrix drawn from a non-Gaussian population with all finite integer moments. We do not assume any particular form for the population correlations between two different returns at two different times. We also speculate on computing percentiles of the distribution of eigenvalues and on generalizing the theory to Lévy stable populations.

2. The model

Consider a portfolio $\underline{\underline{S}} := \{S_{i,t}\}_{i,t=1,1}^{N,T}$ whose returns $X_{i,t} := d_t \log(S_{i,t})$ satisfy an equation:

$$\frac{dS_{i,t}}{S_{i,t}} := X_{i,t} = \underbrace{\alpha(\underline{\underline{S}})}_{=0} dt + \sum_{j,\xi=1}^{N,T} O_{(i,t),(j,\xi)} Y_{j,\xi} \,, \tag{1}$$

where $\alpha(\underline{\underline{S}}) = 0$ (zero drift), the tensor

$$\underline{\underline{O}} := \left\{ O_{(i,t),(j,\xi)} \right\}_{i,j=1,\ldots,N}^{t,\xi=1,\ldots,T} \tag{2}$$

(population covariance tensor) is symmetric, and $\{Y_{j,\xi}\}_{j,\xi=1}^{N,T}$ are iid random variables (base variables), with common even probability density $\rho_Y(y)$ and an analytical log-characteristic function

$$\psi_Y(\lambda) := \log\left(\mathcal{F}_y\left[\rho_Y\right](\lambda)\right) = \sum_{l=1}^{\infty} \frac{c_{2l}}{(2l)!} \lambda^{2l} \,, \tag{3}$$

where c_{2l} for $l = 1, 2, \ldots$ are cumulant of the density of base variables. Note that the model (1) can be written as

$$\vec{X}_t = \sum_{\xi=1}^{T} \underline{\underline{O}}_{t,\xi} \vec{Y}_\xi \,, \tag{4}$$

where $\vec{X}_t := (X_{1,t}, \ldots, X_{N,t})^T$ and $\vec{Y}_t := (Y_{1,t}, \ldots, Y_{N,t})^T$ and $\underline{\underline{O}}_{t,\xi}(i,j) := O_{(i,t),(j,\xi)}$. Whence the model is equivalent to a random vector Moving Average (MA) model with non time-homogeneous coefficients. Since fitting methods for MA models are known (see Durbin Levinson algorithm in [12]) our analysis provides a new approach to the model, an approach whose performance can be tested.

In the population (1) we define following matrices

$$\underline{\underline{\mathcal{E}}}(\underline{\underline{X}}) = \begin{cases} \frac{1}{T} \sum\limits_{t=1}^{T} X_{\boldsymbol{i},t} X_{\boldsymbol{j},t} \,, & \text{space covariances} \\ \frac{1}{N} \sum\limits_{i=1}^{N} X_{i,\boldsymbol{t}} X_{i,\boldsymbol{t}'} \,, & \text{time covariances} \\ \frac{1}{D} \sum\limits_{i=1}^{D} X_{j,\boldsymbol{t}} X_{\boldsymbol{i},j} \,, & \text{time-space cross-covariances} \\ \frac{1}{D} \sum\limits_{i=1}^{D} X_{\boldsymbol{i},j} X_{j,\boldsymbol{t}} \,, & \text{space-time cross-covariances} \end{cases} \tag{5}$$

where $D := \min(N,T)$. We note that these are all possible sample covariance matrices that can be constructed from products of two different random variables $X_{i,t}$. We denote the sample covariance matrices in a compact way as:

$$\underline{\underline{\mathcal{E}}}(\underline{\underline{X}})_{j_1,j_2} = \mathfrak{d}^{\underline{i}}_{j_1,j_2} \prod_{q=1}^{2} X_{i_{2q-1},i_{2q}} \,, \tag{6}$$

where $\underline{i} := (i_1, i_2, i_3, i_4)$ and

$$\mathfrak{d}^{\underline{i}}_{j_1,j_2} := \delta_{j_1, i_{\xi^{(1)}}} \delta_{i_\xi, i_\eta} \delta_{i_{\eta^{(1)}}, j_2} \,, \tag{7}$$

where $(\xi, \eta) = (2,4)$ and $(\xi^{(1)}, \eta^{(1)}) = (1,3)$ for space covariances, $(\xi, \eta) = (1,3)$ and $(\xi^{(1)}, \eta^{(1)}) = (2,4)$ for time covariances, $(\xi, \eta) = (1,4)$ and $(\xi^{(1)}, \eta^{(1)}) = (2,3)$ for time-space cross-covariances and $(\xi, \eta) = (2,3)$ and $(\xi^{(1)}, \eta^{(1)}) = (1,4)$ for space-time cross-covariances respectively. It is our objective to analyze the eigenvalues and eigenvectors of the sample covariance matrices (5) in order to estimate the largest possible set of population parameters $\underline{\underline{Q}}$ of the underlying distribution. For this purpose we define a Greens' function (resolvent) $\underline{\underline{G}}(z)$ viz

$$\underline{\underline{G}}(z) = \left(z - \underline{\underline{\mathcal{E}}}(\underline{\underline{X}})\right)^{-1} \tag{8}$$

for $z \in \mathbb{C}$.

The whole information about the eigenvalues of $\underline{\underline{\mathcal{E}}}(\underline{\underline{X}})$ is contained in the Generalized Density of Eigenvalues (GDE) $\mathfrak{D}_{\Lambda}^{(k)}(z)$. We have

$$\begin{aligned} &\lim_{\epsilon\to 0}\mathrm{Im}\ \mathrm{Op}^{(k)}\left[\underline{\underline{G}}(z+\imath\epsilon)\right] \qquad (9)\\ &= \frac{\pi}{\mathcal{C}_k^D}\sum_{1\le i_1<\ldots<i_k\le D}\sum_{p=1}^{k}\delta\left(z-\lambda_{i_p}\right)\prod_{\substack{q\neq p\\ q=1,\ldots,k}}\left(\lambda_{i_p}-\lambda_{i_q}\right)^{-1} =: \pi\mathfrak{D}_{\Lambda}^{(k)}(z)\,, \end{aligned}$$

where the operator $\mathrm{Op}^{(k)}$ is such that

$$\mathrm{Op}^{(k)}\left[\cdot\right] := \frac{1}{\mathcal{C}_k^D}\sum_{1\le i_1<\ldots<i_k\le D}\mathrm{Min}^{i_1,\ldots,i_k}\left[\cdot\right]\,, \tag{10}$$

where $\mathrm{Min}^{i_1,\ldots,i_k}$ are minors of dimension $1 \le k \le N$, and

$$\mathcal{C}_k^D := D!/(k!(D-k)!).$$

That statement follows from the fact that the numbers $\mathrm{Op}^{(k)}\left[\cdot\right]$ specify the characteristic polynomial of $\cdot$ uniquely since

$$\det[\cdot - \lambda 1] = \sum_{k=0}^{D}\mathcal{C}_k^D\mathrm{Op}^{(k)}\left[\cdot\right](-\lambda)^{D-k}\,. \tag{11}$$

Expanding the resolvent (8) in a Taylor series around $z = 0$ we obtain a following expansion of the GDE:

$$\pi\mathfrak{D}_{\Lambda}^{(k)}(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\ \pi\delta^{(m)}(z)\mathrm{Op}^{(k)}\left[\underline{\underline{\mathcal{E}}}^m(\underline{\underline{X}})\right]\,, \tag{12}$$

where $\delta^{(m)}(z)$ is the m-th derivative of the delta function.

We define a Generalized Spectral Moments (GSM) of order m, k of the sample covariance matrix $\underline{\underline{\mathcal{E}}}(\underline{\underline{X}})$ viz:

$$(\lambda^m)^{(k)} := \int\limits_{\mathbb{C}} dz z^m\mathfrak{D}_{\Lambda}^{(k)}(z) = \frac{1}{\mathcal{C}_k^D}\sum_{1\le i_1<\ldots<i_k\le D}\sum_{p=1}^{k}\frac{\lambda_{i_p}^m}{\prod_{q\neq p}(\lambda_{i_p}-\lambda_{i_q})} \tag{13}$$

and from (10) and (12) we get

$$(\lambda^m)^{(k)} = \mathrm{Op}^{(k)}\left[\underline{\underline{\mathcal{E}}}^m(\underline{\underline{X}})\right] . \tag{14}$$

From (14) and from (6) we get:

$$(\lambda^m)^{(k)} = \frac{1}{\mathcal{C}_k^D}\sum_{\underline{l}}\mathfrak{C}\left(\underline{l}\right)\left(\prod_{q=1}^{k}\prod_{p=1}^{m}\prod_{\xi=1}^{2}X_{l^{q,p}_{2\xi-1},l^{q,p}_{2\xi}}\right) , \tag{15}$$

where $\underline{l} := \left\{l^{q,p}_{\xi}\right\}$, where $q = 1,\ldots,k$, $p = 1,\ldots,m$ and $\xi = 1,2$, and $l^{q,p}_{\xi} = 1,2,\ldots,D$, and the coefficients in (15) read:

$$\begin{aligned}\mathfrak{C}\left(\underline{l}\right) \;:=\; & \sum_{1\le i_1<\ldots<i_k\le D}\sum_{\pi^{(k)}}\mathrm{sign}(\pi^{(k)}) \\ & \times\sum_{\underline{j}}\prod_{q=1}^{k}\mathfrak{d}^{\underline{l}^{q,1}}_{i_q,j^{(q)}_1}\mathfrak{d}^{\underline{l}^{q,2}}_{j^{(q)}_1,j^{(q)}_2}\cdot\ldots\cdot\mathfrak{d}^{\underline{l}^{q,m}}_{j^{(q)}_{m-1},i_{\pi^{(k)}_q}} ,\end{aligned} \tag{16}$$

where $\underline{j} := \{j^q_p\}$, where $q = 1,\ldots,k$, $p = 1,\ldots,m-1$, and $j^q_p = 1,2,\ldots,D$ and the constants $\mathfrak{d}$ are defined in (7). We note that the relation (15) is a quite general relation that is valid also in other statistical populations in particular in those that have infinite integer moments. Relation (15) holds as a equality in distribution. That equality can be used to compute percentiles of the distribution of eigenvalues of the sample covariance matrix.

Now we are going to compute the expectation value of the GSM (13). For this purpose we express the $2m$-correlation function of the variables $X_l := X_{l_1,l_2}$ in (1), where the pair index l is an ordered pair (l_1,l_2), as a decomposition of permutations π^{2m} into $\mathfrak{m}_\theta$, where $\theta = 1,\ldots,m$, cycles of length 2θ. This is termed as the Wick theorem for non-Gaussian distributions and it can be proven in a similar manner as the Gaussian Wick theorem is proven. We have:

$$\begin{aligned}\langle X_{l_1}X_{l_2}\ldots X_{l_{2m}}\rangle \;=\; & \sum_{\substack{\pi^{2m}\\ \sum_{q=1}^m q\mathfrak{m}_q=m}}\left(\prod_{q=1}^{m}c_{2q}^{\mathfrak{m}_q}\right) \\ & \times\left(\prod_{q=1}^{m}\prod_{\xi=1}^{\mathfrak{m}_q}O^{(2q)}_{l_{\pi^{(2m)}_{(\mathcal{A}_q+(\xi-1)2q+1)}},\ldots,l_{\pi^{(2m)}_{(\mathcal{A}_q+(\xi-1)2q+2q)}}}\right) ,\end{aligned} \tag{17}$$

where $\mathcal{A}_q = \sum_{p=1}^{q-1} 2p\mathfrak{m}_p$, the constants c_{2q} are cumulants of base variables and $O^{(2q)}_{l_1,\dots,l_{2q}}$ are $2q$-population covariances defined in (18) where:

$$O^{(2\theta)}_{l_{i_1},\dots,l_{i_{2\theta}}} := \sum_{k=1}^{NT} \prod_{\xi=1}^{2\theta} O_{l_{i_\xi},k} \,. \tag{18}$$

We note that for given $\mathfrak{m}_\theta$ there are

$$\frac{(2m)!}{\prod_{\theta=1}^{m}((2\theta)!)^{\mathfrak{m}_\theta}(\mathfrak{m}_\theta)!} \tag{19}$$

expansion terms in (17) (see Table I) In the case of Gaussian distributed base variables the sum in (17) reduces to cycles of length two only and the number of terms in the sum equals $(2m-1)!!$. This is the Wick theorem known from literature.

We take the expectation value of (15), we use (1), we identify

$$l^{q,p}_\xi = \left(l^{q,p}_{2\xi-1}, l^{q,p}_{2\xi}\right) = l_{(q-1)4m+(p-1)4+\xi} \tag{20}$$

for $q = 1,\dots,4$, $p = 1,\dots,m$, $\xi = 1,2$[1] and we use the Wick theorem (17) and, after lengthy but straightforward transformations, we obtain:

TABLE I

Characterization of permutations π^{2km} according to division into cycles.

km	# of permutations for given $(\mathfrak{m}_1, \mathfrak{m}_2, \dots)$		
1	$1(1)$		
2	$3(2)$	$1(0,1)$	
3	$15(3)$	$15(1,1)$	$1(0,0,1)$
4	$105(4)$	$35(0,2)$	$210(2,1)$
	$28(1,0,1)$	$1(0,0,0,1)$	
5	$945(5)$	$1575(1,2)$	$3150(3,1)$
	$630(2,0,1)$	$210(0,1,1)$	$45(1,0,0,1)$
	$1(0,0,0,0,1)$		
6	$10395(6)$	$5775(0,3)$	$51975(2,2)$
	$51975(4,1)$	$462(0,0,2)$	$13860(3,0,1)$
	$13860(1,1,1)$	$1485(2,0,0,1)$	$495(0,1,0,1)$
	$66(1,0,0,0,1)$	$1(0,0,0,0,0,1)$	

[1] The first and the last index in (20) are ordered pairs of indices.

$$\left\langle (\lambda^m)^{(k)} \right\rangle := \sum_{\substack{\sum_{q=1}^{km} q\mathfrak{m}_q = km \\ \pi^{(2km)}}} \left(\prod_{q=1}^{km} c_{2q}^{\mathfrak{m}_q} \right) \sum_{l_1,\ldots,l_{2km}} \mathfrak{H}_{m,k}(\underline{l})$$

$$\times \prod_{q=1}^{km} \prod_{\xi=1}^{\mathfrak{m}_q} O^{(2q)}_{l_{\pi^{(2km)}_{(\mathcal{A}_q+(\xi-1)2q+1)}},\ldots,l_{\pi^{(2km)}_{(\mathcal{A}_q+(\xi-1)2q+2q)}}}, \tag{21}$$

where

$$\mathfrak{H}_{m,k}(\underline{l}) := \frac{1}{\mathcal{C}_k^D} \sum_{1\le i_1<\ldots<i_k\le D} \det\left(\mathfrak{g}_{i_p,i_q}(\underline{l})_{p,q=1}^k\right), \tag{22}$$

where

$$\mathfrak{g}_{i_p,i_q}(\underline{l}) := \delta_{i_p, l^{q,1}_{\xi^{(1)}}} \delta_{l^{q,m}_{\eta^{(1)}}, i_q} \prod_{p=1}^{m} \delta_{l_\xi^{(q,p)}, l_\eta^{(q,p)}} \prod_{p=1}^{m-1} \delta_{l^{(q,p)}_{\eta^{(1)}}, l^{(q,p+1)}_{\xi^{(1)}}} \tag{23}$$

and $\mathcal{A}_q = \sum_{p=1}^{q-1} 2p\mathfrak{m}_p$.

The outer sum in (21) runs over all possible decompositions of permutation π^{2km} into $\mathfrak{m}_q$ cycles of length $2q$ and the inner sum runs over pair indices $l_1, \ldots, l_{2km} = 1, \ldots, D$. The cycles are ordered in ascending order of their length. The ξ-th cycle of length $2q$:

$$\left(\pi^{(2km)}_{(\mathcal{A}_q+(\xi-1)2q+1)}, \ldots, \pi^{(2km)}_{(\mathcal{A}_q+(\xi-1)2q+2q)}\right) \tag{24}$$

is given a weight c_{2q} and a $2q$-correlation tensor element, defined in (18), endowed with indices corresponding to the cycle. The weight of the permutation is obtained by multiplying the weights of cycles and finally the expression is multiplied by a constant $\mathfrak{H}_{m,k}(\underline{l})$, a constant that is proportional to the $(D-k)$-th coefficient of the characteristic polynomial of the matrix $\{g_{i,j}(\underline{l})\}_{i,j=1}^D$, a coefficient that does not depend on the cumulants and the correlation tensor but does depend on the indices $\underline{l}$. The quantity $\mathfrak{H}_{m,k}(\underline{l})$ is a linear operator acting in the $4km$-dimensional space of $\underline{l}$ indices with integer values. This linear operator is a linear combination of $\mathcal{C}_k^D \cdot k!$ operators, labeled by an ordered sequence $I := 1 \le i_1 < i_2 < \ldots < i_k \le D$ and by a permutation π^k, with a coefficient being the sign of the permutation π^k. The operators are $4km$-dimensional Kronecker delta functions of the form

$$\delta^{(4km)}\left(\underline{\underline{M}}(I;\pi^k)\cdot\underline{l}\right)\,, \tag{25}$$

for some the $4km$-dimensional matrices $M := \underline{\underline{M}}(I;\pi^k)$. In the case $k=1$ the matrix $M := \underline{\underline{M}}(i_1;\pi^1)$ in (25) reads:

$$\begin{aligned}
\underline{\underline{M}}_{(q,p),(q,p)+\xi} &= 1\,, \\
\underline{\underline{M}}_{(q,p),(q,p)+\eta} &= -1\,, \quad \text{for } p=1,\dots,m\,, \\
\underline{\underline{M}}_{(q,p)+km,(q,p)+\eta^{(1)}} &= 1\,, \\
\underline{\underline{M}}_{(q,p)+km,(q,p)+\xi^{(1)}} &= -1\,, \quad \text{for } p=1,\dots,m-1\,, \\
\underline{\underline{M}}_{q+2km-k,(q-1)m+1+\xi^{(1)}} &= 1\,, \\
\underline{\underline{M}}_{q+2km-k,(q-1)m+m+\eta^{(1)}} &= -1\,, \qquad (26) \\
& \qquad (27)
\end{aligned}$$

for $q=1,\dots,k$. Here we denoted $(q,p) := (q-1)m+p$. The matrix is zero otherwise.

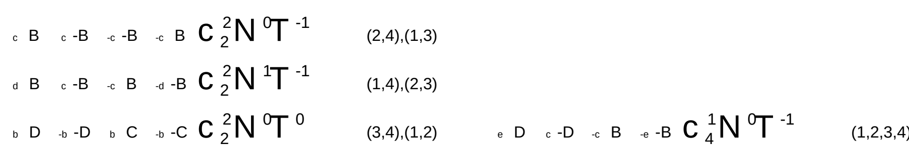

Fig. 1. Graphical representation of expansion terms of the second, $k=1$ spectral moment of the space sample covariance matrix. The (lower) upper case letters are ordered pairs (space, time) of indices of a product of discrete Fourier transforms of the translationally invariant population covariance matrices $\tilde{\mathcal{O}}_k$ (this was not mentioned in text). The columns correspond to given powers of cumulants and the rows correspond, from below to above, given powers of $1/T$. The permutation of a given term is displayed on the right-hand side.

In the generic $k \neq 1$ case the matrix in (25) is also easily found; only the last k rows are modified. Here the units in (27) are two-dimensional identity matrices. The set of $\underline{l}$ indices that are picked out by the operator (25) are obtained as the null space of a matrix $\underline{\underline{M}}$ by means of Singular Value Decomposition. By this means we express the GSM in terms of contractions of the population covariance tensor (see Figs. 1 and 2). Now we show how results known from literature appear as special cases of equation (21).

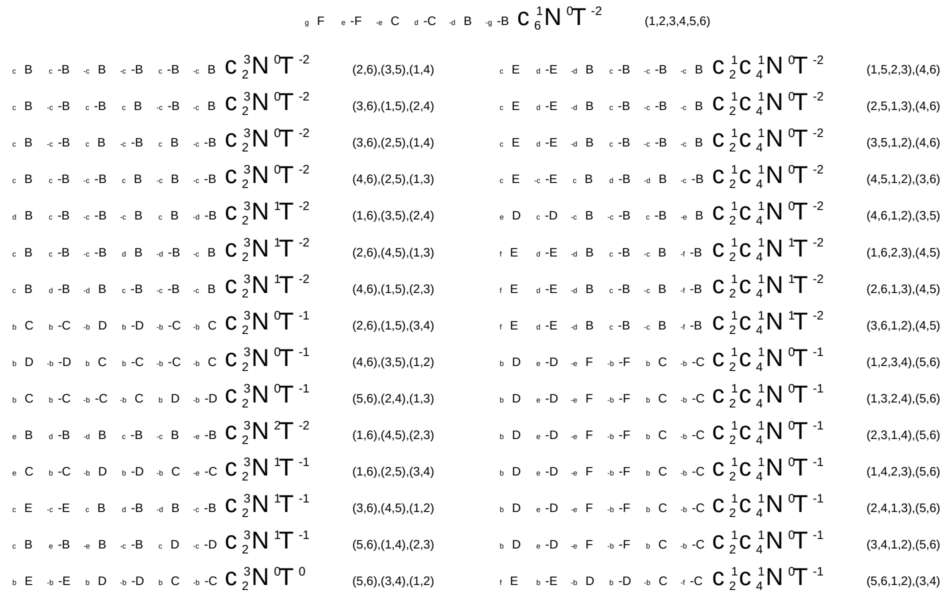

Fig. 2. The same as in Fig. 1 but for third, $k = 1$ spectral moment.

3. Examples

We assume that the correlation tensor factorize-s, meaning that $O_{(i,t),(j,\xi)} = C_{i,j} A_{t,\xi}$, into spatial and temporal parts $C_{i,j}$ and $A_{t,\xi}$, respectively, and that both parts are translationally invariant, meaning that $C_{i,j} = C_{i-j}$ and $A_{t,\xi} = A_{t-\xi}$, respectively. We denote the Spectral Moments of both parts by $\mathfrak{M}1(n) := \frac{1}{N}\mathrm{Tr}[C^n]$ and $\mathfrak{M}2(n) := \frac{1}{T}\mathrm{Tr}[A^n]$, respectively. Then the $k = 1$ spectral moments of the space sample covariance matrix (first in (5)) read:

$$m_{c1} = c_2\, \mathfrak{M}1(2)\, \mathfrak{M}2(2)\,, \tag{28}$$

$$\begin{aligned} m_{c2} &= \frac{c_4\, \mathfrak{M}1(2)^2\, \mathfrak{M}2(2)^2}{T} + {c_2}^2 \left(\mathfrak{M}1(4)\, \mathfrak{M}2(2)^2 + r\, \mathfrak{M}1(2)^2\, \mathfrak{M}2(4) \right. \\ &\quad \left. + \frac{\mathfrak{M}1(4)\, \mathfrak{M}2(4)}{T} \right), \end{aligned} \tag{29}$$

$$\begin{aligned} m_{c3} &= \frac{c_6\, \mathfrak{M}1(2)^3\, \mathfrak{M}2(2)^3}{T^2} + c_2\, c_4 \left(\frac{5\, \mathfrak{M}1(2)\, \mathfrak{M}1(4)\, \mathfrak{M}2(2)\, \mathfrak{M}2(4)}{T^2} \right. \\ &\quad \left. + \frac{7\, \mathfrak{M}1(2)\, \mathfrak{M}1(4)\, \mathfrak{M}2(2)^3 + 3\, r\, \mathfrak{M}1(2)^3\, \mathfrak{M}2(2)\, \mathfrak{M}2(4)}{T} \right) \end{aligned}$$

$$+ {c_2}^3 \left(\mathfrak{M}1(6)\, \mathfrak{M}2(2)^3 + 3\, r\, \mathfrak{M}1(2)\, \mathfrak{M}1(4)\, \mathfrak{M}2(2)\, \mathfrak{M}2(4) \right.$$
$$+ r^2\, \mathfrak{M}1(2)^3\, \mathfrak{M}2(6) + \frac{4\, \mathfrak{M}1(6)\, \mathfrak{M}2(6)}{T^2}$$
$$\left. + \frac{3\, \mathfrak{M}1(6)\, \mathfrak{M}2(2)\, \mathfrak{M}2(4) + 3\, r\, \mathfrak{M}1(2)\, \mathfrak{M}1(4)\, \mathfrak{M}2(6)}{T} \right), \tag{30}$$

where we denoted $m_{cn} := \left\langle (\lambda^n)^{(1)} \right\rangle$. We note that the first term and the next three terms on the right-hand side in (29) correspond to the graph on the right-hand side and to the graphs on the left-hand side from down to above in Fig. 1, respectively. The reader is encouraged to verify that the graphs in Fig. 2 also correspond to particular terms on the right-hand side of equation (30).

In the Gaussian case ($c_{2n} = 0$ for $n > 1$ (higher cumulants)) we retrieve the known relations (equations (34) in [10][2]). The corrections for finite values of higher cumulants in (29) are essential and the question if these corrections tend to zero in the thermodynamic limit (in the limit $T, N \to \infty$) as it is commonly assumed in most papers on this subject, is not simple without an *a priori* knowledge of the population covariance tensor.

4. Summary

We have expressed the generalized spectral moments $\langle \lambda^m \rangle^{(k)}$ of the sample covariance matrices as sums over decompositions of π^{2km} permutations into cycles of all possible lengths.

Terms corresponding to permutations composed entirely of cycles length two ($\mathfrak{m}_\theta = 0$ for $\theta > 1$ and $\mathfrak{m}_1 = km$) are dominant in terms of their absolute value. Other terms are inverse proportional to positive powers of N and T.

5. Open questions

The open questions are:

1. What are other resolvents (8) suitable for parameter estimation in other classes of random distributions?

2. Is it possible to find closed form expressions, like (13), for fractional generalized spectral moments?

[2] The authors have taken $c_2 = 1$.

3. Can we construct, in (9) and (10), operators related to the eigenvectors of sample covariance matrices in order to extract the whole information about the population covariance matrices? Can inverse participation ratios, (next-)nearest neighbor distributions of eigenvalues (universal properties of certain ensembles of random matrices) be used for inference of population covariances?

4. Can the model be generalized to a Lévy stable population?

5. How can the results helpful in risk minimization (minimize the risk measure subject to given return) for different risk measures (variance, VaR, absolute values of the return)?

6. Conclusions

We have obtained a system of non linear equations that relate the generalized spectral moments of the sample covariance matrices to spectral moments of the population covariance matrix in a unique way, ie without any additional assumptions about the parameters of the model. We have derived the Wick theorem for non-Gaussian random variables. This can be helpful for the computation of partition sums of non-harmonic field theories coupled to a thermal bath.

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