SCALAR MESON $f_0(500)$ FROM THE ANALYSIS OF PION SCALAR FORM FACTOR AND THE CORRECT S-WAVE ISO-SCALAR $\pi\pi$ PHASE SHIFT DATA*

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In the recently elaborated fully solvable mathematical problem, to be concerned of a finding of an explicit form of the pion scalar form factor, the inaccurate experimental information on the S-wave iso-scalar $\pi\pi$-scattering phase shift in the elastic region is replaced by the data with theoretical errors to be generated by the Garcia-Martin–Kaminski–Pelaéz–Yndurain Roy-like equations and, as a result, the correct values $m_\sigma = (472 \pm 10)$ MeV and $\Gamma_\sigma = (524 \pm 22)$ MeV of the scalar meson $f_0(500)$ parameters are determined.

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1. Introduction

In paper [1], fully solvable mathematical scheme for finding an explicit form of the experimentally unmeasurable pion scalar form factor (FF) $\Gamma_\pi(t)$ was elaborated and, as a result, an existence of the scalar meson $f_0(500)$ (historically to be called the σ-particle) has been evidently demonstrated.

However, its mass and the width obtained in this way do not agree with values to be determined by different methods in papers [2, 3], which are considered to be the best determinations of the σ-meson parameters up to now. A reason is that there are no data on the S-wave iso-scalar $\pi\pi$-phase shift with very high precision in the elastic region.

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Therefore, in this paper, more precise data on the S-wave iso-scalar $\pi\pi$-scattering phase shift, generated by the Garcia-Martin–Kaminski–Peláéz–Yndurain Roy-like equations in the elastic region, *i.e.* below 1 GeV^2, are used and one finally obtains in the framework of the fully solvable mathematical scheme in [1] the correct values of the σ-meson parameters to be consistent with mass and the width from [2, 3].

2. Pion scalar form factor phase representation

The pion scalar FF $\Gamma_\pi(t)$ is defined by means of the matrix element of the scalar quark density $\bar{u}u + \bar{d}d$

$$\left\langle \pi^i(p_2) \mid \widehat{m}\left(\bar{u}u + \bar{d}d\right) \mid \pi^j(p_1)\right\rangle = \delta^{ij}\Gamma_\pi(t)\,, \tag{1}$$

where $t = (p_2 - p_1)^2$ is the momentum transfer squared and $\widehat{m} = \frac{1}{2}(m_u + m_d)$. The pion scalar FF has similar properties to the pion electromagnetic FF [4].

The starting point of the fully solvable mathematical scheme in [1] is the pion scalar FF phase representation, which is derived from the pion scalar FF dispersion relation with a combination of the pion scalar FF elastic unitarity condition.

Application of any dispersion relation is always more effective if as much subtraction is used as possible. However, the pion scalar FF is not experimentally measurable quantity and we know it only at the normalization point from χPT [5] to be equal to the pion sigma term $\Gamma_\pi(0) = (0.99 \pm 0.02)m_\pi^2$.

Just the latter value $\Gamma_\pi(0) = m_\pi^2$ is used to derive dispersion relation with one subtraction, which together with the pion scalar FF elastic unitarity condition

$$\mathrm{Im}\Gamma_\pi(t) = \Gamma_\pi(t)e^{-i\delta_0^0}\sin\delta_0^0\,, \tag{2}$$

with S-wave isoscalar $\pi\pi$-scattering partial wave amplitude $M_0^0(t) = e^{+i\delta_0^0}\sin\delta_0^0$ and valid for $4m_\pi^2 \leq t \leq 1$ GeV^2, provides the true pion scalar FF phase representation with one subtraction

$$\Gamma_\pi(t) = P_n(t)\exp\left[\frac{t}{\pi}\int\limits_{4m_\pi^2}^{\infty}\frac{\delta_\Gamma(t')}{t'(t'-t)}dt'\right]\,, \tag{3}$$

where $P_n(t)$ is an arbitrary polynomial normalized at $t = 0$ $P_n(0) = 1$, however, its degree must not violate the asymptotic behavior of $\Gamma_\pi(t)$.

3. Most general parametrization of $\tan\delta_\Gamma(t)$ in q variable

The pion scalar FF $\Gamma_\pi(t)$ is analytic in the whole complex t-plane, except for a cut along the positive real axis, from $t = 4m_\pi^2$ up to ∞.

For real values $t < 4m_\pi^2$, $\Gamma_\pi(t)$ is real. The latter implies the so-called reality condition

$$\Gamma_\pi^*(t) = \Gamma_\pi(t^*)\,, \tag{4}$$

i.e. that the values of FF above and below the cut are complex conjugate of each other.

Starting from the unitarity condition (2), one can do analytic continuation of $\Gamma_\pi(t)$ through the upper and lower boundaries of the elastic unitary cut and prove in this way the singularity at $t = 4m_\pi^2$ to be a square root branch point, generating two-sheeted Riemann surface in t-variable on which the pion scalar FF is defined.

So, by an application of the conformal mapping,

$$q = [(t-4)/4]^{1/2}\,, \qquad m_\pi = 1 \tag{5}$$

two-sheeted Riemann surface of $\Gamma_\pi(t)$ in t-variable is mapped into one absolute valued pion c.m. three-momentum q-plane and the elastic cut disappears. Neglecting all branch points beyond 1 GeV^2, there are only poles and zeros of $\Gamma_\pi(t)$ in q-plane. The latter together with the reality condition (4) leads to the completely general parametrization of the tangens of the pion scalar FF phase

$$\tan\,\delta_\Gamma(t) = \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots} \tag{6}$$

with all A_i real coefficients.

There is no knowledge about a behavior of δ_Γ to describe it optimally by finite number of the coefficients A_i in (6). Nevertheless, from the elastic unitarity condition (2), it follows that the phase δ_Γ of $\Gamma_\pi(t)$ coincides with the S-wave iso-scalar $\pi\pi$ phase shift δ_0^0 for which data exist (though very inaccurate and, in some region, even contradicting) and just the latter enabled us to obtain in [1] an explicit form for the pion scalar FF $\Gamma_\pi(t)$ to be valid below 1 GeV^2 and subsequently to specify the $f_0(500)$ meson pole on the second Riemann sheet in t-variable.

4. Analysis of new S_0^0 $\pi\pi$ phase shift data

The inaccurate experimental information on the S-wave iso-scalar $\pi\pi$ phase shift below 1 GeV^2 has been sufficient in [1] to confirm the existence of the σ-particle, however, with no world-wide values of mass and the decay width.

In order to demonstrate obtaining the correct values of the $f_0(500)$-meson parameters, here we use instead of the existing inaccurate experimental information in Fig. 1 the data below 1 GeV^2 with theoretical errors in Fig. 2 to be generated by the Garcia-Martin–Kaminski–Peleáez–Yndurain Roy-like equations. Their best description (see full line in Fig. 2) by the parametrization

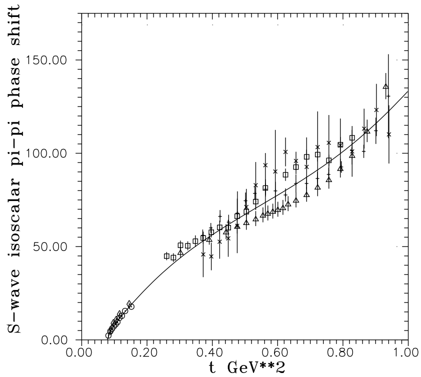

Fig. 1. Experimental information on S-wave iso-scalar $\pi\pi$ phase shift exploited in [1].

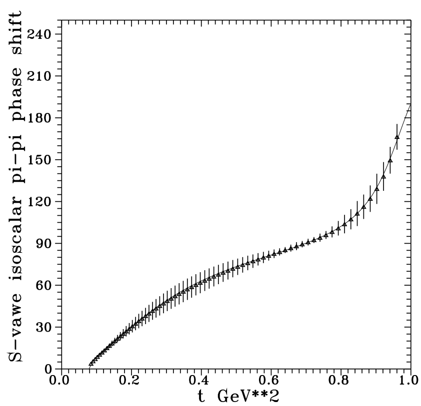

Fig. 2. New S-wave iso-scalar $\pi\pi$ phase shift data with theoretical errors.

$$\delta_0^0(t) = \arctan \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots} \tag{7}$$

is achieved again with first 5 nonzero real coefficients, however, acquiring the following numerical values:

$$\begin{aligned} A_1 &= 0.23456 \pm 0.00778\,; \qquad A_3 = 0.11595 \pm 0.00296\,; \\ A_5 &= -.01180 \pm 0.00031\,; \qquad A_2 = -.10376 \pm 0.00373\,; \\ A_4 &= -.00288 \pm 0.00046 \end{aligned}$$

to be different from those in [1].

5. Calculation of the integral leading to explicit form of $\Gamma_\pi(t)$

The substitution of $\delta_0^0(t)$ (7) with 5 nonzero above-mentioned coefficients into the pion scalar FF phase representation (3) does not lead to the way of an explicit calculation of the corresponding integral.

Therefore, the equivalent form to arctan (well known in the theory of functions of complex variables) the logarithmic representation

$$\delta_0^0(t) = \frac{1}{2i} \ln \frac{\left(1 + A_2 q^2 + A_4 q^4\right) + i\left(A_1 q + A_3 q^3 + A_5 q^5\right)}{\left(1 + A_2 q^2 + A_4 q^4\right) - i\left(A_1 q + A_3 q^3 + A_5 q^5\right)}$$

is utilized, which gives the expression

$$\Gamma_\pi(t) = P_n(t) \exp \frac{\left(q^2+1\right)}{\pi i} \int\limits_0^\infty \frac{q' \ln \frac{\left(1+A_2 q'^2+A_4 q'^4\right)+i\left(A_1 q'+A_3 q'^3+A_5 q'^5\right)}{\left(1+A_2 q'^2+A_4 q'^4\right)-i\left(A_1 q'+A_3 q'^3+A_5 q'^5\right)}}{\left(q'^2+1\right)\left(q'^2-q^2\right)} dq' \quad (8)$$

already leading to our intention.

Really, taking into account the fact that the integrand is even function of its argument, *i.e.* it is invariant under the transformation $q' \to -q'$, the previous expression takes the form

$$\Gamma_\pi(t) = P_n(t) \exp \frac{\left(q^2+1\right)}{2\pi i} \int\limits_{-\infty}^\infty \frac{q' \ln \frac{\left(1+A_2 q'^2+A_4 q'^4\right)+i\left(A_1 q'+A_3 q'^3+A_5 q'^5\right)}{\left(1+A_2 q'^2+A_4 q'^4\right)-i\left(A_1 q'+A_3 q'^3+A_5 q'^5\right)}}{\left(q'^2+1\right)\left(q'^2-q^2\right)} dq' \quad (9)$$

in which the integral is explicitly calculable by means of the theory of residua.

The corresponding integral, considering the case $q^2 < 0$, *i.e.* $q = i\sqrt{\frac{4-t}{4}} \equiv ib$, is transformed into the final form

$$I = \int\limits_{-\infty}^\infty \frac{q' \ln \frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{\left(q'-q_1^*\right)\left(q'-q_2^*\right)\left(q'-q_3^*\right)\left(q'-q_4^*\right)\left(q'-q_5^*\right)}}{\left(q'+i\right)\left(q'-i\right)\left(q'+ib\right)\left(q'-ib\right)} dq' \quad (10)$$

with all singularities of its integrand presented in Fig. 3.

Further, it is convenient to split the integral into sum of two integrals

$$I = \int\limits_{-\infty}^\infty \frac{q' \ln \frac{(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{\left(q'-q_1^*\right)}}{\left(q'+i\right)\left(q'-i\right)\left(q'+ib\right)\left(q'-ib\right)} dq' \quad (11)$$

$$+ \int\limits_{-\infty}^\infty \frac{q' \ln \frac{(q'-q_1)}{\left(q'-q_2^*\right)\left(q'-q_3^*\right)\left(q'-q_4^*\right)\left(q'-q_5^*\right)}}{\left(q'+i\right)\left(q'-i\right)\left(q'+ib\right)\left(q'-ib\right)} dq' = I_1 + I_2 \quad (12)$$

according to singularities to be placed in the upper half-plane or in the lower half-plane, respectively.

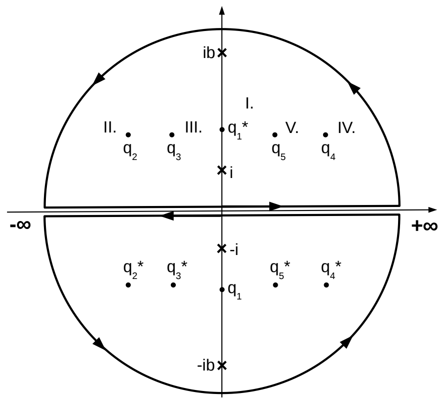

Fig. 3. Poles ($\times$) and branch points ($\bullet$) of the integrands $\phi_1(q', q)$ and $\phi_2(q', q)$ with contours of integrations in the upper and the lower half-planes, respectively.

The simplest way of a calculation of both integrals is the following.

In the first integral I_1,

$$\oint \frac{q' \ln \frac{(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n\,, \tag{13}$$

the contour of integration is closed in the lower half-plane and in the second integral I_2,

$$\oint \frac{q' \ln \frac{(q'-q_1)}{(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n\,, \tag{14}$$

the contour of integration is closed in the upper half-plane.

In a such way, one avoids complicated calculations of the cut contributions (see [1]) to be generated by branch points under logarithms.

Then for I_1, one gets

$$-I_1 = \int\limits_{+\infty}^{-\infty} \phi_1(q') dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n \tag{15}$$

with residua at the poles

$$\mathrm{Res}\,\phi_1(-i, q) = -\frac{1}{2\,(q^2+1)} \ln \frac{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}{-\,(i+q_1^*)}\,,$$

$$\mathrm{Res}\,\phi_1(-ib, q) = \frac{1}{2\,(q^2+1)} \ln \frac{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)}{-\,(q+q_1^*)}\,,$$

and for I_2 as follows

$$I_2 = \int\limits_{-\infty}^{\infty} \phi_2(q')dq' = -2\pi i \sum_{n=1}^{2} \mathrm{Res}_n \tag{16}$$

with residua at the poles

$$\mathrm{Res}\,\phi_2(i,q) = -\frac{1}{2\,(q^2+1)} \ln \frac{(i+q_1^*)}{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}\,,$$
$$\mathrm{Res}\,\phi_2(ib,q) = \frac{1}{2\,(q^2+1)} \ln \frac{(q+q_1^*)}{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)}\,.$$

Then

$$\begin{aligned} I_1 &= \frac{1}{2}\frac{2\pi i}{(q^2+1)} \ln \frac{(q+q_1^*)}{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)} \\ &\quad \times \frac{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}{(i+q_1^*)} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{2}\frac{2\pi i}{(q^2+1)} \ln \frac{(q+q_1^*)}{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)} \\ &\quad \times \frac{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}{(i+q_1^*)}\,. \end{aligned}$$

The sum $I_1 + I_2$ represents the total integral

$$\begin{aligned} I &= \frac{2\pi i}{(q^2+1)} \ln \frac{(q-q_1)}{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)} \\ &\quad \times \frac{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}{(i-q_1)} \end{aligned}$$

and its substitution into the pion scalar FF integral representation (9) gives

$$\begin{aligned} \mathit{\Gamma}_\pi(t) &= P_n(t) \frac{(q-q_1)}{(q+q_2)\,(q+q_3)\,(q+q_4)\,(q+q_5)} \\ &\quad \times \frac{(i+q_2)\,(i+q_3)\,(i+q_4)\,(i+q_5)}{(i-q_1)}\,, \end{aligned}$$

just the explicit form of the pion scalar FF, the graphical behavior of which is presented in Fig. 4.

The $-q_3$ pole of $\mathit{\Gamma}_\pi(t)$ on the second Riemann sheet in t-variable corresponds to $f_0(500)$-meson resonance with the mass and the width $m_{f_0(500)} = (472 \pm 10)$ MeV, $\mathit{\Gamma}_{f_0(500)} = (524 \pm 22)$ MeV to be compatible with world-wide values in [2,3].

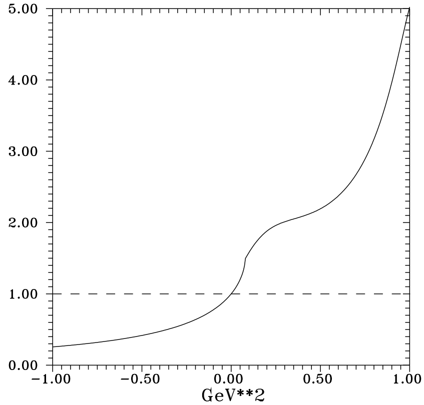

Fig. 4. Behavior of the pion scalar form factor in the region $-1\ \mathrm{GeV}^2 < t < 1\ \mathrm{GeV}^2$.

6. Conclusions

An original method for a prediction of the pion scalar FF behavior has been elaborated by utilization of its phase representation, the δ_0^0 parametrization obtained from general considerations and precise data on it in elastic region with theoretical errors to be generated by the Garcia-Martin–Kaminski–Pelaéz–Yndurain Roy-like equations. As a result, the existence of $f_0(500)$ scalar meson is confirmed again in a completely model independent way, however, with world-wide parameters $m_{f_0(500)} = (472 \pm 10)$ MeV, $\Gamma_{f_0(500)} = (524 \pm 22)$ MeV to be compatible with those in [2, 3] and determined in different ways.

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