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AN EXAMPLE OF A DETERMINISTIC CELLULAR AUTOMATON EXHIBITING LINEAR-EXPONENTIAL CONVERGENCE TO THE STEADY STATE*

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In a recent paper [`arXiv:1506.06649 [nlin.CG]`], we presented an example of a 3-state cellular automaton which exhibits behaviour analogous to degenerate hyperbolicity often observed in finite-dimensional dynamical systems. We also calculated densities of 0, 1 and 2 after n iterations of this rule, using finite state machines to conjecture patterns present in preimage sets. Here, we re-derive the same formulae in a rigorous way, without resorting to any semi-empirical methods. This is done by analysing the behaviour of continuous clusters of symbols and by considering their interactions.

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The general question of finding the iterates of the Bernoulli measure under a given cellular automaton (CA) has been subject of many recent studies, including, among others, [1–8]. A more specific question of this type is sometimes called *the density response problem*: If the probability of occurrence of a certain state in the initial configuration drawn from a Bernoulli distribution is given, what is the probability of occurrence of this state after n iterations of the CA rule?

Of course, one could ask a similar question about the probability of occurrence of longer blocks of symbols after n iterations of the rule. Due to the complexity of CA dynamics, it is clear that questions of this type are rather hopeless if one wants to know the answer for an arbitrary rule. In spite of this, it may still be possible to find the answer if the rule is sufficiently simple.

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One of the methods which can be used to do this is studying the structure of preimages of short blocks and detecting patterns present in them. This approach has been successfully used for a number of deterministic CA rules, such as elementary rules 172, 142, 130 (references [9, 10] and [11], respectively), and several others. It has also been used for a special class of probabilistic CA, known as single-transition α-asynchronous rules [12].

Cellular automata are infinitely-dimensional dynamical systems, yet a behaviour similar to hyperbolicity in finite-dimensional systems has been observed in many of them. In particular, in some binary cellular automata in one dimension, known as *asymptotic emulators of identity*, if the initial configuration is drawn from a Bernoulli distribution, the expected proportion of ones (or zeros) usually tends to its stationary value exponentially fast [13]. This type of behaviour is quite common in many other dynamical systems. For example, in a linear continuous-time dynamical system given by $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, if $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n$ and A is a real $n \times n$ matrix with all eigenvalues distinct and having negative real parts, $\boldsymbol{x}(t)$ tends to zero exponentially fast as $t \to \infty$. Exponential convergence is also observed in nonlinear systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ (where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$) in the vicinity of a hyperbolic fixed point, as long as the Jacobian matrix of $\boldsymbol{f}$ evaluated at the fixed point has only distinct eigenvalues with negative real parts.

If, on the other hand, the matrix A in $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ has degenerate (repeated) eigenvalues, the convergence to the fixed point can be polynomial-exponential, that is, of the form of $P(t)e^{-bt}$, where $P(t)$ is a polynomial and $b > 0$. Finite dimensional discrete-time dynamical systems can exhibit analogous behaviour. Consider, for example, the linear system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \tag{1}$$

The matrix on the right-hand side has a degenerate (double) eigenvalue $\frac{1}{2}$ and, therefore, the convergence to the fixed point $(0, 0)$ is expected to be polynomial-exponential (linear-exponential in this case). Indeed, if we explicitly solve the above equation for x_n and y_n, we obtain

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \left(\frac{1}{2}\right)^n \begin{bmatrix} 1-n & 2n \\ -\frac{n}{2} & 1+n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \tag{2}$$

and we can clearly see the aforementioned linear-exponential convergence.

Very recently, a *probabilistic* CA has been discovered [14] where the density of ones converges to its stationary value in a linear-exponential fashion, just like in the above example of a degenerate hyperbolic fixed point in a finite-dimensional dynamical systems. This probabilistic CA could be viewed as a simple model for diffusion of innovations, spread of rumours, or a similar process involving transport of information between neighbours. More precisely, it consists of an infinite one-dimensional lattice where each site is occupied by an individual who has already adopted the innovation (state 1) or who has not adopted it yet (state 0). Once the individual adopts the innovation, he remains in state 1 forever. Individuals in state 0 can change their states to 1 (adopt the innovation) with probabilities depending on the state of nearest neighbours. This process can be formally described as a radius 1 binary probabilistic CA with the following transition probabilities,

$$\begin{aligned} w(1|000) &= 0\,, \quad w(1|001) = p\,, \quad w(1|010) = 1\,, \quad w(1|011) = 1\,, \\ w(1|100) &= q\,, \quad w(1|101) = r\,, \quad w(1|110) = 1\,, \quad w(1|111) = 1\,, \end{aligned} \tag{3}$$

where p, q, r are fixed parameters of the model, $p, q, r \in [0, 1]$. By transition probability $w(d|abc)$, one means the probability that site in state b with neighbours a and c changes its state to d in one time step. One can show that for a certain choice of parameters p, q and r, the expected value of the density of ones converges to its steady state in a linear-exponential fashion.

Even more recently, we found a *deterministic* rule with three states which exhibits the same kind of behaviour [15]. This rule, to be defined in Sec. 2, will be the subject of our subsequent discussion. In [15], we studied the structure of preimages of 0, 1 and 2 under the action of this rule, and by employing finite state machines, we found some patterns in the preimage sets, which, in turn, allowed us to derive explicit expressions for densities of 0, 1 and 2 after n iterations. The finite state machines used in the derivation were constructed semi-empirically, and no proof of their correctness was given. In the current paper, we wish to fill this gap and present a more formal derivation of the aforementioned expressions, without resorting to finite state machines.

1. Basic definition

We will start from some basic definitions. For $\mathcal{A} = \{0, 1, 2\}$, a finite sequence of elements of $\mathcal{A}$, $\boldsymbol{b} = b_1 b_2 \ldots b_n$, will be called a *block* (or *word*) of length n. The set of all blocks of all possible lengths will be denoted by $\mathcal{A}^\star$.

Let $f : \mathcal{A}^3 \to \mathcal{A}$ be a local function of a nearest-neighbour cellular automaton. A *block evolution operator* corresponding to f is a mapping $\boldsymbol{f} : \mathcal{A}^\star \mapsto \mathcal{A}^\star$ defined as follows. Let $\boldsymbol{a} = a_1 a_2 \ldots a_n \in \mathcal{A}^n$, where $n \geq 3$. Then, $\boldsymbol{f}(\boldsymbol{a})$ is a block of length $n-2$ defined as

$$\boldsymbol{f}(\boldsymbol{a}) = f(a_1, a_2, a_3) f(a_2, a_3, a_4) \ldots f(a_{n-2}, a_{n-1}, a_n)\,. \tag{4}$$

If $\boldsymbol{f}(\boldsymbol{b}) = \boldsymbol{a}$, then we will say that $\boldsymbol{b}$ is a preimage of $\boldsymbol{a}$, and write $\boldsymbol{b} \in \boldsymbol{f}^{-1}(\boldsymbol{a})$. Similarly, if $\boldsymbol{f}^n(\boldsymbol{b}) = \boldsymbol{a}$, then we will say that $\boldsymbol{b}$ is an *n-step preimage* of $\boldsymbol{a}$, and write $\boldsymbol{b} \in \boldsymbol{f}^{-n}(\boldsymbol{a})$.

Let the *density polynomial* associated with a string $\boldsymbol{b} = b_1 b_2 \ldots b_n$ be defined as

$$\Psi_{\boldsymbol{b}}(p,q,r) = p^{\#_0(\boldsymbol{b})} q^{\#_1(\boldsymbol{b})} r^{\#_2(\boldsymbol{b})}\,, \tag{5}$$

where $\#_i(\boldsymbol{b})$ is the number of occurrences of symbol i in $\boldsymbol{b}$. If A is a set of strings, we define the density polynomial associated with A as

$$\Psi_A(p,q,r) = \sum_{\boldsymbol{a}\in A} \Psi_{\boldsymbol{a}}(p,q,r)\,. \tag{6}$$

One can easily show (in a manner similar as done in [13]) that if one starts with a bi-infinite string of symbols drawn from a Bernoulli distribution where probabilities of $0, 1$ and 2 are, respectively, p, q and r, then the expected proportion of sites in state k after n iterations of rule f is given by $\Psi_{\boldsymbol{f}^{-n}(k)}(p,q,r)$. This quantity will be called *density* of symbols k after n iterations of f.

2. The local rule and its properties

Let us now describe the CA rule which will be the subject of this contribution. While studying properties of various 3-state CA rules, we came across an interesting specimen of a nearest-neighbour (radius 1) rule with a local function defined as follows

$$f(x_1,x_2,x_3) = \begin{cases} x_3 & \text{for } x_1 = x_2 > x_3\,,\\ x_2 & \text{otherwise}\,, \end{cases} \tag{7}$$

where $x_1, x_2, x_3 \in \{0,1,2\}$. The origins of this rule have been described in [15]. Here, we will only note that it can be equivalently defined as

$$f(x_1,x_2,x_3) = \begin{cases} 0 & \text{for } (x_1,x_2,x_3) = (1,1,0) \text{ or } (x_1,x_2,x_3) = (2,2,0)\,,\\ 1 & \text{for } (x_1,x_2,x_3) = (2,2,1)\,,\\ x_2 & \text{otherwise}\,, \end{cases} \tag{8}$$

which makes it clear that it differs from the identity rule only on three neighbourhood configurations, $(1,1,0)$, $(2,2,0)$ and $(2,2,1)$.

Figure 1 shows an example of a spatio-temporal pattern generated by this rule, using periodic boundary conditions. It has some important properties which will be relevant to further discussion. First of all, note that $f(x_1,x_2,x_3) \le x_2$, meaning that the state of a given cell cannot increase. This implies that $f(\star,0,\star) = 0$, where $\star$ denotes an arbitrary symbol from the set $\{0,1,2\}$. Zero is thus a quiescent state for this CA.

We also have $f(0,1,\star) = 1$ and $f(2,1,\star) = 1$, which implies that a site in state 1 located at the beginning of a continuous cluster of 1s of any length (even of length 1, meaning isolated 1) remain in state 1 forever. The same is true for 2: since $f(0,2,\star) = 2$ and $f(1,2,\star) = 2$, a site in state 2 located at the beginning of a continuous cluster of 2s stays in state 2 forever. This can be observed in Fig. 1. In fact, words such as 01 or 02 remain unchanged when the rule is iterated, and no information can propagate through a pair sites which are in states 01 or 02. We note in passing that in the CA theory such words are called *blocking words*, and the rules with blocking words are known to be *almost equicontinuous* [16].

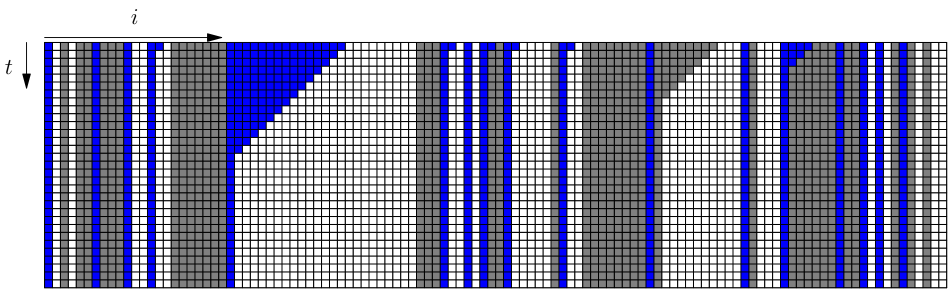

Fig. 1. (Colour on-line) Sample spatio-temporal pattern generated by 3-state rule 140. White, lighter grey and darker grey (blue) cells correspond, respectively, to 0, 1 and 2.

Further inspection of Fig. 1 reveals that a continuous cluster of zeros grows to the left if it is preceded by a cluster of 2s longer than 1, and it also grows to the left if is preceded by a cluster of 1s longer than 1.

3. Structure of preimages of 1

We want to find all strings $\boldsymbol{b}$ of length $2n+1$ such that $\boldsymbol{f}^n(\boldsymbol{b}) = 1$. We know from the definition of the rule that information can propagate only from the right to the left, thus the first $n-2$ entries of b are arbitrary. We will represent n-step preimages of 1 in the form of

$$\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-2} a_1 a_2 \boldsymbol{a}_3 c_1 c_2 \ldots c_n\,,$$

where the allowed values of a_1, a_2, a_3 (to be called a *prefix*) and $c_1 c_2 \ldots c_n$ (to be called a *postfix*) need to be determined. The central site of the preimage string will be, as in the above, denoted by a bold symbol. Since our CA rule has three states, there are $3^3 = 27$ possible values for the prefix $a_1 a_2 a_3$. Not all of them are possible, however. Prefixes 00**0**, 01**0**, 02**0**, 10**0**, 11**0**, 12**0**,

$20\mathbf{0}$, $21\mathbf{0}$ and $22\mathbf{0}$, which can be represented as $\star\star\mathbf{0}$, cannot occur in any preimage of 1. This is because $f(\star, 0, \star) = 0$, meaning that if the central site is in state 0, it will remain in state 0 forever, and consequently $\boldsymbol{f}^n(b) = 0$ for any string b containing one of the above prefixes.

Moreover, prefixes of the type $\star 0\mathbf{2}$ or $\star 1\mathbf{2}$ are not allowed either. This is because for the central site to become 1, as required, the transition $f(2, 2, 1) = 1$ would have to happen somewhere along the way, and for this the central 2 would have to get 2 as the left neighbour. Since states can only decrease, not increase, and the left neighbour of the central site is 0 or 1, this is not possible.

By excluding 9 prefixes of the type $\star\star\mathbf{0}$ and 6 prefixes of the type $\star 0\mathbf{2}$ or $\star 1\mathbf{2}$ we are left with 12 possibilities, $00\mathbf{1}$, $01\mathbf{1}$, $02\mathbf{1}$, $10\mathbf{1}$, $11\mathbf{1}$, $12\mathbf{1}$, $20\mathbf{1}$, $21\mathbf{1}$, $22\mathbf{1}$, $02\mathbf{2}$, $12\mathbf{2}$, $22\mathbf{2}$. All of them are allowed in preimages of 1, providing that an appropriate suffix is added. In what follows, we will find conditions which these suffices need to satisfy. We will divide the possible prefixes into four different types (the reason for this will soon become clear):

1. $\star 1\mathbf{1}$, $22\mathbf{1}$
2. $\star 0\mathbf{1}$, $02\mathbf{1}$, $12\mathbf{1}$
3. $02\mathbf{2}$, $12\mathbf{2}$
4. $22\mathbf{2}$

For prefixes of type 1, $\star 1\mathbf{1}$ and $22\mathbf{1}$, the central site is in the state 1 already, thus we have to make sure that it stays in the same state after n iterations of the rule. The left neighbour of the central 1 is 1 (for $\star 1\mathbf{1}$) or it will become 1 after one iteration (for $22\mathbf{1}$), thus we could potentially be in a danger of the transition $f(1, 1, 0) = 0$. This could happen only if the central 1 belongs to a continuous cluster of ones which is followed by 0 — in such a case, the cluster of ones will shrink one symbol per time step and the transition $f(1, 1, 0) = 0$ will eventually change the central 1 into 0. We have, therefore, two choices in avoiding this scenario: either the central 1 belongs to a cluster of ones followed by 2, which prevents its shrinking due to the fact that $f(1, 1, 2) = 1$, or it belongs to a cluster of ones which extends all the way to the right. In other words, the postfix for type 2 must be of the form

$$c_1 c_2 \ldots c_n = 1^i 2 \underbrace{\star\star\ldots\star}_{n-1-i} \quad \text{or} \quad c_1 c_2 \ldots c_n = 1^n\,,$$

where $i \in \{0, 1, \ldots n-1\}$. In the above and in what follows, 1^n denotes the symbol 1 repeated n times. We will use this convention in the rest of the paper.

For type 2, $\star 0\mathbf{1}$, $02\mathbf{1}$, and $12\mathbf{1}$, note that the central site could become 0 only by utilizing the transition $f(1,1,0)=0$, and for this the left neighbour of the central 1 would have to be 1. This is clearly impossible for $\star 0\mathbf{1}$ as the left neighbour 0 will always remain 0. In the case of $02\mathbf{1}$ and $12\mathbf{1}$, the left neighbour is 2, and it could potentially become 0 by transition $f(2,2,0)=0$. For this, however, we would need the second left neighbour of the central 1 to be in state 2, but this is impossible because site values never increase. Thus all prefixes of type 2 belong to preimages of 1, regardless of the suffix.

For type 3, $02\mathbf{2}$, $12\mathbf{2}$, the central site is in state 2, and its left neighbour is guaranteed to be in state 2 forever. Consequently, the central 2 must change to 1 at some iteration via the $f(2,2,1)=1$ transition, and in order for this to happen, we need to have the right neighbour of the central site in state 1 at some point of time. This can happen when the central 2 belongs to a continuous cluster of 2s followed by 1, meaning that the postfix must be of the form

$$c_1c_2\ldots c_n = 2^i1\underbrace{\star\star\ldots\star}_{n-1-i},$$

where $i \in \{0,1,\ldots n-1\}$.

Type 4 with the prefix $22\mathbf{2}$ is the most complicated one. Similarly as before, the central site must change from 2 to 1, and this can only happen via the transition $f(2,2,1)=1$, but we do not have a guarantee that the left neighbour of the central 2 remains in state 2 forever, as it was the the case of type 3. Nevertheless, the necessary condition for the suffix is the same as for type 3, meaning that the central 2 must belong to a continuous cluster of 2s followed by 1. This in not sufficient, however, because what follows is also important. In order to understand this clearly, consider two strings of length 13 iterated 6 times:

0000222221012	0000122221012
00022221101	00012221101
002221100	001221100
0221100	0121100
21100	12100
100	210
0	1

The first of these strings has prefix 222, and the second one has prefix 122 (thus belonging to type 3). In both of them, the central 2 belongs to a continuous cluster of 2s followed by 1, but the first one does not produce 1 after 6 iterations. This is because the zero which follows propagates to the left and eventually makes the central site to change to 0 via the transition $f(1,1,0)=0$. Any string containing 222 as a prefix must, therefore, satisfy an additional property preventing zeros to propagate to the left. This can be done by making the last part of the suffix to have the same structure as prefix of type 3, so that the entire suffix takes the form

$$c_1c_2\dots c_n = 2^i 1 d_1d_2\dots d_{n-1-i}\,, \tag{9}$$

where $i \in \{0,1,\dots,n-1\}$ and where

$$d_1d_2\dots d_{n-1-i} = 1^j 2 \underbrace{\star\star\dots\star}_{n-2-i-j} \quad \text{or} \quad d_1d_2\dots d_{n-1-i} = 1^{n-1-i}\,, \tag{10}$$

with $j \in \{0,2,\dots,n-2-i\}$.

Finally, for prefix 222, there is one more possibility not covered by the above discussion, namely $c_1c_2\dots c_n = 2^{n-2}10$. Below, we summarize all these findings in a form of a single proposition.

Proposition 3.1 *Block b belongs to $\boldsymbol{f}^{-n}(1)$ if and only if it is one of the following four types.*

Type 1:

$$\boldsymbol{b} = \underbrace{\star\star\dots\star}_{n-2} a_1a_2a_3 1^i 2 \underbrace{\star\star\dots\star}_{n-1-i} \quad \text{or} \quad \boldsymbol{b} = \underbrace{\star\star\dots\star}_{n-2} a_1a_2a_3 1^n\,, \tag{11}$$

where $a_1a_2a_3 \in \{011,111,211,221\}$, $i \in \{0,1,\dots,n-1\}$;

Type 2:

$$\boldsymbol{b} = \underbrace{\star\star\dots\star}_{n-2} a_1a_2a_3 \underbrace{\star\star\dots\star}_{n}\,, \tag{12}$$

where $a_1a_2a_3 \in \{001,101,201,121,021\}$;

Type 3:

$$\boldsymbol{b} = \underbrace{\star\star\dots\star}_{n-2} a_1a_2a_3 2^i 1 \underbrace{\star\star\dots\star}_{n-1-i}\,, \tag{13}$$

where $a_1a_2a_3 \in \{022,122\}$, $i \in \{0,1,\dots,n-1\}$;

Type 4a:

$$\boldsymbol{b} = \underbrace{\star\star\dots\star}_{n-2} a_1a_2a_3 2^i 1 c_1c_2\dots c_{n-1-i}\,, \tag{14}$$

where $a_1a_2a_3 = 222$, $i \in \{0,1,\dots,n-1\}$ and where

$$c_1c_2\dots c_{n-1-i} = 1^j 2 \underbrace{\star\star\dots\star}_{n-2-i-j} \quad \text{or} \quad c_1c_2\dots c_{n-1-i} = 1^{n-1-i} \tag{15}$$

with $j \in \{0,2,\dots,n-2-i\}$;

Type 4b:

$$\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-2} a_1 a_2 a_3 2^{n-2} 10\,, \tag{16}$$

where $a_1 a_2 a_3 = 222$.

4. Density polynomials for preimages of 1

Let us now denote the set of strings of type 1 by T_1, type 2 by T_2 *etc.*, and let us define $\lambda = p + q + r$. Density polynomial for T_1 will be given by

$$\begin{aligned} \Psi_{T_1}(p,q,r) \;&=\; \sum_{i=0}^{n-1} \lambda^{n-2} \left(pq^2 + q^3 + rq^2 + r^2 q\right) q^i r \lambda^{n-i-1} \\ &\quad + \lambda^{n-2} \left(pq^2 + q^3 + rq^2 + r^2 q\right) q^n\,, \end{aligned} \tag{17}$$

which simplifies to

$$\Psi_{T_1}(p,q,r) = \lambda^{2n-3} r \left(\lambda q^2 + r^2 q\right) \sum_{i=0}^{n-1} \lambda^{-i} q^i + \lambda^{n-2} \left(\lambda q^2 + r^2 q\right) q^n\,. \tag{18}$$

By performing summation of the partial geometric sequence in the above, one obtains

$$\Psi_{T_1}(p,q,r) = \lambda^{n-2} r \left(\lambda q^2 + r^2 q\right) \frac{\lambda^n - q^n}{p+r} + \lambda^{n-2} \left(\lambda q^2 + r^2 q\right) q^n\,, \tag{19}$$

which further simplifies to

$$\Psi_{T_1}(p,q,r) = r \left(\lambda q^2 + r^2 q\right) \frac{\lambda^{2n-2}}{p+r} + \lambda^{n-2} \left(\lambda q^2 + r^2 q\right) \frac{pq^n}{p+r}\,. \tag{20}$$

Similar calculations (omitted here) yield

$$\Psi_{T_2}(p,q,r) \;=\; \left(\lambda pq + q^2 r + pqr\right) \lambda^{2n-2}\,, \tag{21}$$

$$\Psi_{T_3}(p,q,r) \;=\; \lambda^{n-2} r^2 q \left(\lambda^n - r^n\right)\,. \tag{22}$$

The type 4a is the most complicated. Let us first compute the density polynomial for the set of strings of the form of

$$c_1 c_2 \ldots c_k = 1^j 2 \underbrace{\star\star\ldots\star}_{k-1-j} \quad \text{or} \quad c_1 c_2 \ldots c_k = 1^k\,, \tag{23}$$

with $j \in \{0, 1, \ldots, k-1\}$. The density polynomial for the above, to be denoted by $h_k(p, q, r)$, is given by

$$h_k(p,q,r) = \sum_{j=0}^{k-1} q^j r \lambda^{k-1-j} + q^n = r\frac{\lambda^k - q^k}{p+r} + q^k = \frac{r\lambda^k}{p+r} + \frac{pq^k}{p+r}\,. \quad (24)$$

Having this result, we can write the density polynomial for the entire set T_{4a}

$$\begin{aligned}
\Psi_{T_{4a}}(p,q,r) &= \lambda^{n-2} r^3 \sum_{i=0}^{n-1} r^i q h_{n-1-i}(p,q,r) \\
&= \lambda^{n-2} r^3 \sum_{i=0}^{n-1} r^i q \frac{r\lambda^{n-1-i}}{p+r} + \lambda^{n-2} r^3 \sum_{i=0}^{n-1} r^i q \frac{pq^{n-1-i}}{p+r} \\
&= \frac{\lambda^{n-2} q r^4}{p+r} \sum_{i=0}^{n-1} r^i \lambda^{n-1-i} + \frac{\lambda^{n-2} pqr^3}{p+r} \sum_{i=0}^{n-1} r^i q^{n-1-i} \\
&= \frac{\lambda^{n-2} q r^4 (\lambda^n - r^n)}{(p+r)(p+q)} + \frac{\lambda^{n-2} pqr^3}{p+r} \sum_{i=0}^{n-1} r^i q^{n-1-i}\,. \qquad (25)
\end{aligned}$$

When $q \neq r$, we thus obtain

$$\Psi_{T_{4a}}(p,q,r) = \frac{\lambda^{n-2} q r^4 (\lambda^n - r^n)}{(p+r)(p+q)} + \frac{\lambda^{n-2} pqr^3 (q^n - r^n)}{(p+r)(q-r)}\,. \quad (26)$$

When $q = r$, the last sum becomes $\sum_{i=0}^{n-1} r^i q^{n-1-i} = \sum_{i=0}^{n-1} q^{n-1} = q^{n-1} n$, therefore,

$$\Psi_{T_{4a}}(p,q,q) = \frac{\lambda^{n-2} q^5 (\lambda^n - q^n)}{(p+q)^2} + \frac{\lambda^{n-2} pq^3}{p+q} n q^n\,. \quad (27)$$

Finally, type 4b is straightforward,

$$\Psi_{T_{4b}}(p,q,r) = \lambda^{n-2} r^{n+1} pq\,. \quad (28)$$

We are now ready to compute the density polynomial of preimages of 1, by summing density polynomials for T_1, T_2, T_3, T_{4a} and T_{4b}. This yields, for $q \neq r$,

$$\begin{aligned}
\Psi_{\boldsymbol{f}^{-n}(1)}(p,q,r) &= r\left(\lambda q^2 + r^2 q\right) \frac{\lambda^{2n-2}}{p+r} + \lambda^{n-2}\left(\lambda q^2 + r^2 q\right) \frac{pq^n}{p+r} \\
&\quad + \left(\lambda pq + q^2 r + pqr\right) \lambda^{2n-2} + \lambda^{n-2} r^2 q \left(\lambda^n - r^n\right) \\
&\quad + \frac{\lambda^{n-2} q r^4 (\lambda^n - r^n)}{(p+r)(p+q)} + \frac{\lambda^{n-2} pqr^3 (q^n - r^n)}{(p+r)(q-r)} + \lambda^{n-2} r^{n+1} pq\,.
\end{aligned} \quad (29)$$

Collecting together terms for $(q\lambda)^n$, $(r\lambda)^n$, and λ^{2n}, we obtain, after some algebra

$$\begin{aligned}\Psi_{\boldsymbol{f}^{-n}(1)}(p,q,r) &= \frac{pq^2\left(-pr+pq+q^2\right)(q\lambda)^n}{\lambda^2\,(p+r)\,(q-r)} \\ &\quad+\frac{qr\left(-p^2r+p^2q+pq^2-2\,pqr+r^3-q^2r\right)(r\lambda)^n}{\lambda^2\,(p+q)\,(q-r)} \\ &\quad+\frac{q\left(p^3+p^2q+2\,p^2r+pr^2+3\,pqr+r^3+r^2q+q^2r\right)\lambda^{2n}}{\lambda\,(p+r)\,(p+q)}\,,\end{aligned} \tag{30}$$

which is the same formula as derived in [15].

Similarly, for $q=r$, we obtain

$$\begin{aligned}\Psi_{\boldsymbol{f}^{-n}(1)}(p,q,q) &= q\left(\lambda q^2+q^3\right)\frac{\lambda^{2n-2}}{p+q}+\lambda^{n-2}\left(\lambda q^2+q^3\right)\frac{pq^n}{p+q} \\ &\quad+\left(\lambda pq+q^3+pq^2\right)\lambda^{2n-2}+\lambda^{n-2}q^3\left(\lambda^n-q^n\right) \\ &\quad\times\frac{\lambda^{n-2}q^5\left(\lambda^n-q^n\right)}{(p+q)^2}+\frac{\lambda^{n-2}pq^3}{p+q}nq^n+\lambda^{n-2}q^{n+1}pq\,.\end{aligned} \tag{31}$$

After simplification and reordering of terms, this yields

$$\begin{aligned}\Psi_{\boldsymbol{f}^{-n}(1)}(p,q,q) &= \frac{pq^3\,(n+1)\,(q\lambda)^n}{\lambda^2\,(q+p)}+\frac{q^2\left(2\,p^3+4\,p^2q+pq^2-2\,q^3\right)(q\lambda)^n}{(q+p)^2\,\lambda^2} \\ &\quad+\frac{\left(p^3+3\,p^2q+4\,pq^2+3\,q^3\right)q\lambda^{2n}}{\lambda\,(q+p)^2}\,,\end{aligned} \tag{32}$$

which, again, agrees with the result "guessed" in [15] using finite state machines.

5. Preimages of 2 and their density polynomials

From the definition of the rule, we know that a site can be in state 2 only if it was in that state at the beginning, that is, sites in state 0 or 1 cannot change to 2. Moreover, $f(2,2,0)=0$, $f(2,2,1)=1$, and in all other cases $f(a_1,2,a_3)=2$. This means that a site in state 2 remains in that state forever if it is preceded by 0 or 1. Therefore, any string of the form

$$\boldsymbol{b}=\underbrace{\star\star\ldots\star}_{n-1}0\mathbf{2}\underbrace{\star\star\ldots\star}_{n}\quad\text{or}\quad\boldsymbol{b}=\underbrace{\star\star\ldots\star}_{n-1}1\mathbf{2}\underbrace{\star\star\ldots\star}_{n}$$

will be an n-step preimage of 2, $\boldsymbol{f}^n(b)=2$.

What if 2 is preceded by 2? In this case, it must be followed by a sufficient number of 2s before the first 0 or 1 appears, as any 0 or 1 at the end of a cluster of 2s shortens such cluster by one on each iteration. Therefore, for n iterations, we need n 2s. We thus need, in order for $\boldsymbol{f}^n(b) = 2$ to hold in this case,

$$\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-1} 2\boldsymbol{2}2^n\,.$$

The above observations can be summarized as follows.

Proposition 5.1 *Block b belongs to $\boldsymbol{f}^{-n}(2)$ if and only if it is one of the following three types:*

1. $\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-1} 0\boldsymbol{2} \underbrace{\star\star\ldots\star}_{n}$,
2. $\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-1} 1\boldsymbol{2} \underbrace{\star\star\ldots\star}_{n}$,
3. $\boldsymbol{b} = \underbrace{\star\star\ldots\star}_{n-1} 2^{n+2}$.

This yields the density polynomial

$$\Psi_{\boldsymbol{f}^{-n}(2)}(p,q,r) = (p+q)r\lambda^{2n-1} + \lambda^{n-1}r^{n+2}\,. \tag{33}$$

6. Preimages of 0 and their density polynomials

Since everything what is not a preimage of 1 or 2 must be a preimage of 0, we have

$$\Psi_{\boldsymbol{f}^{-n}(0)}(p,q,r) = \lambda^{2n+1} - \Psi_{\boldsymbol{f}^{-n}(1)}(p,q,r) - \Psi_{\boldsymbol{f}^{-n}(2)}(p,q,r)\,. \tag{34}$$

After simplification, this yields, for $r \neq q$,

$$\begin{aligned}
\Psi_{\boldsymbol{f}^{-n}(0)}(p,q,r) &= \frac{\left(-pr+pq+q^2\right)pq^2\left(q\lambda\right)^n}{\lambda^2\left(p+r\right)\left(r-q\right)}\\
&+\frac{pr\left(-r^2p+q^2p+q^3-r^3-qr^2\right)\left(r\lambda\right)^n}{\lambda^2\left(p+q\right)\left(r-q\right)}\\
&+\frac{\left(p^3+2\,p^2q+2\,p^2r+2\,r^2p+3\,qpr+2\,q^2p+r^3+2\,qr^2+q^3+q^2r\right)p\lambda^{2n}}{\left(p+q\right)\left(p+r\right)\lambda}\,,
\end{aligned} \tag{35}$$

and for $r = q$,

$$\Psi_{\boldsymbol{f}^{-n}(0)}(p,q,q) = \frac{\left(p^3+4\,p^2q+7\,q^2p+5\,q^3\right)p\lambda^{2n}}{\lambda\,(p+q)^2} - \frac{pq^3\,(n+1)\,(q\lambda)^n}{\lambda^2\,(p+q)} - \frac{q^2p\left(3\,p^2+8\,pq+6\,q^2\right)(q\lambda)^n}{(p+q)^2\,\lambda^2}\,. \tag{36}$$

Again, similarly as in the density polynomial for 1, the linear-exponential dependence of the form $(n+1)(q\lambda)^n$ is present in the second term.

7. Density of ones

As already stated, density polynomials $\Psi_{\boldsymbol{f}^{-n}(k)}(p,q,r)$ represent probability of occurrence of k after n iterations starting from a Bernoulli distribution with probabilities of 0, 1 and 2 equal to, respectively, p, q, and r, where $p+q+r=1$. If one starts with a symmetric Bernoulli distribution where $r=q$, the probability of occurrence of 1 after n steps, to be denoted by $P_n(1)$, will be given by Eq. (32) as long as one substitutes $r=q$ and $q=(1-p)/2$. This yields, after simplification,

$$P_n(1) = P_\infty(1) - (An+B)\left(\frac{1-p}{2}\right)^n\,, \tag{37}$$

where

$$A = \frac{(p-1)^2}{4\,(1+p)^2}\left(p^3-p\right), \tag{38}$$

$$B = \frac{(p-1)^2}{4\,(1+p)^2}\left(-p^3-5\,p-3\,p^2+1\right), \tag{39}$$

$$P_\infty(1) = \frac{(1-p)\left(p^3+5\,p^2-p+3\right)}{4\,(1+p)^2}\,. \tag{40}$$

One can see that for $0<p<1$, $P_n(1)$ tends to $P_\infty(1)$ as $n\to\infty$, and that the convergence is linear-exponential in n. Such "degenerate" convergence takes place for probability of occurrence of 0 as well, as seen in Eq. (36).

On the other hand, when $r\neq q$ in the initial Bernoulli distribution, the convergence is purely exponential, as in Eqs. (30) and (35).

8. Conclusions

We presented an example of a 3-state rule exhibiting, under certain conditions, linear-exponential convergence to the steady state. This phenomenon is remarkably similar to degenerate hyperbolicity in finite dimensional dynamical systems. It is not clear, however, what is the origin of this analogy. One can speculate that "simple" CA rules, such as those which are equicontinuous or almost equicontinuous, can be somewhat approximated by finite-dimensional systems. Local structure theory could possibly be applicable in this case, as it allows to construct finite-dimensional systems approximating orbits of Bernoulli measure under the action of a given CA. It is hoped that this contribution inspires further research on this subject.

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REFERENCES

[1] D.A. Lind, *Phys. D* **10**, 36 (1984); *Cellular Automata*, Los Alamos, N.M., 1983.

[2] P.A. Ferrari, A. Maass, S. Martínez, P. Ney, *Ergod. Theor. Dyn. Syst.* **20**, 1657 (2000).

[3] A. Maass, S. Martínez, *Biol. Res.* **36**, 113 (2003).

[4] B. Host, A. Maass, S. Martínez, *Discrete Cont. Dyn. Syst.* **9**, 1423 (2003).

[5] M. Pivato, R. Yassawi, *Ergod. Theor. Dyn. Syst.* **22**, 1269 (2002).

[6] M. Pivato, R. Yassawi, *Ergod. Theor. Dyn. Syst.* **24**, 1961 (2004).

[7] A. Maass, S. Martínez, M. Sobottka, *Nonlinearity* **19**, 2137 (2006).

[8] A. Maass, S. Martínez, M. Pivato, R. Yassawi, *Ergod. Theor. Dyn. Syst.* **26**, 1203 (2006).

[9] H. Fukś, `arXiv:1007.1026 [nlin.CG]`.

[10] H. Fukś, *Complex Syst.* **16**, 123 (2006).

[11] H. Fukś, A. Skelton, *Int. J. Natural Comp. Res.* **1**, 85 (2010) [`arXiv:1108.1987 [nlin.CG]`].

[12] H. Fukś, A. Skelton, *Dis. Math. Theor. Comp. Science* **AP**, 95 (2011).

[13] H. Fukś, J.-M.G. Soto, *Complex Syst.* **23**, 1 (2014) [`arXiv:1306.1189 [nlin.CG]`].

[14] H. Fukś, "An Example of Computation of the Density of Ones in Probabilistic Cellular Automata by Direct Recursion", submitted for publication, 2015.

[15] H. Fukś, J. Midgley-Volpato, vol. 24 of, *TUCS Lect. Notes* **24**, 47 (2015) [`arXiv:1506.06649 [nlin.CG]`].

[16] P. Kůrka, *Topological Dynamics of Cellular Automata*, in: *Encyclopedia of Complexity and System Science*, R.A. Meyers, ed., Springer, 2009.