THE EINSTEIN-CARTAN EQUATIONS IN ASTROPHYSICALLY INTERESTING SITUATIONS. I. THE CASE OF SPHERICAL SYMMETRY

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The Einstein-Cartan equations (general relativity plus spin in a manifold with curvature and torsion) are written explicitly for the case of spherical symmetry. In this case there may exist, at most, eight non-vanishing independent components of the torsion tensor when one does not assume the "classical description" of spin. It is shown explicitly, by giving various non-equivalent ways to do it, how exact solutions of spherical symmetry for matter-filled regions may be generalized from general relativity into the Einstein-Cartan theory. A classification of cosmological models with the Robertson-Walker metric in the Einstein-Cartan theory is given.

1. The Einstein-Cartan equations

Recently an extension of original Einstein's theory of gravitation which has been proposed first by Cartan (1922, 1923) more than half a century ago, has been revived and reformulated. At the present observational level, it appears to be practically indistinguishable in its predictions from general relativity, because the field equations in empty space (i.e. filled with no other than the gravitational field) are the same in both theories, and the tests for general relativity are based mostly on equations for empty space. Now, the predictions of the ECT (as the Einstein-Cartan theory will be called briefly from now on) may differ from those of general relativity for regions of space filled with matter. (This is due to the algebraic relation between the physical quantity of spin density and the geometric quantity — the torsion of the underlying manifold.) The three obvious areas of a possible study to be made are: (a) cosmology, (b) static stellar configurations, and (c) collapse. A physically new feature of the ECT, which makes the theory highly attractive, is the characteristic spin-spin repulsive interaction which dominates the behaviour of matter at extremely high densities (above, say, 10^{54} g cm^{-3}) and is able to prevent the occurrence of singularities both in cosmology and in collapse.

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The occurrence of singularities in general relativity was shown (the famous Hawking-Penrose theorems!) to be general and unavoidable feature of this theory. Even the pressure of matter, though obviously it is being regarded as a characteristic of the mutual repulsion, is unable to prevent singularity. In the standard theory of general relativity only the introduction of a positive cosmological constant Λ may provide in some cases regular models of the Universe, but this does not help us in the case of local collapse. Some other sophisticated ways to cope with the singularity are based on the introduction of the bulk viscosity terms into the frame of the Friedmannian cosmology (Heller et al. 1973) or on a modification of the gravitational Lagrangian (Nariai and Tomita 1971). We do not go into details of these treatments, since we are interested only in presenting the application of the ECT to the avertion of singularities. While the two other approaches may appear too phenomenological, as based on certain ad hoc assumptions, we have in the case of the ECT a firm theory which is a most natural and simplest modification of general relativity, not rival to it, but following only an extension of the previous theory to the uttermost frontiers. Einstein's concept of the energy-momentum tensor, as a source of the geometry, is extended to the spin density tensor as another physical quantity that influences the structure of the space-time (necessarily extending it beyond the Riemannian limit). The resulting spin-spin interaction, which is a straightforward consequence of the geometrization of the angular momentum in the ECT, is able to occur only within the underlying mathematical structure of the ECT. One is unable to introduce it by some kind of phenomenological approach into general relativity, it may appear only in the somehow richer geometry of the manifold that is used in the ECT (not only curvature but also torsion exists, in general). It is the torsion of the geometric structure which prevents, for a series of the ECT models, the singularity. It is evident that for general relativity, based on a geometry without torsion, this mechanism cannot work. On the other hand, in general relativity the spin of matter influenced the geometry only very indirectly, through its contribution to the energy-momentum tensor. Now, in the ECT, it influences the geometric structure in a direct mode, through its algebraic relation to the torsion.

Cosmological models of an expanding universe without any initial singularity have been constructed explicitly by this author and others (Kopczyński 1972, 1973; Kuchowicz 1975 a, b; Tafel 1973; Trautman 1973 a). We are going now to present some new cosmological (and also other) models in the series of papers with which we start. For established details of the ECT theory, the reader is referred to the fundamental papers of Trautman (1972 a, b, c; 1973 b) and Hehl (1973, 1974); an abridged introduction into the ECT is given by this author (Kuchowicz 1975 c). We use Trautman's formulation with the help of the calculus of tensor-valued exterior forms, when all geometric objects are described by their components with respect to the field of frames θ^i (i, 1, ... 4) in the cotangent spaces of the 4-dimensional differentiable manifold. The linear connection is given by a set of 1-forms ω^i_k defining the covariant derivative. The curvature form Ω^i_k and the torsion from Θ^i are given by the two formulae

$$\Omega^i_k = d\omega^i_k + \omega^i_l \wedge \omega^l_k, \qquad \Theta^i = d\theta^i + \omega^i_l \wedge \theta^l, \tag{1.1}$$

and the associated torsion tensor Q^i_{jk} and curvature tensor R^i_{jkl} are

$$\Theta^i = \tfrac{1}{2} Q^i_{jk}\theta^j \wedge \theta^k, \quad \Omega^i_j = \tfrac{1}{2} R^i_{jkl}\theta^k \wedge \theta^l. \tag{1.2}$$

Following Trautman, we use the forms η, η^i, etc. as the duals of 1, θ^i, $\theta^i \wedge \theta^j$, etc. In the following, the formula for the covariant exterior derivative will be needed;

$$D\eta_i = Q^j_{ij}\eta. \tag{1.3}$$

The symbol "D" is used to denote a covariant exterior derivative which, when applied to a tensor-valued 0- form (usual tensor), reduces to the usual covariant derivative $\theta^k\nabla_k$.

Now the system of the Einstein-Cartan equations has the following simple form:

$$e_i \equiv \tfrac{1}{2} \eta_{ijk} \wedge \Omega^{jk} = -8\pi\tilde{G}t_i, \tag{1.4}$$

$$c_{ij} \equiv -\eta_{ijk} \wedge \Theta^k = -8\pi\tilde{G}s_{ij}. \tag{1.5}$$

It may be written also in terms of the components:

$$R^j_i - \tfrac{1}{2}\delta^j_i R = -8\pi\tilde{G}t^j_i, \tag{1.6}$$

$$Q^k_{ij} - \delta^k_i Q^l_{lj} - \delta^k_j Q^l_{il} = 8\pi\tilde{G}s^k_{ij}. \tag{1.7}$$

The vector-valued 3-form t_i is related to the "canonical", asymmetric energy-momentum tensor t^j_i:

$$t_i = \eta_j t^j_i, \tag{1.8}$$

while the antisymmetric tensor-valued 3-form s_{ij} is related to the tensor of spin s^k_{ij}:

$$s_{ij} = \eta_k s^k_{ij}. \tag{1.9}$$

R^i_j is the Ricci tensor, R — the curvature scalar (both for our general manifold with torsion). $\tilde{G}$ is the gravitational constant, and we assume the light velocity in vacuum $c = 1$.

The canonical 3-form t_i of energy-momentum is related to the symmetric 4-form T^j_i by Trautman's identity:[1]

$$T^j_i = \theta^j \wedge t_i - \tfrac{1}{2} Ds^j_i. \tag{1.10}$$

2. *The general case of spherical symmetry*

In the systematic search for exact solutions of the Einstein-Cartan equations, it might be useful to start with the most simple case of spherical symmetry. Now, this would lead to the unphysical assumption of a spherically symmetric spin distribution, but it is possible to argue that this may constitute a minor trouble when compared with the standard singularities in general relativity. Already in this oversimplified case we may get an insight into the role of torsion and spin in overcoming the singularities, just as it was with the first

[1] Thanks to this identity we are able to replace the system (1.4), (1.5) by another with T^j_i, and the dilemma of the symmetric vs. canonical energy-momentum tensors finds an interesting solution in the ECT.

exact cosmological solution of the ECT (Kopczyński 1972). In our previous study of this problem (Kuchowicz 1975b) we have already emphasized that the cosmological models given there may be regarded only as an introduction to more complicated, especially axially symmetric models.

Before we go over to a study of axial symmetry, it may be appropriate to look for a possible generalization of the results of the previous study, still within the framework of spherical symmetry. All our previous results have been obtained within a simplified treatment in which the validity of the so-called "classical description" of spin is assumed. This assumption is based on the extension of the properties of intrinsic angular momentum from special relativity. The spin angular momentum density tensor s^i_{jk} is factorized into the antisymmetric spin tensor S_{jk} and the 4-velocity vector u^i

$$s^k_{ij} = S_{ij}u^k \quad \text{with} \quad s^k_{ik} = 0. \tag{2.1}$$

In the case of spherical symmetry, this leads to the existence of only one component of S_{jk}, and hence to only one nonvanishing component of s^i_{jk} (the quantity s^4_{23}, defined with respect to the basis set of 1-forms given by Eq. (2.2)). Also for other symmetries we have a very limited number of components of the spin angular momentum tensor, and of torsion.

We are using throughout this paper the most general metric of spherical symmetry, corresponding to the following set of orthogonal basis 1-forms

$$\theta^1 = e^{\lambda/2}dr, \quad \theta^2 = re^{\sigma/2}d\theta, \quad \theta^3 = re^{\sigma/2}\sin\theta d\varphi, \quad \theta^4 = e^{\nu/2}dt, \tag{2.2}$$

where λ, σ and ν may depend on r and on t. The most general form of the torsion tensor for such a space-time is derived in Appendix I. In the following, we apply large Roman letters to denote the independent non-vanishing components of the torsion tensor

$$\begin{aligned} A &= Q^1_{14}, & E &= Q^2_{24} = Q^3_{34}, \\ B &= Q^1_{23}, & F &= Q^2_{34} = -Q^3_{24}, \\ C &= Q^2_{12} = Q^3_{13}, & G &= Q^4_{14}, \\ D &= Q^2_{13} = -Q^3_{12}, & H &= Q^4_{23}. \end{aligned} \tag{2.3}$$

Equation (1.7) may be used to express the independent, non-vanishing components of the spin angular momentum density tensor s^i_{jk} in terms of the geometric quantities

$$8\pi\tilde{G}s^1_{14} = -2E,$$

$$8\pi\tilde{G}s^1_{23} = B,$$

$$8\pi\tilde{G}s^2_{12} = 8\pi\tilde{G}s^3_{13} = -C-G,$$

$$8\pi\tilde{G}s^2_{13} = -8\pi\tilde{G}s^3_{12} = D,$$

$$8\pi\tilde{G}s^2_{24} = 8\pi\tilde{G}s^3_{34} = -A-E,$$

$$8\pi\tilde{G}s^2_{34} = -8\pi\tilde{G}s^3_{24} = F,$$

$$8\pi\tilde{G}s^4_{14} = -2C,$$
$$8\pi\tilde{G}s^4_{23} = H. \tag{2.4}$$

Thanks to the simple algebraic relation between torsion and spin tensors, we may use alternatively either of these two tensors. In the following, as in our preceding study (Kuchowicz 1975b), we prefer to use the torsion components $A \ldots H$. Formulae for the connection 1-forms ω^i_k and curvature 2-forms Ω^i_k in the manifold under study are given in Appendix II. These formulae constitute a generalization of the respective formulae from the preceding study (Kuchowicz 1975b), where there appeared only one torsion component Q^4_{23}. We do not give separately the components of the curvature tensor R^i_{jkl}, as these may be read off directly from the resulting expressions with the use of the second formula of Eq. (1.2). We use the signature $(-,-,-,+)$, and the following definition of the Ricci tensor: $R^j{}_i = R^{kj}_{ik}$, and Einstein's geometric tensor $G^j{}_i = R^j{}_i - \delta^j{}_i R^l{}_l$. Trautman's identity (1.10) is used to express the "canonical" energy-momentum tensor $t^j{}_i$ which appears in Eq. (1.6) in terms of the symmetric energy-momentum tensor $\tilde{\mathcal{T}}^j_i$, and which is related to the symmetric 4-form: $T^j_i = \eta\tilde{\mathcal{T}}^j_i$. For the aims of this paper we assume that $\tilde{\mathcal{T}}^j_i$ is the energy-momentum tensor of a perfect fluid

$$\tilde{\mathcal{T}}^j_i = (p+\varrho)u_i u^j - p\delta^j_i, \tag{2.5}$$

where p denotes pressure, ϱ — energy density, and we use co-moving co-ordinates for matter, so that u^4 is the only non-vanishing component of the 4-velocity of matter.

We find that the Ricci tensor, in general, beside its diagonal components, has the following non-vanishing components: $R_{23} = -R_{32}$, R_{14} and R_{41}. In general relativity we have $R_{14} = R_{41}$ and $R_{23} = 0$. Now, we get a set of 6 equations generalizing equations (3.6) ... (3.9) of the preceding paper (Kuchowicz 1975b). We write down immediately the three diagonal equations

$$-8\pi\tilde{G}\tilde{\mathcal{T}}^1_1 = 8\pi\hat{G}p = \left[\tfrac{1}{4}\sigma'(\sigma'+2\nu')+\frac{\sigma'+\nu'}{r}+\frac{1}{r^2}\right]e^{-\lambda}$$
$$+\left[-\ddot{\sigma}-\tfrac{3}{4}\dot{\sigma}^2+\frac{\dot{\sigma}\dot{\nu}}{2}\right]e^{-\nu}-\frac{e^{-\sigma}}{r^2}-\left[C\nu'+(2C+2G)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}$$
$$-2(\dot{E}+E\dot{\sigma})e^{-\nu/2}+\frac{B^2}{4}+C^2-E^2+\frac{H^2}{4}+2CG+\frac{HF}{2}, \tag{2.6}$$

$$-8\pi\tilde{G}\mathcal{T}^2_2 = -8\pi\tilde{G}\mathcal{T}^3_3 = 8\pi\tilde{G}p = \left[\frac{\sigma''+\nu''}{2}+\frac{(\sigma')^2+(\nu')^2}{4}+\frac{\nu'\sigma'}{4}-\frac{\lambda'\sigma'}{4}\right.$$
$$\left.-\frac{\lambda'\nu'}{4}+\frac{\nu'-\lambda'}{2r}+\frac{\sigma'}{r}\right]e^{-\lambda}+\left[-\frac{\ddot{\sigma}+\ddot{\lambda}}{2}-\frac{\dot{\sigma}^2+\dot{\lambda}^2}{4}+\frac{\dot{\nu}\dot{\sigma}}{4}+\frac{\dot{\lambda}\dot{\nu}}{4}-\frac{\dot{\lambda}\dot{\sigma}}{4}\right]e^{-\nu}$$
$$-\left[C'+G'+C\left(\frac{\nu'}{2}+\frac{\sigma'}{2}+\frac{1}{r}\right)+G\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}$$
$$-\left[\dot{E}+\dot{A}+(A+E)\left(\frac{\dot{\sigma}}{2}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}+\frac{H^2}{4}-\frac{B^2}{4}-AE-\frac{BD}{2}+CG+\frac{HF}{2}, \tag{2.7}$$

$$8\pi\tilde{G}\mathscr{T}^4_4 = 8\pi\tilde{G}\varrho = \left[-\sigma''-\tfrac{3}{4}(\sigma')^2+\frac{\lambda'\sigma'}{2}+\frac{\lambda'-3\sigma'}{r}-\frac{1}{r^2}\right]e^{-\lambda}$$

$$+\left[\frac{\dot\lambda\dot\sigma}{2}+\frac{\dot\sigma^2}{4}\right]e^{-\nu}+\frac{e^{-\sigma}}{r^2}+\left[2C'+4C\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+[A\dot\sigma+E(\dot\sigma+\dot\lambda)]e^{-\nu/2}$$

$$+\frac{B^2}{4}-C^2+E^2+\frac{H^2}{4}+2AE+BD+\frac{HF}{2}. \tag{2.8}$$

The equation for the "$^1_4$" component is as follows

$$\left[\dot\sigma'-\frac{\dot\sigma\nu'}{2}+(\dot\sigma-\dot\lambda)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\frac{\lambda+\nu}{2}}+\left[E'+E\left(\frac{1}{r}+\frac{\sigma'-\nu'}{2}\right)-A\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}$$

$$+\left[-\dot C+C\left(\frac{\dot\lambda}{2}-\frac{\dot\sigma}{2}\right)+G\frac{\dot\sigma}{2}\right]e^{-\nu/2}+3AC+\frac{BF}{2}+\frac{BH}{2}+\frac{DH}{2}-EG=0, \tag{2.9}$$

while the equation for the "$^4_1$" component has the following form

$$\left[\dot\sigma'-\frac{\dot\sigma\nu'}{2}+(\dot\sigma-\dot\lambda)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\frac{\lambda+\nu}{2}}+\left[E'+E\left(\frac{1}{r}+\frac{\sigma'-\nu'}{2}\right)-A\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}$$

$$+\left[-\dot C+C\left(\frac{\dot\lambda}{2}-\frac{\dot\sigma}{2}\right)+G\frac{\dot\sigma}{2}\right]e^{-\nu/2}-AC+\frac{BF}{2}+\frac{BH}{2}+\frac{DH}{2}+3EG=0. \tag{2.10}$$

Finally, the equations for the "$^2_3$" and "$^3_2$" components yield the following algebraic relation

$$B(2C+G) = H(2E+A). \tag{2.11}$$

It is natural to impose pressure isotropy upon a spherically symmetric configuration of matter. This leads to an equality of the two expressions for pressure given by Eq. (2.6) and (2.7). This leads to the following equation which must be fulfilled for all λ, ν, σ, A, B, C, D, E and G (it imposes only no conditions upon the torsion components F and H)

$$g_{GR}+g_{EC} = 0, \text{ where } g_{EC} \stackrel{\text{def}}{=} g_1+g_2+g_3, \tag{2.12}$$

and the expressions appearing in this equation are defined as follows

$$g_{GR} = \left[\frac{\sigma''+\nu''}{2}+\frac{(\nu')^2}{4}-\frac{\lambda'\sigma'}{4}-\frac{\nu'\sigma'}{4}-\frac{\lambda'\nu'}{4}-\frac{\lambda'+\nu'}{2r}-\frac{1}{r^2}\right]e^{-\lambda}$$

$$+\left[\frac{\ddot\sigma-\ddot\lambda}{2}+\frac{\dot\sigma^2}{2}-\frac{\dot\lambda^2}{2}-\frac{\dot\lambda\dot\sigma}{4}-\frac{\dot\nu\dot\sigma}{4}+\frac{\dot\lambda\dot\nu}{4}\right]e^{-\nu}+\frac{e^{-\sigma}}{r^2},$$

$$g_1 = \left[\dot E-\dot A+E\left(\tfrac{3}{2}\dot\sigma-\frac{\dot\lambda}{2}\right)-\frac{A}{2}(\dot\sigma+\dot\lambda)\right]e^{-\nu/2}-AE,$$

$$g_2 = \left[-C' - G' + C\left(\frac{1}{r} + \frac{\sigma' + \nu'}{2}\right) + G\left(\frac{1}{r} + \frac{\sigma'}{2}\right)\right] e^{-\lambda/2} - C^2 - CG,$$

$$g_3 = -\frac{B}{2}(B+D).$$

In all our equations a dot denotes differentiation with respect to t, while a prime denotes differentiation with respect to r. We have introduced the indexes "GR" and "EC" in Eq. (2.12) in order to characterize a typical expression in general relativity, and the characteristic term arising in the Einstein-Cartan theory, respectively. The pressure isotropy condition (2.12) reduces to the well-known constraint $g_{GR} = 0$ in general relativity.

It is possible to apply the same kind of reasoning (and notation) to two equations (2.9) and (2.11); they may be replaced, after some algebraic manipulations, by the equivalent set

$$AC = EG, \tag{2.13}$$

and

$$f_{GR} + f_{EC} = 0, \text{ with } f_{EC} \overset{\text{def}}{=} f_1 + f_2 + f_3. \tag{2.14}$$

The functions f_i are defined as follows

$$f_{GR} = \left[\dot{\sigma}' - \frac{\dot{\sigma}\nu'}{2} + (\dot{\sigma} - \dot{\lambda})\left(\frac{1}{r} + \frac{\sigma'}{2}\right)\right] e^{-\frac{\lambda+\nu}{2}},$$

$$f_1 = \left[E' + E\left(\frac{1}{r} + \frac{\sigma' - \nu'}{2}\right) - A\left(\frac{1}{r} + \frac{\sigma'}{2}\right)\right] e^{-\lambda/2},$$

$$f_2 = \left[-\dot{C} + C\left(\frac{\dot{\lambda}}{2} - \frac{\dot{\sigma}}{2}\right) + G\frac{\dot{\sigma}}{2}\right] e^{-\nu/2},$$

$$f_3 = 2AC + \tfrac{1}{2}BF + \tfrac{1}{2}BH + \tfrac{1}{2}DH.$$

Now the whole set of the Einstein-Cartan equations which is to be used for our aims of deriving exact solutions of the ECT, consists of two expressions (2.6) and (2.8) (or (2.7) and (2.8)) for pressure and energy density, the pressure isotropy condition (2.12), and relation (2.14). All these conditions remain, though in a reduced form, without the torsion terms, in general relativity. In addition, we have two algebraic conditions (2.11) and (2.13) which are characteristic features of the Einstein-Cartan theory.

There are more degrees of freedom to deal with solutions in a space-time with curvature and torsion than in a space-time with curvature only. It is therefore not astonishing that the increase in the number of functions introduced into our treatment is faster than the increase in the number of conditions imposed thereupon. Let us consider, e. g. the geometric structure of the manifold as being given a priori, and the physical quantities as being defined by the geometric quantities. Thus, while dealing with a perfect fluid of spinning particles, we use three equations: (2.6), (2.8), and (1.7) as the expressions yielding pressure, energy density and spin angular momentum density of the fluid, in terms of λ, σ, ν, Q^i_{jk} and their derivatives. These expressions should give us, of course, such matter configurations in which p and ϱ should be positively defined, the corresponding equation of state of matter should be physically admissible, etc. These physical conditions are to be imposed upon any set of geometric quantities which fulfils differential conditions (2.12) and (2.14), and algebraic conditions (2.11) and (2.13). In general relativity, the algebraic conditions do not enter, and the differential conditions simplify, thus, e. g. in considering static spheres there remains actually a single differential condition which may be regarded as the constraint upon the form of one of the metric functions provided the other two are given. This approach was exploited in a previous series of papers by this author (see e. g. Kuchowicz 1968, 1973) for a systematic derivation of new exact solutions of the Einstein equations. This concept, which can be traced back to an old paper of Tolman (1939) and possibly earlier, may be applied directly to the ECT. While in general relativity we had, for spherical symmetry, two differential conditions imposed upon three functions of two variables (r, and t), we have now to fulfil four conditions while the number of available geometric quantities has increased to eleven. We can admit that certain physically reasonable assumptions, added to the theory, may reduce the number of independent non-vanishing torsion components, but our freedom of choice of the underlying geometry is always greater than in general relativity, provided we do not introduce too restrictive assumptions (cf. the discussion on the so-called spin-conservation in the preceding paper (Kuchowicz 1975b)).

3. *A generalization of the correspondence theorem*

We have enough solutions of the Einstein equations at our disposal, and it would appear most appropriate to generalize them to solutions of the Einstein-Cartan equations. In the preceding paper (Kuchowicz 1975b) we have presented the so-called correspondence theorem between general relativity and the Einstein-Cartan theory, according to which it is possible to apply every spherically symmetric exact solution of general relativity in the Einstein-Cartan theory. This correspondence was proved for the simple case of a perfect fluid with the "classical description" of spin, when only the torsion component $Q^4_{23} = -Q^4_{32}$ does not vanish. Now this may be extended upon the general situation we are studying here, with the eight non-zero independent components of the torsion tensor.

The *generalized correspondence theorem* may be formulated in the following way:

Every spherically symmetric metric of a Riemannian space-time of general relativity may be regarded at the same time as the metric of a family of space-times with torsion, where the torsion tensor components have to fulfil two algebraic conditions (2.11) and (2.13), and the two differential conditions:

$$f_{EC} = 0, \text{ and } g_{EC} = 0. \tag{3.1}$$

Also the set of boundary or asymptotic conditions of general relativity has to be replaced by the corresponding set for the Einstein-Cartan theory. General relativistic formulae for the pressure and energy density expressed in terms of the metric tensor components and their derivatives have to be replaced, finally, by our formulae (2.6) (or (2.7)) and (2.8) from the preceding section of this paper. The four conditions (2.11), (2.13) and (3.1) for the eight independent components of the torsion tensor can be fulfilled in many different ways. We are thus able to generate various non-equivalent solutions of the Einstein--Cartan equations with the same metric tensor. When we go back from the Riemann-Cartan geometry to the standard Riemanian geometry, all these solutions reduce to the same solution of the general theory of relativity.

This applies to all kinds of spherically symmetric solutions. In some situations it is necessary to take into account also the effect of boundary conditions; such situations arise, e. g. in the treatment of spheres of perfect fluid, or of spherical layers of such fluid. Now, at the sphere boundary $r = r_b$ the pressure should be zero. We see that formulae (2.6) and (2.7) for pressure in the ECT involve a general relativistic part p_{GR} which is zero at the boundary when we take some metric from general relativity to a generalization in the ECT. Now we demand that also the other term p_{add} which is characteristic for the ECT should vanish at the boundary

$$p_{add}(r_b) = 0. \tag{3.2}$$

We may decompose

$$p_{add} = h_1+h_2+h_3 = k_1+k_2+k_3, \tag{3.3}$$

where

$$h_1 = -2(\dot{E}+E\dot{\sigma})e^{-\nu/2}-E^2,$$

$$h_2 = -\left[C\nu'+2(C+G)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+C^2+2CG,$$

$$h_3 = \frac{B^2}{4}+\frac{H^2}{4}+\frac{FH}{2},$$

$$k_1 = -\left[\dot{E}+\dot{A}+(A+E)\left(\frac{\dot{\sigma}}{2}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}-AE,$$

$$k_2 = -\left[C'+G'+C\left(\frac{\nu'}{2}+\frac{\sigma'}{2}+\frac{1}{r}\right)+G\left(\frac{\sigma'}{2}+\frac{1}{r}\right)\right]e^{-\lambda/2}+CG,$$

$$k_3 = -\frac{B^2}{4}+\frac{H^2}{4}-\frac{BD}{2}+\frac{FH}{2}.$$

Eq. (3.3) is significantly less restrictive than the conditions discussed in the preceding section, nevertheless it may bring additional difficulties in adapting a general relativistic solution to the ECT.

Let us illustrate the role of the boundary conditions by considering the case when we have $s^i_{ki} = 0$, i. e. the characteristic "trace" of the spin 3-index tensor should be zero. This is half the way toward the "classical description" of spin (without the factorization of s^i_{jk}). This leads to the conditions

$$2E+A = 0,\ 2C+G = 0 \tag{3.4}$$

thanks to which two algebraic conditions (2.11) and (2.13) are automatically satisfied. Now we can distinguish two cases according to whether B is zero or not. With $B \neq 0$, the four torsion components B, C, E, H are completely arbitrary, and D and F are completely determined by them and by the metric

$$D = \frac{2}{B}\left\{\left[3\dot{E}+E\left(\tfrac{5}{2}\dot{\sigma}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}+\left[C'+C\left(\frac{\nu'}{2}-\frac{\sigma'}{2}-\frac{1}{r}\right)\right]e^{-\lambda/2}\right.$$

$$\left.+4\frac{E^2}{B}+2\frac{C^2}{B}-\right\}B,$$

$$F = \frac{1}{B^2}[8BCE-2C^2H-4E^2H]+\frac{2}{B}\left\{\left[\dot{C}+C\left(\tfrac{3}{2}\dot{\sigma}-\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}\right.$$

$$\left.-\left[E'+E\left(\frac{3}{r}+\tfrac{3}{2}\sigma'-\frac{\nu'}{2}\right)\right]e^{-\lambda/2}\right\}+\frac{2H}{B^2}\left\{\left[C'+C\left(\frac{\nu'-\sigma'}{2}-\frac{1}{r}\right)\right]e^{-\lambda/2}\right.$$

$$\left.-\left[3\dot{E}+E\left(\tfrac{5}{2}\dot{\sigma}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}\right\}.$$

With $B = 0$, the situation is more complicated, as Eq. (3.1) gives two conditions upon four functions C, D, E, H, and we cannot give a general solution of these equations

$$\left[3\dot{E}+E\left(\tfrac{5}{2}\dot{\sigma}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}+\left[C'+C\left(\frac{\lambda'-\sigma'}{2}-\frac{1}{r}\right)\right]e^{-\lambda/2}+2E^2-C^2 = 0,$$

$$\left[E'+E\left(\frac{3}{r}+\tfrac{3}{2}\sigma'-\frac{\nu'}{2}\right)\right]e^{-\lambda/2}+\left[-\dot{C}+C\left(\frac{\dot{\lambda}}{2}-\tfrac{3}{2}\dot{\sigma}\right)\right]e^{-\nu/2}-4CE+\tfrac{1}{2}DH = 0.$$

Now the torsion component F is completely free. In addition, the two equations above are fulfilled when the three torsion components (E, C, and either D or H) are identically zero. Thus we have the following simple result: Any spherically symmetric metric of general relativity describes at the same time the Riemenn-Cartan geometry in which: (a) either the two torsion components D and F are arbitrary, while all the other ones have to be zero, or (b) the two torsion components H and F may be arbitrary while all other ones have to vanish.

In case (a), the expressions for density and pressure in the ECT are the same as in general relativity. In case (b), the expressions for these physical quantities in the ECT are related to the corresponding expressions in the general relativistic theory with the same metric

$$8\pi\tilde{G}p_{EC} = 8\pi\tilde{G}p_{GR}+\frac{FH}{2}+\frac{H^2}{4}, \quad 8\pi\tilde{G}\varrho_{EC} = 8\pi\tilde{G}\varrho_{GR}+\frac{FH}{2}+\frac{H^2}{4}. \tag{3.5}$$

The additional terms on the right-hand side are able to exert a significant influence on the behaviour of matter under such circumstances when they are comparable with the other terms. Thus, e. g. the H^2 term is just the repulsive spin-spin interaction term which makes it impossible in cosmology to approach the point singularity (Kopczyński 1972; Kuchowicz 1975b).

With a boundary condition for pressure, we may insert either directly our solution into Eq. (3.2) to get some restrictions upon the hitherto arbitrary torsion components, or we may start with this condition from the beginning. Especially simple cases arise when some of the torsion components are zero:

I. $A = E = 0$, $C = -\frac{1}{2}G$ and H — are completely arbitrary, and B has to fulfil only a boundary condition. Then D and F are determined by them and by the metric.

II. With $C = G = 0$, the situation is similar to the preceding case, only C has to be replaced by $E = -\frac{1}{2}A$. Thus A and H are now arbitrary, etc.

III. With $A = C = E = G = 0$ we have, at most, two independent non-vanishing torsion components $B = -D$, and H. They both should be such as to vanish at the surface of the sphere.

IV. $A = E = B = 0$, leads to the general condition $DH \neq 0$. C is the solution of the equation: $g_2(C) = 0$, Either D or H may be chosen arbitrary, and F has to satisfy a boundary condition.

V. $C = G = B = 0$. This is quite analogous to the preceding case, with the torsion component E in the role of C.

VI. With $A = B = C = E = G = 0$ we have only the constraint $DH = 0$, and the boundary condition: $H(H+2F)|_{r=r_b} = 0$. Generally, two torsion components may be non-zero, but if these two are F and H, then one of the alternative boundary conditions must be filfilled: either $H(r_b) = 0$, or $H(r_b) = -2F(r_b)$.

TABLE I

Simple choices of the Einstein-Cartan geometry for a given spherically symmetric metric of general relativity

Starting conditions			Arbitrary	Other
			components of the torsion tensor	
$2E+A \neq 0$	$2C+G \neq 0$	$B \neq 0$	$B \neq 0$ and 3 out of the four (A, C, E, G)	D, F, H, and one of the four to the left
		$B = 0$	D, F, and one of the four components listed in the right-hand side column	A, C, E, G — related to each other by 4 equations, $H = 0$
	$C = G = 0$	$B \neq 0$	$B \neq 0$, A, E	D, F, $H = 0$
		$B = 0$	D, F	E, A, $H = 0$
$E = A = 0$, $2C+G \neq 0$, $B = 0$			F, and two out of the four (C, D, G, H, but not C and G simultaneously!)	Two of the four components listed to the left are determined by the other ones

The situation in the case when at least one of the algebraic conditions (3.4) does not hold is illustrated in Table I (without the restriction by Eq. (3.2). It is important to mention here that in Table I we pointed only to the simplest assignment, and our identification proposals do not exhaust the possibilities of various assumptions of torsion components corresponding to the same spherically symmetric metric. In the situation with the highest number of non-vanishing torsion components (the first line from top in Table I) we get, e. g. the following form of those torsion components which are not arbitrary

$$H = \frac{2C+D}{2E+A}B, \quad D = \frac{2(g_1+g_2)}{B} - B, \quad F = \frac{4AC}{B} + \frac{2}{B}\left(f_1+f_2-\frac{H}{B}(g_1+g_2)\right),$$

$$G = \frac{AC}{E}.$$

Similar expressions (or constraints to be solved) may be given in the other cases. In general, due to the interpretation in the framework of the "classical description" of spin, the most important cases are those with non-zero H. Though they are not numerous in the Table, it is necessary to mention that we got $H \neq 0$ while considering earlier the case with $s^i_{ki} = 0$, which is not covered by Table I.

4. *Cosmological models with the Robertson-Walker metric*

In general relativity, the Robertson-Walker metric:

$$e^\nu = 1, \quad e^\lambda = e^\sigma = \frac{R^2(t)}{(1+\frac{1}{4}kr^2)^2}, \quad k = 0, \pm 1 \tag{4.1}$$

corresponds to homogeneous and isotropic cosmological models. These models have, of course, spherical symmetry though they have no distinguished central point. One might look for their possible generalization into the framework of the Einstein-Cartan theory. The first model of Kopczyński (1972) belongs to this general class. It is not spatially homogeneous though the metric tensor fulfils the homogeneity postulate; this is due to the torsion which distinguishes a point in space. This kind of violation of the cosmological principle is a general feature of cosmological models of spherical symmetry within the framework of the ECT. To get rid of this feature, one needs to go over to axially symmetric models of the Universe; this has been done by some authors (Kopczyński 1973, Tafel 1973), and will also be studied systematically in the subsequent parts of this paper. Let us start, however, with simple extensions of the standard Friedman models, as this may give us a first insight into the question of avoiding the cosmological singularity which motivates the work in this area. In order not to repeat the arguments, we are referring to comments on this problem in two other papers (Kuchowicz 1975a, b). A lot of cosmological models has been derived with the "classical description" of spin. We do not attempt here to derive some more particular models of the Universe. Our aim is only to look for a generalization of the equations which were derived in the preceding paper (Kuchowicz 1975b).

With the substitution of the metric (4.1) into the pressure isotropy condition (2.12) and the differential condition (2.14), these two are reduced to the following form

$$\frac{1-\frac{1}{4}kr^2}{rR(t)}(C+G)-\frac{1+\frac{1}{4}kr^2}{R(t)}(C'+G')+\dot{E}-A$$

$$+\frac{2\dot{R}(t)}{R(t)}(E-A)-AE-\tfrac{1}{2}B^2-\tfrac{1}{2}BD-C^2-CG=0 \tag{4.2}$$

$$\frac{1-\frac{1}{4}kr^2}{rR(t)}(E-A)+\frac{1+\frac{1}{4}kr^2}{R(t)}E'+\frac{\dot{R}(t)}{R(t)}G-\dot{C}$$

$$+2AC+\tfrac{1}{2}BF+\tfrac{1}{2}BH+\tfrac{1}{2}DH=0 \tag{4.3}$$

Now the group of motions is extended by a fourth generator $X_4 = \partial/\partial r$ which gives the independence of the torsion components from the radial coordinate r. The expressions given by the left-hand sides of Eq. (4.2) and (4.3) should vanish independently of r. Hence, the following conditions should be satisfied

$$E = A, \qquad C = -G.$$

With these conditions, algebraic equations (2.11) and (2.13) reduce to

$$AG = 0, \; -BG = 3AH, \tag{4.5}$$

and we get only five following different possibilities to fulfil at the same time all equations (4.2) ... (4.5):

I. $A = B = C = D = E = G = 0$, no constraints upon F and H.

II. $A = B = C = E = G = H = 0$, no constraints on D and F.

III. $A = C = E = F = G = 0$, $D = -B \neq 0$, no constraint on H.

IV. $C = F = G = H = 0$, $E = A \neq 0$, $B \neq 0$, $D = -\dfrac{2A^2}{B} - B(\neq 0)$.

V. $A = B = E = 0$, $C = -G \neq 0$, no constraint on F. A single constraint involving G, D and H: $G\dfrac{\dot{R}}{R}+\dot{G}+\frac{1}{2}DH = 0$.

With all these conditions listed above, we are able to enter the expressions for matter density and pressure which have the following form for the Riemann-Cartan space-time with the Robertson-Walker metric

$$8\pi\tilde{G}\varrho = \frac{3}{R^2}(\dot{R}^2+k)+8\pi\tilde{G}\varrho_{\text{add}},$$

$$8\pi\tilde{G}p = -\frac{2\ddot{R}}{R}-\frac{\dot{R}^2+k}{R^2}+8\pi\tilde{G}p_{\text{add}}. \tag{4.6}$$

The additional terms $8\pi\tilde{G}\varrho_{\text{add}}$ and $8\pi\tilde{G}p_{\text{add}}$ on the right-hand sides are characteristic of the ECT. Upon inspection of the additional density term for the possibility V, we find that G

(which should be non-zero) is multiplied by a term involving explicitly r. Now a dependence of such physical quantities as density — on r is inadmissible, otherwise the cosmological model under study ceases to be at least "metrically" homogeneous. We must thus discard the fifth possibility, and we are left with the four classes of extensions of the Robertson-Walker models in the Einstein-Cartan theory which are listed in Table I. The standard Robertson-Walker models from general relativity (with all torsion components equal to zero) may be considered as the limiting case of models of classes I and II. Actually, the expressions for energy density and pressure for models of class II, even if both torsion components D and F do not vanish, are the same as in general relativity. It is thus easy to see that a singularity cannot be averted in the models of class II. In models belonging to class I, a singularity may be averted thanks to the new terms; it is necessary that $H \neq 0$ in order that a singularity disapears. Models of classes III and IV are very specific models, going beyond the "classical description" of spin in which only the H component of torsion may remain. Models of the first three classes fall within the generalized concept of a classical spin description, when we assume the validity of the condition: $s^k_{ik} = 0$, while model IV does not allow this condition to be fulfilled. The introduction of the torsion components with the upper index differing from "4" is, in our co-moving system, equivalent to introducing some new components of the spin angular momentum density tensor in addition to the spin density. These may be called the spin flux densities and can be conceived as the 4-dimensional generalizations of the moment stresses (Hehl 1973, 1974).

Let us discuss briefly some peculiarities of the four classes of the extended Robertson-Walker models.

Models of class I include all the models with a "classical description" of spin we presented in the previous paper (Kuchowicz 1975b). With $F \neq 0$, we introduce the spin stresses s^2_{34} and s^2_{24}. But since the term involving F appears in the same combination with $\frac{1}{4}H^2$ in the expressions for density and for pressure, the linear equations of state of the type $p = (\gamma - 1)\varrho$ are the same as in the case of the classical description of spin, and all conclusions with respect to the behaviour of the radius function $R(t)$ which were derived in the previous study (Kuchowicz 1975b) remain valid.

The two possibly non-vanishing spin stresses s^2_{13} $(= s^3_{12})$ and s^2_{34} $(= s^3_{24})$ for type II models do not introduce any difference with respect to the standard general relativistic behaviour of the radius function $R(t)$, energy density and pressure.

We may compare universe models filled with dust, with radiative models possessing the same scale factor $R(t)$. Let the indexes "d" and "r" denote dust and radiation, respectively. Then it is possible to generalize a result which was derived in the preceding paper for the "classical description" of spin only. From the straightforward calculations we derive a remarkable results, comparing energy densities of dust and radiation universes at the same instant of cosmic time:

$$\frac{\varrho_r}{\varrho_d} = \begin{cases} \frac{3}{2} \text{ for type I and type III models,} \\ \frac{1}{2} \text{ for type IV models.} \end{cases}$$

This means that provided we have at a given time a certain scale factor $R(t)$ corresponding to a certain energy density ϱ_d in a dust universe, the corresponding energy density ϱ_r for

TABLE II

Four classes of the extended Robertson-Walker models in the Einstein-Cartan theory

Class	$8\pi\tilde{G}\varrho_{add}$	$8\pi\tilde{G}p_{add}$	Non-zero torsion components
I	$\frac{H^2}{4}+\frac{FH}{2}$	$\frac{H^2}{4}+\frac{FH}{2}$	Possibly F, H
II	0	0	Possibly D, F
III	$\frac{H^2}{4}-\frac{3}{4}B^2$	$\frac{B^2}{4}+\frac{H^2}{4}$	Necessarily $D = -B$, possibly H
IV	$2A^2+6A\frac{\dot{R}}{R}-\frac{3}{4}B^2$	$\frac{B^2}{4}-A^2-2\left(\dot{A}+2A\frac{\dot{R}}{R}\right)$	Necessarily: $A = E$, B, and $D = -\frac{2A^2}{B}-B$

a universe filled with radiation[2], is by 50 percent higher or by 50 percent lower (depending on the model type) for the same expansion stage characterized by the same value of $R(t)$. The peculiar behaviour of type IV models is exhibited by the fact that we have $\varrho_r < \varrho_d$ for it. Our result remains valid as long as the additional terms from Table I are not zero. It allows us to consider only one type of the dust or radiation models, and then to formulate the physical implications for the other type. This is a specific feature of the ECT which has no analogue in general relativity.

Finally, let us briefly state that in type III models both of a dust and of a radiative type, singularity can be averted in spite of the negative contribution from B in the expression for energy density. It may be shown that with the two assumptions of energy conservation ($8\pi\tilde{G}\varrho R^3 = M$ for dust, and $8\pi\tilde{G}\varrho R^4 = M$ for radiation models) and spin conservation ($H = H_0R^{-3}$) we get from the first integral of the respective equation of state for dust matter

$$(3\dot{R}^2-C)R+\frac{H_0^2}{4R^3} = M, \tag{4.7}$$

for radiative matter

$$(\tfrac{3}{2}\dot{R}^2-C)R^2+\frac{H_0^2}{4R^2} = M. \tag{4.8}$$

The appearance of the repulsive spin-spin interaction term with H_0 makes, in each case, the region around $r = 0$ inaccessible, just as in the first model of Kopczyński (1972). We see thus that also among expanding models of type III there may exist such which exhibit no singularity.

[2] Matter with the equation of state $p = \frac{1}{3}\varrho$ is defined as radiation.

5. *Conclusions*

Though spherical symmetry may appear irreconcilable with the presence of torsion, it constitutes the most simple case which can be investigated within the framework of the Einstein-Cartan theory. It was our aim to point to the possibilities which arise for astrophysical and cosmological applications of the Einstein-Cartan theory. The spherically symmetric case enables us to take over numerous exact solutions of general relativity into the new theory. Various inequivalent ways to perform such a generalization have been indicated here. Such generalized solutions may be now generated for the most general case, with eight non-vanishing components of the torsion tensor, and some of their features may be treated as an indication of the properties of more general and physically reasonable solutions lacking spherical symmetry.

It would be therefore useful to go over to the case of axial symmetry, at least for cosmological models. This will be done in the next paper of this series, dealing with those models described by the 4-parametric Lie group, possessing a subgroup of the Bianchi types VII, VIII or IX. Also further generalizations of the physical assumptions (e. g. a transition to viscous fluids) will follow.

APPENDIX I

General form of the torsion tensor in the case of spherical symmetry

It is well known that a geometric object is spherically symmetric when its Lie derivative, with respect to any of the three Killing vectors $\underset{(i)}{\xi}$ for spherical symmetry, is equal to zero. We give here these vectors in terms of components with respect to the spherical coordinates (r, θ, φ, t) we are using

$$\underset{(1)}{\xi} = (0, 0, 1, 0), \qquad \underset{(2)}{\xi} = (0, \sin\varphi, \operatorname{cotg}\theta\cos\varphi, 0),$$

$$\underset{(3)}{\xi} = (0, \cos\varphi, -\operatorname{cotg}\theta\sin\varphi, 0). \tag{AI.1}$$

Alternatively we can give the three generators of the group of motion in this case

$$X_1 = \frac{\partial}{\partial\varphi}, \qquad X_2 = \sin\varphi\frac{\partial}{\partial\theta} + \operatorname{cotg}\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$X_3 = \cos\varphi\frac{\partial}{\partial\theta} - \operatorname{cotg}\theta\sin\varphi\frac{\partial}{\partial\varphi} \tag{AI.2}$$

with the commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2 \tag{AI.3}$$

corresponding to the Bianchi type IX.

The Lie derivative of the torsion tensor Q^i_{jk} with respect to any of the three vectors

$\underset{(i)}{\xi}$ has the form

$$\underset{\xi}{\pounds} Q^i_{jk} = \xi^l Q^i_{jk,l} + Q^i_{lk}\xi^l_{,j} + Q^i_{jl}\xi^l_{,k} - Q^l_{jk}\xi^i_{,l}, \tag{AI.4}$$

where comma denotes ordinary differentiation. As we insert $\underset{(1)}{\xi}$ into Eq. (AI.4) and equate the latter to zero, we get

$$\frac{\partial Q^i_{jk}}{\partial \varphi} = 0, \tag{AI.5}$$

which means that the Q^i_{jk} do not depend on φ. By inserting $\underset{(2)}{\xi}$ into Eq. (AI.4) and equating it to zero we obtain the following set of equations:

$$\sin\varphi \frac{\partial Q^i_{12}}{\partial\theta} + \frac{\cos\varphi}{\sin^2\theta} Q^1_{31} - \cos\varphi Q^3_{12}\delta^i_2 + \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{12} + \operatorname{cotg}\theta \sin\varphi Q^3_{12}\right)\delta^i_3 = 0,$$

$$\sin\varphi \frac{\partial Q^i_{13}}{\partial\theta} - \cos\varphi Q^i_{21} + \operatorname{cotg}\theta\sin\varphi Q^i_{31} - \cos\varphi Q^3_{13}\delta^i_2$$

$$+ \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{13} + \operatorname{cotg}\theta\sin\varphi Q^3_{13}\right)\delta^i_3 = 0,$$

$$\sin\varphi \frac{\partial Q^i_{14}}{\partial\theta} - \cos\varphi Q^3_{14}\delta^i_2 + \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{14} + \operatorname{cotg}\theta\sin\varphi Q^3_{14}\right)\delta^i_3 = 0,$$

$$\sin\varphi \frac{\partial Q^i_{23}}{\partial\theta} + \operatorname{cotg}\theta\sin\varphi Q^i_{32} - \cos\varphi Q^3_{23}\delta^i_2 + \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{23} + \operatorname{cotg}\theta\sin\varphi Q^3_{23}\right)\delta^i_3 = 0,$$

$$\sin\varphi \frac{\partial Q^i_{24}}{\partial\theta} - \frac{\cos\varphi}{\sin^2\theta} Q^i_{34} - \cos\varphi Q^3_{24}\delta^i_2 + \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{24} + \operatorname{cotg}\theta\sin\varphi Q^3_{24}\right)\delta^i_3 = 0,$$

$$\sin\varphi \frac{\partial Q^i_{34}}{\partial\theta} + \cos\varphi Q^i_{24} - \sin\varphi\operatorname{cotg}\theta Q^i_{34} - \cos\varphi Q^3_{34}\delta^i_2$$

$$+ \left(\frac{\cos\varphi}{\sin^2\theta} Q^2_{34} + \operatorname{cotg}\theta\sin\varphi Q^3_{34}\right)\delta^i_3 = 0. \tag{AI.6}$$

This set is easily changed into a set for $\underset{(3)}{\xi}$ by the substitution of $\cos\varphi$ for $\sin\varphi$, and $(-\sin\varphi)$ for $\cos\varphi$. Then, by some algebraic manipulations, we obtain

$$Q^i_{31} + Q^2_{12}\delta^i_3 - \sin^2\theta Q^3_{12}\delta^i_2 = 0,$$

$$Q^2_{13}\delta^i_3 - \sin^2\theta(Q^i_{21} + Q^3_{13}\delta^i_2) = 0,$$

$$Q^2_{14}\delta^i_3 - \sin^2\theta Q^3_{14}\delta^i_2 = 0,$$

$$Q^2_{23}\delta^i_3 - \sin^2\theta Q^3_{23}\delta^i_2 = 0,$$

$$Q^i_{34} - Q^2_{24}\delta^i_3 + \sin^2\theta Q^3_{24}\delta^i_2 = 0,$$

$$Q^2_{34}\delta^i_3 + \sin^2\theta(Q^i_{24} - Q^3_{34}\delta^i_2) = 0,$$

$$\frac{\partial Q^i_{12}}{\partial\theta} = -\mathrm{cotg}\,\theta Q^3_{12}\delta^i_3,$$

$$\frac{\partial Q^i_{13}}{\partial\theta} = -\mathrm{cotg}\,\theta[Q^i_{31} + Q^3_{13}\delta^i_3],$$

$$\frac{\partial Q^i_{14}}{\partial\theta} = -\mathrm{cotg}\,\theta Q^3_{14}\delta^i_3,$$

$$\frac{\partial Q^i_{23}}{\partial\theta} = -\mathrm{cotg}\,\theta[Q^i_{32} + Q^3_{23}\delta^i_3],$$

$$\frac{\partial Q^i_{24}}{\partial\theta} = -\mathrm{cotg}\,\theta Q^3_{24}\delta^i_3,$$

$$\frac{\partial Q^i_{34}}{\partial\theta} = \mathrm{cotg}\,\theta[Q^i_{34} - Q^3_{34}\delta^i_3]. \tag{AI.7}$$

From an analysis of this system of equations, we arrive at the following interesting result which may be formulated best in terms of the components with respect to the set of 1-forms θ^i: There may exist no more than 8 independent nonvanishing components of the torsion tensor: Q^1_{14}, Q^1_{23}, Q^4_{14}, Q^4_{23}, $Q^2_{12} = Q^3_{13}$, $Q^2_{24} = Q^3_{34}$, $Q^2_{13} = -Q^3_{12}$, $Q^2_{34} = -Q^3_{24}$. The 1-forms θ^i are defined as in Chapter 1 of the paper. The Q^i_{jk} depend on r and on t.

As for the components with respect to the coordinates (such components appear actually in Eq. (A1.7)), the result appears to be slightly more complicated. Now, we have to write: $Q^1_{23} = F_1 \sin\theta$, $Q^4_{23} = F_2 \sin\theta$, $Q^2_{13} = -Q^3_{12}\sin^2\theta = F_3\sin\theta$, $Q^2_{34} = -Q^3_{24}\sin^2\theta = F_4\sin\theta$, and to demand that the F_i and the remaining non-zero Q-components (Q^1_{14}, Q^4_{14}, Q^2_{12}, Q^2_{24}) are functions of r and of t.

The same result applies to the existence of independent components of the spin angular momentum density tensor s^i_{jk}.

APPENDIX II

The connection 1-forms ω^i_k, curvature 2-forms Ω^i_j, and curvature scalar R

A. Components of the connection 1-forms with respect to the basis 1-forms θ^i

$$\omega^1{}_2 = -\omega^2{}_1 = \left[C - \left(\frac{1}{r} + \frac{\sigma'}{2}\right)e^{-\lambda/2}\right]\theta^2 - \tfrac{1}{2}B\theta^3,$$

$$\omega^1{}_3 = -\omega^3{}_1 = \tfrac{1}{2}B\theta^2 + \left[C - \left(\frac{1}{r} + \frac{\sigma'}{2}\right)e^{-\lambda/2}\right]\theta^3,$$

$$\omega^2{}_3 = -\omega^3{}_2 = (D+\tfrac{1}{2}B)\theta^1 - \frac{e^{-\sigma/2}}{r}\operatorname{cotg}\theta\theta^3 - (F+\tfrac{1}{2}H)\theta^4,$$

$$\omega^1{}_4 = \omega^4{}_1 = \left(A+\frac{\dot\lambda}{2}e^{-\nu/2}\right)\theta^1 + \left(-G+\frac{\nu'}{2}e^{-\lambda/2}\right)\theta^4,$$

$$\omega^2{}_4 = \omega^4{}_2 = \left(E+\frac{\dot\sigma}{2}e^{-\nu/2}\right)\theta^2 - \tfrac{1}{2}H\theta^3,$$

$$\omega^3{}_4 = \omega^4{}_3 = \tfrac{1}{2}H\theta^2 + \left(E+\frac{\dot\sigma}{2}e^{-\nu/2}\right)\theta^3.$$

Here, as in all the following formulae, we get the expressions corresponding to the case with the classical description of spin (Kuchowicz 1975b) when we put all the torsion components equal to zero with the exception of H.

B. Frame components of the curvature 2-forms

$$\Omega^2{}_1 = -\Omega^1{}_2 = \left\{\left(\frac{\sigma''}{2} + \frac{(\sigma')^2}{4} + \frac{\sigma'}{r} - \frac{\lambda'}{2r} - \frac{\lambda'\sigma'}{4}\right)e^{-\lambda}\right.$$

$$\left. - \frac{\dot\lambda\dot\sigma}{4}e^{-\nu} - \left[C' + \left(\frac{1}{r}+\frac{\sigma'}{2}\right)C\right]e^{-\lambda/2} - \left(E\frac{\dot\lambda}{2} + A\frac{\dot\sigma}{2}\right)e^{-\nu/2} - \tfrac{1}{4}B^2 - AE - \tfrac{1}{2}BD\right\}\theta^1\wedge\theta^2$$

$$+\left\{\frac{1}{2}\left[B'+B\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2} + \tfrac{1}{2}H\left(A+\frac{\dot\lambda}{2}e^{-\nu/2}\right)\right.$$

$$\left. -(\tfrac{1}{2}B+D)\left[C - \left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\right]\right\}\theta^1\wedge\theta^3 + \left\{-\left[\frac{\dot\sigma'}{2} + \frac{\dot\sigma-\dot\lambda}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right) + \frac{\dot\sigma\nu'}{4}\right]e^{-\frac{\nu+\lambda}{2}}\right.$$

$$\left. +(\dot C + \tfrac{1}{2}C\dot\sigma - \tfrac{1}{2}G\dot\sigma)e^{-\nu/2} + \tfrac{1}{2}E\nu'e^{-\lambda/2} - \tfrac{1}{2}BF - \tfrac{1}{4}BH - EG\right\}\theta^2\wedge\theta^4$$

$$+\left\{-\tfrac{1}{2}(B+\tfrac{1}{2}B\dot\sigma)e^{-\nu/2} - \left[\tfrac{1}{4}H\nu' + (F+\tfrac{1}{2}H)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}\right.$$

$$\left. -CF - \tfrac{1}{2}CH + \tfrac{1}{2}GH\right\}\theta^3\wedge\theta^4.$$

$$\Omega^3{}_1 = -\Omega^1{}_3 = \left\{-\tfrac{1}{2}\left[B'+B\left(\frac{2}{r}+\sigma'\right)\right]e^{-\lambda/2} - D\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\right.$$

$$\left. -\tfrac{1}{4}H\dot\lambda e^{-\nu/2} - \tfrac{1}{2}AH + \tfrac{1}{2}BC + CD\right\}\theta^1\wedge\theta^2 + \left\{\left(\frac{\sigma''}{2} + \frac{(\sigma')^2}{4} + \frac{\sigma'}{r} - \frac{\lambda'}{2r} - \frac{\lambda'\sigma'}{4}\right)e^{-\lambda}\right.$$

$$\left. - \frac{\dot\lambda\dot\sigma}{4}e^{-\nu} - \left[C' + C\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2} - \tfrac{1}{2}(A\dot\sigma + E\dot\lambda)e^{-\nu/2} - AE - \tfrac{1}{4}B^2 - \tfrac{1}{2}BD\right\}\theta^1\wedge\theta^3$$

$$+\left\{\tfrac{1}{2}(\dot{B}+\tfrac{1}{2}B\dot{\sigma})e^{-\nu/2}+\left[\frac{H\nu'}{4}-\left(F+\frac{H}{2}\right)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+CF+\tfrac{1}{2}CH-\tfrac{1}{2}GH\right\}\theta^2\wedge\theta^4$$

$$+\left\{\left[\frac{\dot{\lambda}-\dot{\sigma}}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)-\frac{\dot{\sigma}'}{2}+\frac{\dot{\sigma}\nu'}{4}\right]e^{-\frac{\nu+\lambda}{2}}+(\dot{C}+\tfrac{1}{2}C\dot{\sigma}-\tfrac{1}{2}G\dot{\sigma})e^{-\nu/2}+\tfrac{1}{2}E\nu'e^{-\lambda/2}\right.$$

$$\left.-\tfrac{1}{2}BF-\tfrac{1}{4}BH-EG\right\}\theta^3\wedge\theta^4.$$

$$\Omega^3{}_2=-\Omega^2{}_3=\left\{\left[\dot{D}+\frac{\dot{B}}{2}+\frac{\dot{\lambda}}{2}\left(D+\frac{B}{2}\right)\right]e^{-\nu/2}+\left[F'+\frac{H'}{2}+\frac{\nu'}{2}\left(F+\frac{H}{2}\right)\right]e^{-\lambda/2}\right\}\theta^1$$

$$\wedge\theta^4+\left\{-\frac{e^{-\sigma}}{r^2}+\left[C-\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\right]^2-\left(E+\frac{\dot{\sigma}}{2}e^{-\nu/2}\right)^2+\tfrac{1}{4}B^2-\tfrac{1}{4}H^2\right\}\theta^2\wedge\theta^3.$$

$$\Omega^1{}_4=\Omega^4{}_1=\left\{\left[\frac{\dot{\lambda}\dot{\nu}}{4}-\frac{\ddot{\lambda}}{2}-\frac{\dot{\lambda}^2}{4}\right]e^{-\nu}+\left(\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{(\nu')^2}{4}\right)e^{-\lambda}\right.$$

$$\left.-G'e^{-\lambda/2}-\left(\dot{A}+\frac{\dot{\lambda}}{2}A\right)e^{-\nu/2}\right\}\theta^1\wedge\theta^4$$

$$+\left\{\left(\frac{1}{r}+\frac{\sigma'}{2}\right)He^{-\lambda/2}+\tfrac{1}{2}B\dot{\sigma}e^{-\nu/2}+BE-CH\right\}\theta^2\wedge\theta^3.$$

$$\Omega^2{}_4=\Omega^4{}_2=\left\{\left[\frac{\dot{\sigma}'}{2}-\frac{\dot{\sigma}\nu'}{4}+\frac{\dot{\sigma}-\dot{\lambda}}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\frac{\nu+\lambda'}{2}}\right.$$

$$\left.+\left[E'+(E-A)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+\tfrac{1}{2}C\dot{\lambda}e^{-\nu/2}+AC+\tfrac{1}{4}BH+\tfrac{1}{2}DH\right\}\theta^1\wedge\theta^2$$

$$+\left\{\left[(D+\tfrac{1}{2}B)\frac{\dot{\sigma}}{2}-B\frac{\dot{\lambda}}{4}\right]e^{-\nu/2}-\tfrac{1}{2}\left[\left(\frac{1}{r}+\frac{\sigma'}{3}\right)H+H'\right]e^{-\lambda/2}-\tfrac{1}{2}AB+\tfrac{1}{2}BE+DE\right\}\theta^1\wedge\theta^3$$

$$+\left\{\left(-\frac{\ddot{\sigma}}{2}+\frac{\dot{\sigma}\dot{\nu}}{4}-\frac{\dot{\sigma}^2}{4}\right)e^{-\nu}+\frac{\nu'}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda}-(\dot{E}+\tfrac{1}{2}E\dot{\sigma})e^{-\nu/2}\right.$$

$$\left.-\left[\tfrac{1}{2}C\nu'+G\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+CG+\tfrac{1}{4}H^2+\tfrac{1}{2}HF\right\}\theta^2\wedge\theta^4$$

$$+\{\tfrac{1}{2}(\dot{H}+H\dot{\sigma}+F\dot{\sigma})e^{-\nu/2}+\tfrac{1}{4}B\nu'e^{-\lambda/2}-\tfrac{1}{2}BG+EF+\tfrac{1}{2}EH\}\theta^3\wedge\theta^4.$$

$$\Omega^3{}_4=\Omega^4{}_3=\left\{\tfrac{1}{2}\left[H\left(\frac{1}{r}+\frac{\sigma'}{2}\right)+H'\right]e^{-\lambda/2}+\tfrac{1}{2}[\tfrac{1}{2}B(\dot{\lambda}-\dot{\sigma})-D\dot{\sigma}]e^{-\lambda/2}\right.$$

$$\left.+\tfrac{1}{2}AB-\tfrac{1}{2}BE-DE\right\}\theta^1\wedge\theta^2+\left\{\left[\frac{\dot{\sigma}'}{2}-\frac{\dot{\sigma}\nu'}{4}+\frac{\dot{\sigma}-\dot{\lambda}}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\frac{\nu+\lambda}{2}}\right.$$

$$+\left[E'+(E-A)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+\tfrac{1}{2}\,C\dot{\lambda}e^{-\nu/2}+AC+\tfrac{1}{4}\,BH+\tfrac{1}{2}\,DH\Big\}\theta^1\wedge\theta^3$$

$$+\{-\tfrac{1}{2}(\dot{H}+H\dot{\sigma}+F\dot{\sigma})e^{-\nu/2}-\tfrac{1}{4}\,B\nu' e^{-\lambda/2}+\tfrac{1}{2}\,BG-EF-\tfrac{1}{2}\,EH\}\theta^2\wedge\theta^4$$

$$+\left\{\left(\frac{\dot{\sigma}\dot{\nu}}{4}-\frac{\ddot{\sigma}}{2}-\frac{\dot{\sigma}^2}{4}\right)e^{-\nu}+\frac{\nu'}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda}-(\dot{E}+\tfrac{1}{2}\,E\dot{\sigma})e^{-\nu/2}\right.$$

$$\left.-\left[\tfrac{1}{2}\,C\nu'+G\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda/2}+CG+\tfrac{1}{4}\,H^2+\tfrac{1}{2}\,HF\right\}\theta^3\wedge\theta^4.$$

C. Tensor components and the curvature scalar

The tensor components of R^i_{jkl} may be read off from the set of expressions for Ω^i_j with the help of Eq. (1.2). Then we may calculate the Ricci tensor components with the simple rule: $R^i{}_j = R^{ki}{}_{jk}$; we do not give them here because they are used directly in the formulae of Section 2. Here, we give only the curvature scalar $R = R^i_i$ which appears directly nowhere in the formulae

$$R=\left[-2\sigma''-\nu''-\tfrac{3}{2}(\sigma')^2-\tfrac{1}{2}(\nu')^2+\lambda'\sigma'+\tfrac{1}{2}\,\lambda'\nu'-\nu'\sigma'+\frac{2(\lambda'-\nu')-6\sigma'}{r}-\frac{2}{r^2}\right]e^{-\lambda}$$

$$+[2\ddot{\sigma}+\ddot{\lambda}+\tfrac{3}{2}\,\dot{\sigma}^2+\tfrac{1}{2}\,\dot{\lambda}^2+\dot{\lambda}\dot{\sigma}-\tfrac{1}{2}\,\dot{\lambda}\dot{\nu}-\dot{\nu}\dot{\sigma}]e^{-\nu}+\frac{e^{-\sigma}}{r^2}$$

$$+2\left[2C'+G'+(2C+G)\left(\frac{2}{r}+\sigma'\right)+C\nu'\right]e^{-\lambda/2}+2\left[\dot{A}+2\dot{E}+(A+2E)\left(\dot{\sigma}+\frac{\dot{\lambda}}{2}\right)\right]e^{-\nu/2}$$

$$+4AE+\tfrac{1}{2}\,B^2+2BD-2C^2-4CG+2E^2-FH-\tfrac{1}{2}\,H^2.$$

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