ON A CLASSICAL LIMIT OF THE DIRAC EQUATION WITH AN EXTERNAL ELECTROMAGNETIC FIELD. PART I. PROPER-TIME EVOLUTION, LORENTZ COVARIANT EXPECTATION VALUES AND CLASSICAL EQUATIONS OF MOTION

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A framework is proposed for studying in a Lorentz covariant manner the Ehrenfest-type classical limit of the Dirac equation with an external electromagnetic field. We reformulate the Dirac equation using a Lorentz-invariant proper-time coordinate in Minkowski space-time and we propose a new kind of expectation values which are Lorentz covariant. Next, we give an example of a Lorentz- and gauge-covariant, bispinorial wave packet moving along a given classical trajectory. Finally, we obtain Lorentz-covariant classical equations of motion for a spin-$\frac{1}{2}$ particle in an inhomogeneous electromagnetic field. The classical equations of motion are a result of a consistency condition.

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1. *Introduction*

Since its discovery in 1928 the Dirac equation has been one of the most important equations in relativistic quantum physics. Nowadays this linear, first order partial differential equation is commonly regarded as something very simple. However, it is a fact that solutions of this equation are known only in rather few cases of particularly symmetric external electromagnetic fields. For more general external fields we can only find approximate solutions. There exist two important classes of approximate solutions of the Dirac equation with an external electromagnetic field: WKB type solutions, [1], and wave packet solutions, [2, 3]. Investigations of the both types of approximate solutions lead to equations which can be interpreted in the language of classical mechanics as classical equations of motion. For this reason such investigations are usually referred to as classical limit of the Dirac equation of the WKB type or Ehrenfest-type, respectively.

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In this paper we present the first part of our investigations of the Ehrenfest-type classical limit of the Dirac equation. Below we develop a special formalism whose aim is to preserve relativistic invariance in the classical limit of the Ehrenfest-type. The formalism guarantees Lorentz covariance of expectation values, and it enables us to write explicitly examples of Lorentz covariant wave packets moving along a given classical trajectory. Moreover, it leads to a rather interesting derivation of classical equations of motion for relativistic, spin $\frac{1}{2}$ particle moving in an external electromagnetic field. In this paper we do not consider the problem of actually finding an approximate, wave packet type solutions for the Dirac equation. This will be the subject of a forthcoming paper.

Our considerations can be regarded as a continuation of the investigations of wave packets for Dirac particle carried out in the papers [2, 3]. The considerations in [2] are restricted to the free Dirac particle. In reference [3] in fact only a constant external electromagnetic field is considered, and covariance properties of expectation values are not investigated in the most interesting case of nonzero external field.

Apart from the importance from a practical point of view our considerations about wave-packet-type solutions of the Dirac equation have another interesting aspect. In contradistinction to the nonrelativistic Schrödinger equation, the Dirac equation has been invented without much emphasis on classical mechanics. Relativistic invariance was the guiding principle. Therefore, it is not obvious what kind of classical mechanics underlies the Dirac equation. The Pauli equation and the Foldy-Wouthuysen representation and numerous subsequent investigations, see, e.g. [4–6] for results and further references, have thrown some light on this problem Nevertheless, the problem still is far from being fully clarified, and it is one of our goals to improve this situation by avoiding in our derivation of the underlying classical equations of motion some of the shortcomings present in other derivations of the classical equations of motion from the Dirac equation.

Thus, the contents of our paper is closely related to the topic of classical, spinning particles. This topic has a rather long history, and it is abounding in the literature. The derivations of classical equations of motion for a relativistic, spinning particle moving in an external electromagnetic field presented in the literature can roughly be divided into the following three groups:

— the purely classical approaches dealing with the so-called macroscopic spin, see, e.g. [4, 7][1];
— the derivations utilising anticommuting dynamical variables, see, e.g. [8];
— the derivations starting from the Dirac equation or another relativistic wave equation.

Obviously, our derivation belongs to the third group. Let us describe this group in more detail. The main problem which the approaches belonging to the third group have to solve is how to preserve Lorentz covariance while passing to classical mechanics. In order to overcome this difficulty, many authors introduce so-called "proper time", see, e.g. [4, 9], which actually is a fifth dimension in space-time. Another "proper-time" is introduced

[1] The literature on classical, spinning particle counts over one hundred papers written during a period of more than fifty years. Therefore, in the review of the literature given below we only quote sample papers written relatively recently. References to many older papers can be found in the papers [4, 6].

in [5]. Here it is not the fifth dimension, however its relation to the usual space and time coordinates is not clear.

Quite a different attempt is presented in [6]. It relies on the relativistic version of the Foldy-Wouthuysen transformation and on a Lorentz covariant position operator. Both are given only in an approximate form. In the approximation considered this approach does not involve any notions which are exterior to the quantum mechanics of a Dirac particle. Unfortunately, it is not clear whether the appropriate operators exist in the next order approximation. The same remark applies also to the approach presented in [10].

Moreover, there is a shortcoming which all the above mentioned derivations based on relativistic wave equations have in common. Namely, they do not specify the actual state of the particle, i.e., the wave function. The classical equations of motion are deduced from the Heisenberg equations of motion for operators by merely a formal dequantization, consisting of the replacement of the operators by real-number-valued classical variables. Therefore, it is not clear how to interpret these classical variables from the quantum mechanical point of view.

The derivations of the classical equations of motion presented in the WKB papers [1] also belong to the third group of approaches. The authors of these papers use only those notions which are explicitly or implicitly present in the quantum mechanics of a Dirac particle. Also, a relevant wave function of the particle is given. Thus, these classical equations have rather sound quantum mechanical foundations. They certainly are the right classical equations for the WKB-type classical limit of the Dirac equation. These classical equations of motion were obtained only for the lowest order approximation with respect to powers of $\hbar$. Probably for this reason they do not contain derivatives of the external electromagnetic field $F_{\mu\nu}$. In fact, the classical equations coincide with the well-known B-M-T equations, [11]. In our paper we show that the same equations appear within the Ehrenfest approach, again in lowest order approximation (in an expansion with respect to the inverse of the mass of the Dirac particle.) Also within the Ehrenfest approach we calculate the next order contributions — containing terms with derivatives of $F_{\mu\nu}$.

Now, let us briefly sketch the contents of the present paper. However, before doing this we would like to recall that the main idea of the Ehrenfest approach to the classical limit, [12], consists of assuming that the solution of a wave equation can approximately be given in the form of a wave packet moving along a definite classical trajectory with the corresponding world-line $\xi^\mu(\tau)$. Then, the classical variables are defined as expectation values of quantum mechanical operators. The classical equations of motion are nothing but equations of motion for expectation values of quantum observables. However, when proceeding along these lines in the case of the Dirac equation, we have to solve the problem already mentioned of preserving the Lorentz covariance. This means that we have to develop a rather unusual formulation of quantum mechanics of a Dirac particle which explicitly utilizes the classical trajectory followed by the wave packet.

First, in Section 2 we present the new formulation of quantum mechanics of a Dirac particle. Here the evolution of the wave function is not parametrized by time but by the proper time for the world-line $\xi^\mu(\tau)$. This proper-time formulation is valid for wave functions which do not vanish in a certain finite vicinity of the classical trajectory. The Dirac equation acquires a new form such that the corresponding Hamiltonian is not Hermitean. In spite of that the proper-time evolution of the wave function is unitary. The wave functions depend on the proper time and also on three other variables which parametrize the hyperplane perpendicular, in the Lorentz covariant 4-dimensional sense, to the world-line of the classical particle. We also find that this new form of the Dirac equation suggests a new scalar product for Dirac bispinors which is different from the standard

$$\langle\psi|\varphi\rangle = \int d^3x\psi^\dagger\varphi.$$

The new scalar product implies a new formula for expectation values, Section 3. The new formula guarantees that the expectation values (and therefore also the classical variables) have a clear transformation law with respect to Lorentz transformations. This is in sharp contrast to the traditional formula for expectation values. Also in Section 3 we briefly discuss the observables for the Dirac particle from the viewpoint of the new form of the Dirac equation and of the new scalar product.

In Section 4 we apply the proper-time formulation in order to construct an example of the relativistic wave packet moving along the given classical trajectory. This wave packet is explicitly covariant with respect to Lorentz transformations and also with respect to gauge transformations of the external potential $A_\mu(x)$.

In Section 5 we exploit the fact that in the proper-time formulation of the Dirac equation it is possible to introduce a rest-frame for the quantum mechanical system. We show that the new scalar product reduces to the ordinary one in the rest-frame. The rest-frame form of the Hamiltonian is Hermitean. We also show that the Foldy-Wouthuysen transformation performed on the rest-frame Hamiltonian is Lorentz invariant. Also the resulting transformed Hamiltonian is Lorentz invariant. This we regard as one of the more interesting advantages of our formalism. In the standard formulation of the Foldy-Wouthuysen transformation the Lorentz invariance is very obscure.

In Section 6 we present a definition of the classical trajectory $\xi^\mu(\tau)$ on the basis of the Dirac equation. We show that the adopted definition implies a number of relations which we shall call consistency conditions. It turns out that these conditions require that $\xi^\mu(\tau)$ obeys certain classical equations of motion. We obtain these classical equations of motion in the approximation linear with respect to $F_{\mu\nu}$, neglecting the second and higher derivatives of $F_{\mu\nu}$, and also neglecting terms which are proportional to m^{-3} or a higher power of m^{-1}, where m is the mass of the Dirac particle. In the order m^{-1} these classical equations coincide with the B-M-T equations. We think that it is an interesting feature of our approach that the classical equation for $\xi^\mu(\tau)$ appears on a rather general level, without actually assuming any explicit form of the wave packet.

In Section 7 we compare our classical equations of motion with some other classical equations presented in the literature. We also point out that within the Ehrenfest approach to the classical limit the length of the classical spin in general is not constant. Finally, we present a restriction on the applicability of the proper-time formulation of the Dirac equation.

2. The proper-time formulation of the Dirac equation

We shall consider a well-localised wave packet moving in the ordinary R^3 space, along a given trajectory $\vec{\xi}(t)$. The trajectory $\vec{\xi}(t)$ gives rise to the world-line $\xi^\mu(\tau)$. The wave function does not vanish in a certain finite vicinity of this world-line.

In this vicinity of the world-line we shall introduce new coordinates (τ, z^i). They are defined by the following formulae.

$$x^\mu = \xi^\mu(\tau)+z^\mu, \tag{2.1}$$

$$\dot{\xi}_\mu(\tau)z^\mu = 0, \tag{2.2}$$

where the dot denotes the differentiation with respect to τ. From (2.2) it follows that

$$z^0 = \vec{v}(\tau)\vec{z}, \tag{2.3}$$

where $v^i(\tau) = \dot{\xi}^i(\tau)/\dot{\xi}_0(\tau)$.

Because we would like to interpret $\xi^\mu(\tau)$ as the world-line of a classical particle, we assume that

$$\dot{\xi}_\mu\dot{\xi}^\mu > 0, \qquad \dot{\xi}_0 > 0. \tag{2.4}$$

In the following the Cartesian coordinates x^μ will be referred to as the lab-frame coordinates.

Let us present an explicit example. For a constant electric field $\vec{E} = (E, 0, 0)$ the world-line corresponding to the motion along the x-axis has the form $(c = 1)$

$$\xi^0(\tau) = \tau_0 \sinh \frac{\tau}{\tau_0},$$

$$\xi^1(\tau) = \tau_0\left(\cosh \frac{\tau}{\tau_0} - 1\right),$$

$$\xi^2(\tau) = \xi^3(\tau) = 0, \tag{2.5}$$

where $\tau_0 = \dfrac{m}{eE}$. Solving the Eqs. (2.1), (2.2) with respect to $\tau, \vec{z}$ we obtain the following formulae

$$\tau(x^\mu) = \tau_0 \operatorname{arc\,tgh} \frac{x^0}{x^1+\tau_0},$$

$$z^1(x^\mu) = \tau_0+x^1-\tau_0\left(1-\left(\frac{x_0}{x^1+\tau_0}\right)^2\right)^{-\frac{1}{2}},$$

$$z^2 = x^2, \qquad z^3 = x^3.$$

It is clear that the $\tau, \vec{z}$ coordinates are defined by these formulae only when

$$x^0 < x^1+\tau_0$$

(for $x^0 > 0$, $x^1 > 0$). The boundary hyperplane is $x^0 = x^1 + \tau_0$. From (2.5) it follows that in the (x^0, x^1) plane the trajectory is given by the equation

$$\tau_0 + x^1 = (\tau_0^2 + x_0^2)^{\frac{1}{2}}.$$

It is easy to see that for large x_0 the distance parallel to the x^1 axis between the boundary line and the trajectory decreases like $\frac{1}{2}\tau_0^2/x_0$. The width d of the wave packet (in the x^1 direction) seen from the lab-frame should be smaller than this distance. It is easy to check that this condition is equivalent to the following condition for the rest-frame width $d_R = d\gamma$, where $\gamma \equiv (1-\vec{v}^2)^{-\frac{1}{2}}$,

$$d_R < \tfrac{1}{2}\tau_0(1+\tau_0^2 x_0^{-2})^{\frac{1}{2}}.$$

Thus, the rest-frame width of the wave packet (in the x^1 direction) has to be smaller than $\dfrac{m}{2eE}$. On the other hand, d_R has to be greater than the Compton wave length $\lambda_C = \dfrac{\hbar}{m}$, otherwise the quantum mechanical description has to be replaced by a quantum field theoretical one. Therefore, the electric field cannot be too large,

$$eE < eE_C \sim \frac{m^2}{\hbar}.$$

The field E_C is by several orders of magnitude stronger than typical electric fields.

In the above example the velocity $\vec{v}$ and the acceleration $\dot{\xi}_0\dot{\vec{v}}$ are parallel. In the general case, discussed in Section 7, we obtain a more restrictive bound on the strength of the external electromagnetic field.

The coordinate τ is, by assumption, a scalar with respect to Lorentz transformations. In particular, we may choose τ to be the proper-time. z^μ is a four-vector with respect to Lorentz transformations. z^i are the coordinates in the directions transverse to the world-line $\xi^\mu(\tau)$ in the 4-dimensional sense defined by the formula (2.2).

Let us state explicitly that we regard Lorentz transformations in a passive way, i.e. as describing a change of Cartesian coordinates in Minkowski space-time. This implies that the Lorentz transformations act simultaneously on x^μ, $\xi^\mu(\tau)$ and z^μ.

The fact that τ is a Lorentz-scalar constitutes the main advantage of the new coordinate system.

The coordinates τ, $\vec{z}$ are closely related to the so-called Fermi coordinates parametrizing a vicinity of a line in space-time [13].

In general, the coordinates τ, $\vec{z}$ are local, i.e. the formulae (2.1), (2.2) define the functions $\tau = \tau(x^\mu)$, $z^i = z^i(x^\mu)$ uniquely only in certain finite vicinity of the world-line $\xi^\mu(\tau)$. Only in the trivial case of $\xi^\mu(\tau)$ being a straight-line are τ, z^i well-defined on the whole Minkowski space. We shall assume that the wave function essentially vanishes outside of that vicinity of the world-line. We shall discuss this assumption in more detail at the end of this paper.

Let us introduce the notation

$$s^0 = \tau, \quad s^i = z^i, \quad (s^0, s^i) = (s^\alpha). \tag{2.6}$$

Of course, (s^α) is not a four-vector.

Then,

$$dx^\mu = \frac{\partial x^\mu}{\partial s^\alpha}\, ds^\alpha,$$

where

$$\frac{\partial x^\mu}{\partial \tau} = \dot{\xi}^\mu + \delta_0^\mu \dot{\vec{v}}\vec{z}, \qquad \frac{\partial x^\mu}{\partial z^i} = \delta_i^\mu + \delta_0^\mu v^i. \tag{2.7}$$

Thus, we have

$$\eta_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} ds^\alpha ds^\beta, \tag{2.8}$$

where $(\eta_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, -1)$ is the metric tensor in Minkowski space time, and

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial s^\alpha} \frac{\partial x^\nu}{\partial s^\beta} \tag{2.9}$$

is the metric tensor written in the coordinates s^α. From (2.9) and (2.7) we see that the coordinates s^α are not Cartesian. In the following we shall need

$$g = |\det (g_{\alpha\beta})|.$$

It is not difficult to show that

$$g = \left(\frac{\dot{\xi}^2 + \dot{\xi}_0 \dot{\vec{v}}\vec{z}}{\dot{\xi}_0}\right)^2. \tag{2.10}$$

The contravariant metric tensor $g^{\alpha\beta}$ is given by the formula

$$g^{\alpha\beta} = \frac{\partial s^\alpha}{\partial x^\mu} \frac{\partial s^\beta}{\partial x^\nu}\, \eta^{\mu\nu}. \tag{2.11}$$

From (2.1), (2.2) it follows that

$$\frac{\partial \tau}{\partial x^\mu} = \frac{\dot{\xi}_\mu}{\dot{\xi}^2 + \dot{\xi}_0 \dot{\vec{v}}\vec{z}}, \tag{2.12}$$

where

$$\dot{\xi}^2 = \dot{\xi}_\mu \dot{\xi}^\mu, \qquad \dot{\vec{v}} = \frac{d\vec{v}}{d\tau}.$$

Also,

$$\frac{\partial z^i}{\partial x^\mu} = \delta_\mu^i - \frac{\dot{\xi}^i \dot{\xi}_\mu}{\dot{\xi}^2 + \dot{\xi}_0 \dot{\vec{v}}\vec{z}}. \tag{2.13}$$

Now, let us discuss the effects of a Lorentz transformation. We have

$$x'^\mu = L^\mu{}_\nu x^\nu, \qquad \xi'^\mu(\tau) = L^\mu{}_\nu \xi^\nu(\tau), \qquad z'^\mu = L^\mu{}_\nu z^\nu. \tag{2.14}$$

From the formulae (2.14) and (2.3) it follows that

$$z'^i = N^i{}_k z^k, \tag{2.15}$$

where

$$N^i{}_k = L^i_k + L^i_0 v^k(\tau). \tag{2.16}$$

Let us stress that the components of the velocity v^k present on the r.h.s. of the formula (2.16) are taken with respect to the initial reference frame. Because

$$z'^\mu z'_\mu = z^\mu z_\mu,$$

we have the relation

$$\delta_{sp} - v^s v^p = (\delta_{ik} - v'^i v'^k) N^i{}_s N^k{}_p, \tag{2.17}$$

where v'^i are the components of the velocity in the new reference frame ($v'^i = \dot{\xi}'^i/\dot{\xi}'_0$).

Let us also stress that the velocity v^k present on the r.h.s. of (2.16) is, in general, τ dependent. Therefore,

$$dz'^i = N^i{}_k dz^k + \dot{N}^i{}_k d\tau, \tag{2.18}$$

where, in the case of τ independent Lorentz transformations,

$$\dot{N}^i{}_k = L^i_0 \dot{v}^k.$$

We also have

$$\frac{\partial}{\partial z^i} = \frac{\partial z'^k}{\partial z^i} \frac{\partial}{\partial z'^k} = N^k{}_i \frac{\partial}{\partial z'^k}; \tag{2.19}$$

again it should be remembered that $N^k{}_i$ depends on the components of the velocity with respect to the initial reference frame.

Finally, let us state explicitly that under Lorentz transformations

$$\tau' = \tau, \quad d\tau' = d\tau. \tag{2.20}$$

However,

$$\frac{\partial}{\partial \tau} = \frac{\partial s'^\alpha}{\partial \tau} \frac{\partial}{\partial s'^\alpha} = \frac{\partial}{\partial \tau'} + \dot{N}^i{}_k (N^{-1})^k{}_p z'^p \frac{\partial}{\partial z'^i}. \tag{2.21}$$

This transformation law does not contradict (2.20) — the derivative $\partial/\partial\tau$ is calculated under the assumption that $z^i = \text{const}$, while for $\partial/\partial\tau'$ the assumption is that $z'^i = \text{const}$. From (2.15), (2.16) it follows that in the former case z'^i is τ dependent. This explains the second term on the right hand side of Eq. (2.21).

Let us write the Dirac equation,

$$\left[i\gamma^\mu \left(\hbar \frac{\partial}{\partial x^\mu} + ieA_\mu \right) - m \right] \psi = 0, \tag{2.22}$$

using the coordinates s^α. We will use B_α and Γ^α defined by

$$A_\mu = \frac{\partial s^\alpha}{\partial x^\mu} B_\alpha, \quad \Gamma^\alpha = \frac{\partial s^\alpha}{\partial s^\mu} \gamma^\mu. \tag{2.23}$$

Then the Dirac equation takes the form

$$\left[i\Gamma^\alpha\left(\hbar \frac{\partial}{\partial s^\alpha} + ieB_\alpha\right) - m\right]\psi = 0. \tag{2.24}$$

We will use the following representation for the γ matrices:

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{2.25}$$

where σ_0 is the 2×2 unit matrix and σ_i are the Pauli matrices. Thus,

$$\gamma^0 = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i,$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I. \tag{2.25'}$$

Using (2.25′) it is easy to write an equation for the conjugate bispinor $\bar\psi = \psi^\dagger\gamma^0$,

$$i\left(\hbar \frac{\partial}{\partial s^\alpha}\bar\psi - ieB_\alpha\bar\psi\right)\Gamma^\alpha + m\bar\psi = 0. \tag{2.26}$$

The Γ^α matrices are s^α dependent. They have a very important property

$$\frac{\partial}{\partial s^\alpha}(\sqrt{g}\,\Gamma^\alpha) = 0. \tag{2.27}$$

Using a well-known formula for Christoffel symbols

$$\Gamma^\delta_{\beta\delta} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial s^\beta}\sqrt{g},$$

we can write (2.27) in the form

$$\nabla_\alpha \Gamma^\alpha = 0,$$

where $\nabla_\alpha\Gamma^\beta = \frac{\partial}{\partial s^\alpha}\Gamma^\beta + \Gamma^\beta_{\alpha\gamma}\Gamma^\gamma$ is the covariant derivative of Γ^β (this formula implies that in our case the spinorial connection vanishes).

The equations (2.24), (2.26), together with the fact that

$$\Gamma^0 = \frac{1}{\dot\xi_0\sqrt{g}}\gamma^\mu\dot\xi_\mu$$

is a nonsingular matrix for small $\vec{z}$, suggest that for the wave functions which do not vanish only in a sufficiently small vicinity of the classical trajectory we can consider from now on the τ-evolution of the wave function instead of the time evolution. This change of the evolution parameter constitutes the main step in our considerations of the Ehrenfest-type classical limit of the Dirac equation.

Using (2.24), (2.26) and (2.27) it is easy to check that

$$\frac{\partial}{\partial s^\alpha}(\sqrt{g}\,\bar{\psi}\Gamma^\alpha\psi) = 0. \tag{2.28}$$

Assuming that ψ vanishes for $|\vec{z}| \to \infty$ and applying a standard reasoning we obtain from (2.28) that

$$(\psi|\psi) \overset{\mathrm{df}}{=} \int d^3\vec{z}\,\sqrt{g}\,\bar{\psi}\Gamma^0\psi \tag{2.29}$$

is a τ-independent quantity provided that ψ is a solution of the Dirac equation (2.24) which vanishes in the region where $\vec{z}$ is not defined.

Using the formulae (2.23), (2.12), (2.10) we obtain that

$$(\psi|\psi) = \int \frac{d^3\vec{z}}{\dot{\xi}_0}\,\bar{\psi}\gamma^\mu\dot{\xi}_\mu\psi. \tag{2.30}$$

We also can write $(\psi|\psi)$ in the form

$$(\psi|\psi) = \int d^3\vec{z}\psi^\dagger(1-\vec{\alpha}\vec{v})\psi, \tag{2.31}$$

where $\alpha^i = \gamma^0\gamma^i$ are Hermitean matrices. From the assumption (2.4) it follows that $|\vec{v}| < 1$. Moreover, $(\vec{\alpha}\vec{v})^2 = \vec{v}^2 < 1$. Thus,

$$M = 1-\vec{\alpha}\vec{v} \tag{2.32}$$

is a 4×4, Hermitean, positive definite matrix. Therefore, the expression

$$(\psi|\varphi) = \int d^3\vec{z}\psi^\dagger M(\tau)\varphi \tag{2.33}$$

can be used as a scalar product in the space of Dirac bispinors. We have checked that the norm $(\psi|\psi)$ of the wave function is constant with respect to τ if ψ is a solution of the Dirac equation. In order to distinguish the scalar product (2.30) (which, in general, is defined only for wave functions nonvanishing in a finite vicinity of the world line $\xi^\mu(\tau)$) from the ordinary scalar product of Dirac bispinors

$$\langle\psi|\varphi\rangle = \int d^3x\psi^\dagger\varphi,$$

we shall call (2.30) "the bilinear form".

It is a rather nice feature of the bilinear form (2.30) that it is invariant with respect to Lorentz transformations (2.14) for any ψ, φ. Let us recall that in the new reference frame defined by (2.14) one has to use

$$\psi'(x') = S(L)\psi(x), \tag{2.34}$$

where $S(L)$ obeys the condition

$$S^{-1}(L)\gamma^{\mu}S(L) = L^{\mu}{}_{\nu}\gamma^{\nu}.$$

For the conjugate bispinor $\bar{\psi}$ we have

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(L). \tag{2.35}$$

Thus,

$$\bar{\psi}'(\tau, \vec{z}')\gamma^{\mu}\dot{\xi}'_{\mu}\psi'(\tau, \vec{z}') = \bar{\psi}(\tau, \vec{z})\gamma^{\mu}\dot{\xi}_{\mu}\psi(\tau, \vec{z}). \tag{2.36}$$

The integration measure $d^3z/\dot{\xi}_0$ also is Lorentz invariant. This follows from the fact that

$$\delta(\bar{\tau}-\tau)d\bar{\tau}\,\frac{d^3\vec{z}}{\dot{\xi}_0(\tau)} = \delta((x^{\mu}-\xi^{\mu}(t))\dot{\xi}_{\mu})d^4x. \tag{2.37}$$

The r.h.s. of (2.37) is explicitly Lorentz invariant. On the l.h.s. of (2.37) $\delta(\tau-\tau_0)d\tau$ is also Lorentz invariant because of (2.20).

Another way of seeing the Lorentz invariance of the bilinear form (2.30) is to use the formula (2.29). We have

$$(\psi|\varphi) = \int d^3z d\bar{\tau}\,\sqrt{g}\,\delta(\bar{\tau}-\tau)\bar{\psi}\Gamma^0\varphi. \tag{2.38}$$

The integration measure in (2.38) is invariant under general coordinate transformations, while $\delta(\bar{\tau}-\tau)$ is invariant under Lorentz transformations because of (2.20). Finally,

$$\bar{\psi}'(\tau, \vec{z}')\Gamma^0\varphi'(\tau, \vec{z}') = \bar{\psi}(\tau, \vec{z})S^{-1}\Gamma^0 S\varphi(\tau, \vec{z}),$$

and

$$\Gamma^0(\dot{\xi}') = \frac{\partial s'^0}{\partial x'^{\mu}}(\dot{\xi}')\gamma^{\mu} = \frac{\dot{\xi}'_{\mu}\gamma^{\mu}}{\dot{\xi}'^2+\dot{\xi}'_0\dot{\vec{v}}'\vec{z}'}.$$

In the last formula we have explicitly indicated that Γ^0 depends on the four-velocity $\dot{\xi}^{\mu}$, and that in the new reference frame (denoted by the prime) we have to use the new components of the four velocity. It is easy to see that

$$\dot{\xi}_0\dot{\vec{v}}\vec{z} = -\ddot{\xi}^{\mu}z_{\mu}.$$

Therefore, this quantity is a Lorentz scalar. Thus,

$$\Gamma^0(\dot{\xi}') = \frac{\dot{\xi}_{\mu}\gamma^{\mu}}{\dot{\xi}^2+\dot{\xi}_0\dot{\vec{v}}\vec{z}}.$$

Using the condition (2.35) we obtain that

$$S^{-1}\Gamma^0(\dot{\xi}')S = \frac{\dot{\xi}'_{\mu}L^{\mu}{}_{\nu}\gamma^{\nu}}{\dot{\xi}^2+\dot{\xi}_0\dot{\vec{v}}\vec{z}} = \frac{\dot{\xi}_{\nu}\gamma^{\nu}}{\dot{\xi}^2+\dot{\xi}_0\dot{\vec{v}}\vec{z}} = \Gamma^0(\dot{\xi}).$$

This ends the second proof of the Lorentz invariance of the scalar product.

From (2.37) it follows that the bilinear form (2.30) can also be written in the form

$$(\psi|\varphi) = \int d\Sigma_\mu \bar{\psi}\gamma^\mu\varphi,$$

where $\int d\Sigma_\mu$ is the integral over the a space-like hyperplane perpendicular to $\dot{\xi}(\tau)$ and passing through the point $\xi(\tau)$.

Let us recall that the usual scalar product for Dirac bispinors, i.e. $\langle\psi \mid \varphi\rangle = \int d^3x\psi^\dagger\varphi$, is Lorentz invariant only if ψ and φ are solutions of the Dirac equation. In fact, proofs of Lorentz invariance of the scalar product $\langle\psi \mid \varphi\rangle$ are based on the continuity equation

$$\partial_\mu(\bar{\psi}\gamma^\mu\varphi) = 0,$$

which is valid if ψ and φ obey the Dirac equation. In the case of the bilinear form (2.30) the proofs of Lorentz invariance presented above do not require that ψ, φ obey the Dirac equation. However, in order to prove that the bilinear form (2.30) is τ-independent we have to use the continuity equation (2.28), hence also the Dirac equation.

The Dirac equation (2.24) can be written in a Hamiltonian form,

$$i\hbar\frac{\partial}{\partial\tau}\psi = H\psi, \tag{2.39}$$

where

$$H = \left(1+\frac{\dot{\xi}_0(\vec{v}\vec{z})}{\dot{\xi}^2}\right)(\gamma^\mu\dot{\xi}_\mu)(m-\gamma^i\pi_i)+\dot{\xi}^i\pi_i+eB_0, \tag{2.40}$$

and

$$\pi_i \overset{\text{df}}{=} i\hbar\frac{\partial}{\partial z^i} - eB_i. \tag{2.41}$$

Because $\frac{\partial}{\partial\tau} \neq \frac{\partial}{\partial\tau'}$, the Hamiltonian H is not a Lorentz scalar. Moreover, it is easy to see that H is not a Hermitean operator with respect to the bilinear form (2.30), because the operator $z^k\pi_i$ is not Hermitean for $k = i$.

In spite of the non-Hermiticity of H, the τ-evolution of the wave function ψ defined by the equation (2.39) is unitary. Actually, it is easy to check explicitly that

$$\frac{d}{d\tau}(\psi|\psi) = 0$$

(by using the formulae (2.30), (2.39), (2.40)). Then one sees that the contribution coming from the non-Hermiticity of H cancels with the τ derivative of the matrix M, see (2.32), (2.33).

Because H is a non-Hermitean operator, it cannot be regarded as an observable. In particular, it should not be related to an energy. The Hermitean operator corresponding to the energy will be presented in the next Section. Thus, the only role of H is to generate the τ-evolution of the wave function.

The rather unusual fact that a non-Hermitean operator generates a unitary τ-evolution is due to the τ-dependence of the matrix M defining the bilinear form, see the formula (2.33). For the same reason an operator which is Hermitean at $\tau = \tau_0$ does not have to be Hermitean at $\tau \neq \tau_0$, even if this operator is τ-independent. Also, if P is a Hermitean operator for all τ, then $\partial P/\partial\tau$ does not have to be a Hermitean operator. Such peculiar features of this proper time form of the Dirac equation should be regarded as a certain disadvantage of this formalism. However, from our point of view this is more than compensated by the fact that in this formulation one can clearly see how expectation values of quantum observables transform under Lorentz transformations. This will be discussed in the next Section.

In order to convince oneself that a non-Hermitean Hamiltonian can lead to a consistent quantum theory one can consider the following simple model of one-dimensional quantum mechanics of n-component wave functions. The Hamiltonian and the scalar product are

$$H = \hat{g}^{-1}\dot{\hat{g}}ix\frac{d}{dx}; \qquad (\psi|\varphi) = \int dx\psi^{\dagger}(x,t)\hat{g}(t)\varphi(x,t),$$

where $\hat{g}(t)$ is a $n\times n$, Hermitean, positive-definite, time-dependent matrix. It can be checked that the Schrödinger equation $i\hbar\dot{\psi} = H\psi$ gives a unitary evolution of the wave functions.

It is also rather easy to extend the proper time formulation in order to include the possibility that the particle has an anomalous magnetic moment. This can be done by adding to H the expression which corresponds to the term $\frac{1}{2}(g_0-2)\mu_B S^{\mu\nu}F_{\mu\nu}$ added to the l.h.s. of the Dirac Eq. (2.22), i.e.

$$\Delta H = -\tfrac{1}{2}(g_0-2)\mu_B\gamma^{\varrho}\dot{\xi}_{\varrho}S^{\mu\nu}F_{\mu\nu}(1+(\dot{\xi})^{-2}\dot{\xi}_0\vec{v}\vec{z}), \tag{2.42}$$

where $S^{\mu\nu} = \frac{1}{4i}[\gamma^{\mu},\gamma^{\nu}]$ is the spin tensor, $\mu_B = \frac{e\hbar}{2m}$, and $F_{\mu\nu} = \partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the electromagnetic field strength tensor. The operator ΔH is explicitly Lorentz invariant. It is also Hermitean with respect to the bilinear form (2.30). Therefore, the operator $H+\Delta H$ generates a unitary τ evolution.

Let us end this Section by recalling that the form of the Dirac equation presented above is in general valid only in a finite vicinity of the world-line $\xi^{\mu}(\tau)$ because of the local character of the coordinates z^i. However, this is sufficient for our purpose because we are going to consider only wave packets which essentially vanish outside of this vicinity of the world-line, i.e. we assume that the far ends of the wave packets which reach beyond that vicinity of $\xi^{\mu}(\tau)$ give a negligibly small contribution to expectation values. In the particular case of $\xi^{\mu}(\tau)$ being a straight-line the proper time form coincides with the standard form of the Dirac equation in the rest-frame of the particle, see Section 5. In this case the co-ordinates τ, $\vec{z}$ are global.

Finally, let us note that the proper time form of the Dirac equation, as well as the bilinear form (2.30) are form-invariant with respect to reparametrizations of the world-line $\xi^{\mu}(\tau)$.

3. A new definition of classical variables for the Dirac particle

The most essential feature of the Ehrenfest approach to the correspondence between classical and quantum mechanics is that classical dynamical variables are identified with expectation values of the corresponding quantum mechanical operators. When the quantum mechanical state of the particle is described by a wave packet narrow in both position space and momentum space, then the quantum-mechanical description of the time-evolution of the state of the particle essentially reduces to classical equations of motion for real parameters characterising the wave packet.

Straightforward application of this approach in the case of the Dirac equation leads to well-known difficulties with preserving the Lorentz covariance, [14]. Namely, in accordance with the probabilistic interpretation of the wave function, the expectation value of an observable P in the state ψ should be calculated as

$$\langle \psi | P\psi \rangle = \int d^3x \psi^\dagger(\vec{x}, t) P \psi(\vec{x}, t). \tag{3.1}$$

From the Minkowski space-time point of view, this integral can be written in the form

$$\langle \psi | P\psi \rangle = \int d\Sigma_\mu \bar{\psi} \gamma^\mu P \psi, \tag{3.2}$$

where

$$(d\Sigma_\mu) = (d^3x, 0, 0, 0)$$

is the integration measure over the hyperplane $x^0 = t$. Now, let us consider two observers related to each other by a Lorentz transformation. Each of them calculates the expectation value with respect to his reference frame. However, because the hyperplanes of constant time in both reference frames do not coincide, the integrals (3.2) calculated by the two observers are in general different even if P is a Lorentz scalar. The integrals will be equal if

$$\partial_\mu (\bar{\psi} \gamma^\mu P \psi) = 0. \tag{3.3}$$

Then, using a standard reasoning one can prove that the integral (3.2) does not depend on the choice of the hyperplane (for ψ sufficiently quickly vanishing at infinity for any spatial direction). However, the continuity equation (3.3) is not valid in general. Only in the trivial case when $P = 1$ the validity of (3.3) is guaranteed by the Dirac equation (2.22) for arbitrary A_μ. Therefore, the classical variables defined by (3.1) have a rather complex transformation law even if the quantum operator P is a simple Lorentz tensor. Actually, in order to find this transformation law one has to know the Dirac bispinor $\psi(\vec{x}, t)$ for all t (where ψ is a solution of (2.22)). The transformation law will probably depend on the external potentials A_μ and on the initial value of the Dirac function $\psi(\vec{x}, t = t_0)$.

One could try to solve this problem of Lorentz covariance by using very particular quantum mechanical operators like, e.g., the relativistic position operator [15]. These operators are constructed in such a manner that the transformation laws of their expectation values, calculated with respect to the ordinary scalar product $\langle \psi | \varphi \rangle$, have the required simple form. Unfortunately, in the case of the Dirac equation with an external electromagnetic field the new operators can be constructed only approximately [6]. In fact, it is not clear whether exact operators of this type exist at all when the external field is non--zero.

On the other hand, it is easy to see that the quantity

$$p \equiv (\psi|P\psi) \stackrel{\mathrm{df}}{=} \int \frac{d^3\vec{z}}{\dot{\xi}_0}\, \bar{\psi}(\gamma^\mu \dot{\xi}_\mu) P\psi \tag{3.4}$$

has the transformation law determined only by the transformation law of the observable P. For instance, if $P = (P^{\mu\nu})$ is a Lorentz tensor, i.e. if

$$P'^{\mu\nu}(\tau, \vec{z}') = L^\mu{}_\sigma L^\nu{}_\eta P^{\sigma\eta}(\tau, \vec{z})$$

when $x'^\varrho = L^\varrho{}_\sigma x^\sigma$, then a reasoning essentially identical with the proof of Lorentz invariance of $(\psi \mid \psi)$ presented in the previous Section shows that

$$p'^{\mu\nu} = L^\mu{}_\sigma L^\nu{}_\eta p^{\sigma\eta},$$

where

$$p'^{\mu\nu} = \int \frac{d^3 z'}{\dot{\xi}'_0}\, \bar{\psi}'(\tau, \vec{z}')\, (\gamma^\mu \dot{\xi}'_\mu) P'^{\mu\nu}(\tau, \vec{z}')\psi(\tau, \vec{z}').$$

In the proof we do not have to use any continuity equation.

For this reason we propose to define the classical dynamical variables by utilising the bilinear form (2.30) instead of the ordinary scalar product $\langle \psi \mid \varphi \rangle$. Thus, we will assume that the classical variable p associated with the quantum observable P is given by the formula (3.4). The classical variables defined in this manner transform under the Lorentzian changes of the reference frame in Minkowski space-time in the same way as the corresponding quantum operators. We will call the quantity $(\psi \mid P\psi)$ the "covariant expectation value".

This new definition of the classical variables resembles the "hyperplane formalism" of Ref. [16]. In our case there exist distinguished hyperplanes, namely the ones defined for each τ by (2.1), (2.2). In the paper [16] no explicit expression for the expectation values is given.

The prescription (3.4) for the classical variables explicitly contains the trajectory $\xi_\mu(\tau)$ and the solution $\psi(x)$ of the Dirac equation (2.22). Actually, the trajectory $\xi_\mu(\tau)$ is to be deduced from $\psi(x)$. The formalism presented so far is designed for the case when $\psi(x)$ is a wave packet which moves along a given trajectory. A precise definition of $\xi^\mu(\tau)$ could be

$$(\psi|x^\mu\psi) = \xi^\mu(\tau). \tag{3.5}$$

Some other possible definitions of $\xi^\mu(\tau)$ are described at the beginning of Section 6.

The l.h.s. of (3.5) depends on $\xi^\mu(\tau)$ in a very complicated, implicit manner, so that the definition (3.5), as well as the definitions given in Section 6, probably are not useful for a direct computation of $\xi^\mu(\tau)$. However, we shall show in Section 6 that these formal definitions are sufficient in order to obtain the classical equation of motion for $\xi^\mu(\tau)$.

Basic classical variables are usually represented by real numbers. Therefore, as basic quantum observables we shall choose operators which are Hermitean with respect to the bilinear form $(\psi \mid \varphi)$.

The τ-derivative of the classical variable p can be obtained by differentiating both sides of the definition (3.4). It is easy to check that

$$\frac{d(\psi|P\psi)}{d\tau} = \left(\psi \,\middle|\, \left(\frac{i}{\hbar}[H, P] + \frac{\partial P}{\partial\tau}\right)\psi\right). \tag{3.6}$$

Here, again we have used the identity $\nabla_\alpha \Gamma^\alpha = 0$. The partial derivative $\partial P/\partial\tau$ is present because of a possible explicit τ-dependence of the operator P.

Let us remark that neither the commutator $[H, P]$ nor the operator $\partial P/\partial\tau$ are in general Hermitean, or Lorentz tensors, even if P is a Hermitean, Lorentz tensor operator. This is due to the fact that H is not a Hermitean operator, and that H and $\partial/\partial\tau$ are not Lorentz scalars. However, the sum $\frac{i}{\hbar}[H, P]+\frac{\partial P}{\partial\tau}$ is a Hermitean, Lorentz tensor operator because $dp/d\tau$ is a real-valued Lorentz tensor for any ψ (because p is real, and because $d/d\tau$ is a Lorentz scalar due to (2.20)).

One can introduce the Heisenberg picture in the proper time formalism by the following definitions. The Heisenberg picture counterpart of an operator P is defined by the relation

$$(\psi(\tau) \mid P\psi(\tau))_\tau = (\psi(\tau = 0) \mid P_{\mathrm{H}}(\tau)\psi(\tau = 0))_{\tau=0}. \tag{3.7}$$

From (3.7) it follows that

$$P_{\mathrm{H}} = U^{-1}(\tau, 0)PU(\tau, 0), \tag{3.8}$$

where $U(\tau, 0)$ is the τ-evolution operator corresponding to the equation (2.39). In order to deduce (3.8) from (3.7) we have used the relation

$$\int d^3z(U(\tau, 0)\psi(\tau = 0)) \quad M(\tau)U(\tau, 0)\varphi(\tau = 0) = \int d^3z\psi(\tau = 0)M(0)\varphi(\tau = 0), \tag{3.9}$$

which reflects the fact that the τ-evolution is unitary. The equation (3.6) is equivalent to the following operational equation

$$\frac{dP_{\mathrm{H}}}{d\tau} = \frac{i}{\hbar}[H_{\mathrm{H}}, P_{\mathrm{H}}] + \left(\frac{\partial P}{\partial\tau}\right)_{\mathrm{H}}. \tag{3.10}$$

Now we would like to discuss briefly observables for the Dirac particle. For such a particle any observable is a function (in general, nonlinear) of the basic observables x^i, $p_i = i\hbar\frac{\partial}{\partial x^i}$, and also it is a linear function of γ^μ, $\gamma^\mu\gamma^5$, $S^{\mu\nu}$, γ^5, where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$

In the proper time formalism we insist on having an explicit Lorentz covariance. Therefore, we will use as the basic observables the following operators

$$x^\mu,\ \tilde{\pi}_\mu = i\hbar \frac{\partial}{\partial x^\mu} - eA_\mu,$$

$$\Gamma \equiv \gamma^\mu \dot{\xi}_\mu,\ i\Gamma\gamma^5,\ \Gamma\gamma^\mu,\ \Gamma\gamma^5\gamma^\mu,\ \Gamma S^{\mu\nu}, \tag{3.11}$$

where $S^{\mu\sigma}$ is given by (2.43). Notice that Γ and γ^5 are Lorentz scalars. $\tilde{\pi}_0$ is related to the kinetic energy of the classical particle.

Passing to the (τ, z^i) variables we obtain

$$x^\mu = \xi^\mu(\tau) + z^\mu, \tag{3.12}$$

$$\tilde{\pi}_\mu = \delta^i_\mu \pi_i + \frac{\dot{\xi}_\mu}{\dot{\xi}^2} \Gamma(m - \gamma^i \pi_i), \tag{3.13}$$

where π_i is given by (2.41), and we have used (2.39).

It is easy to check that the observables (3.11) are Hermitean with respect to the bilinear form $(\psi \mid \varphi)$.

The set (3.11) of observables is not minimal. For instance x_0 can be expressed by x^i,

$$x^0 = \xi^0 + \vec{v}\vec{z} = \xi^0 + \vec{v}(\vec{x} - \vec{\xi})$$

(thus we have not introduced any time operator).

We use the nonminimal set in order to deal with full Lorentz tensors — the set (3.11) is closed with respect to Lorentz transformations.

The classical variables corresponding to the operators (3.11) are defined by (3.4). Their τ dependence is determined by the τ dependence of ψ which follows from the Dirac equation (2.22) (x^μ is related to τ, z^i by (3.12)). In practice we can exactly solve the Eq. (2.22) only for rather particular external potentials $A_\mu(x)$.

4. Lorentz- and gauge-covariant wave packet

In this Section we shall utilize the new variables τ, $\vec{z}$ in order to construct an example of a relativistic wave packet moving along a given classical trajectory $\vec{\xi}(t)$. We do not expect that this wave packet is a good approximation to an exact solution of the Dirac equation, apart from the very basic feature that it is localised around the classical trajectory. In order to construct a better approximation to the exact solution it is necessary to apply a technique for approximate by solving the Dirac equation. One of the possibilities is to construct an approximate wave packet propagator following the ideas presented in, e.g. [17].

We shall require that the wave packet $\psi(x)$ obeys the following three conditions
— its center moves along the classical trajectory $\vec{\xi}(t)$,
— the wave function $\psi(x^\mu)$ transforms under Lorentz transformations according to (2.34),

— $\psi(x^\mu)$ transforms in the usual manner under gauge transformations, i.e.

$$A'_\mu(x) = A_\mu(x)+\partial_\mu\lambda(x), \qquad \psi'(x) = \exp\left[-\frac{ie}{\hbar}\lambda(x)\right]\psi(x). \tag{4.1}$$

The last two requirements are imposed by the fact that exact solutions of the Dirac equation (2.22) have these properties.

Let us first consider the gauge transformations. Then it is convenient to pass to the B_α potentials, see (2.23). It is easy to see that under the gauge transformation (4.1)

$$B'_\alpha(\tau, \vec{z}) = B_\alpha(\tau, \vec{z})+\frac{\partial}{\partial s^\alpha}\lambda. \tag{4.2}$$

Now, let us introduce a new bispinor ψ_0, defined by the formula

$$\psi(x) = \exp\left[-\frac{ie}{\hbar}\Lambda_0(\tau, \vec{z})\right]\psi_0(\tau, \vec{z}), \tag{4.3}$$

where

$$\Lambda_0(\tau, \vec{z}) = \int_{-\infty}^{\tau} B_0(\bar{\tau}, 0)d\bar{\tau}+z^iB_i(\tau, 0)+\tfrac{1}{2}z^iz^k\frac{\partial B_k}{\partial z^i}(\tau, 0)$$

$$+\frac{1}{3!}z^iz^kz^s\frac{\partial^2 B_s}{\partial z^i\partial z^k}(\tau, 0)+\ \ldots \tag{4.4}$$

It is easy to see that under the gauge transformation (4.2)

$$\Lambda'_0(\tau, \vec{z}) = \Lambda_0(\tau, \vec{z})+\lambda(\tau, \vec{z})+\lambda(-\infty, 0). \tag{4.5}$$

The constant $\lambda(-\infty, 0)$ does not matter because it gives rise to a constant phase of the wave function $\psi(x)$. Thus, the Ansatz (4.3), (4.4) essentially secures that under the gauge transformations $\psi(x)$ transforms in the manner required by (4.1), provided that $\psi_0(\tau, \vec{z})$ is gauge-invariant. Thus, ψ_0 defined by (4.3) can be regarded as gauge-invariant.

Let us note that Λ_0 is a scalar with respect to Lorentz transformations. For instance,

$$z^iB_i = z^i\frac{\partial x^\mu}{\partial z^i}A_\mu = z^iA_i+z^iv^iA_0 = z^\mu A_\mu.$$

Notice also that

$$B_0(\tau, \vec{z}) = \delta^\mu_0\dot{\vec{v}}\vec{z}A_\mu+\dot{\xi}^\mu A_\mu$$

is not a Lorentz scalar in general. However, $B_0(\tau, 0)$ is a Lorentz scalar because of the fact that $\vec{z} = 0$ is invariant under Lorentz transformations.

Now, let us turn to Lorentz transformations. Because Λ_0 is a scalar, $\psi_0(\tau, \vec{z})$ has to transform according to (2.34). An analysis of wave packets for free Dirac particle in the positive energy sector suggests the following Ansatz for ψ_0

$$\psi_0(\tau, \vec{z}) = \sum_{\alpha=1,2} \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \chi_\alpha(\vec{k}, \tau) U^{(\alpha)}(\vec{k}) \exp\left[-\frac{i}{h}\left(k_\mu z^\mu + \frac{(k_\varrho \dot{\xi}^\varrho)(\xi^\mu \dot{\xi}_\mu)}{\dot{\xi}^2} \right)\right] f(k, \dot{\xi}). \tag{4.6}$$

Here $k_0 = +\sqrt{m^2+\vec{k}^2}$, $U^{(\alpha)}(\vec{k})$ are two linearly independent positive energy solutions of the free Dirac equation, $f(k, \dot{\xi})$ is a shape function (in momentum space) for the wave packet, (χ_α) is a 2-component spinor. In the representation adopted in (2.25) the bispinors $U^{(\alpha)}$ have the form [18]

$$U^{(\alpha)}(\vec{k}) = [2m(m+k_0)]^{-\frac{1}{2}} \begin{pmatrix} (k_0+m) & e^{(\alpha)} \\ (\vec{\sigma}\vec{k}) & e^{(\alpha)} \end{pmatrix}, \tag{4.7}$$

where $e^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Their normalization is

$$U^{\dagger(\alpha)} U^{(\beta)} = \frac{k_0}{m} \delta_{\alpha\beta}, \qquad \overline{U}^{(\alpha)} U^{(\beta)} = \delta_{\alpha\beta}. \tag{4.8}$$

We shall assume that the shape function f is a Lorentz scalar function on the positive energy part of the hyperboloid $k_\mu k^\mu = m^2$, clearly peaked at the momentum $k_\mu^{(0)} = m\dot{\xi}_\mu$. Such a function is easy to construct with the help of boosts. A boost which transforms the standard 4-momentum $(m, 0, 0, 0) \equiv q^{(0)}$ into the 4-momentum $q = (q^\mu)$, i.e., $H_q q^{(0)} = q$, has the form, [19]

$$(H_q)^i{}_0 = (H_q)^0{}_i = \frac{q^i}{m}, \qquad (H_q)^0{}_0 = \frac{q_0}{m},$$

$$(H_q)^i{}_j = \delta^i{}_j + \frac{q^i q^j}{m(m+q_0)}. \tag{4.9}$$

Now, let us consider a four-component object $p \overset{\mathrm{df}}{=} H_q^{-1} k$ and its transformation law under Lorentz transformation of q and k. We have

$$p' \equiv H_{Lq}^{-1} L k = H_{Lq}^{-1} L H_q p \equiv R(L, q) p,$$

where $R(L, q) = H_{Lq}^{-1} L H_q$ is a rotation. In fact, it is the Wigner rotation associated with the Lorentz transformation L and the momentum q. Thus, $\vec{p}^2$ is a Lorentz scalar. It is easy to see that $\vec{p} = 0$ if and only if $k = q$. Now it is clear that as the scalar shape function f we can take, e.g.

$$f(k, \dot{\xi}) = \exp\left[-\frac{\vec{p}^2}{a_0^2} \right], \tag{4.10}$$

where $p = (H_{\bar{q}})^{-1}k$ and $\bar{q} = m\dot{\xi}/\sqrt{\dot{\xi}^2}$. Instead of the exponential we can also take any other function of $\vec{p}^2$ which is clearly peaked at $\vec{p} = 0$. Using (4.9) it is easy to show that

$$\vec{p}^2 = \left[\frac{1}{2m}(k - m(\dot{\xi}^2)^{-\frac{1}{2}}\dot{\xi})^2 - m\right]^2 - m^2. \tag{4.11}$$

The parameter a_0 measures the width of the wave packet in the momentum space. We shall assume that a_0 is a constant Lorentz scalar.

The bispinor ψ_0 will satisfy the condition (2.34) if

$$\chi'_\alpha(Lk, \tau)U^{(\alpha)}(Lk) = S(L)U^{(\beta)}(k)\chi_\beta(k, \tau). \tag{4.12}$$

From (4.12) it follows that

$$\chi'_\alpha(Lk, \tau) = \hat{R}_{\alpha\beta}(L, k)\chi_\beta(k, \tau), \tag{4.13}$$

where $\hat{R}(L, k) = [\hat{R}_{\alpha\beta}(L, k)]$ is a SU(2) matrix corresponding to the Wigner rotation $R(L, k)$. In order to prove the relation (4.13) we use the correspondence between the Lorentz group and SL(2, C). This correspondence is given by the relation, [19],

$$A\sigma_\nu A^\dagger = L^\mu{}_\nu\sigma_\mu, \tag{4.14}$$

where $A \in$ SL(2, C). The SL(2, C) counterpart of the boost H_q is given by

$$\hat{H}_q = [2m(q_0 + m)]^{-\frac{1}{2}}((q_0 + m)\sigma_0 + \vec{q}\vec{\sigma}). \tag{4.15}$$

It is easy to see that $\hat{H}_q^2 = \frac{1}{m} q^\mu\sigma_\mu$, $\hat{H}_q^\dagger = \hat{H}_q$. With the help of (4.14) one can check that the matrix

$$S(L) = \frac{1}{2}\begin{pmatrix} A + (A^\dagger)^{-1} & A - (A^\dagger)^{-1} \\ A - (A^\dagger)^{-1} & A + (A^\dagger)^{-1} \end{pmatrix} \tag{4.16}$$

obeys the relation (2.35) for γ^μ given by (2.25). Using (4.15), (4.16) we can write $U^{(\alpha)}$ in the form

$$U^{(\alpha)}(k) = S(H_k)\begin{pmatrix} e^{(\alpha)} \\ 0 \end{pmatrix}. \tag{4.17}$$

Then, it follows from (4.8), (4.17), (4.16) that

$$\hat{R}(L, k) = \hat{H}_{Lk}^{-1} A\hat{H}_k. \tag{4.18}$$

From (4.18) we see that $\hat{R}(L, k)$ is the SL(2, C) counterpart of the Wigner rotation $R(L, k)$.

The k-dependence of the χ_α spinor is still arbitrary. Actually, the k-dependence of the transformation matrix in (4.13) implies that it is not possible to assume that χ_α does not depend on k without breaking the Lorentz covariance of ψ_0. Therefore, we shall introduce a new spinor $\zeta(k, \tau)$ which is defined by the formula

$$\chi(k, \tau) = M(k, \dot{\xi})\zeta(k, \tau), \tag{4.19}$$

where

$$M(k,\dot{\xi}) = \tfrac{1}{2}\hat{H}_k^{-1}(\hat{H}_k^2+\hat{H}_{m\dot{\xi}}^2)\hat{H}_{m\dot{\xi}}^{-1}. \tag{4.20}$$

The transformation matrix for the ζ spinor does not depend on k. Namely, we have

$$\zeta'(Lk,\tau) = M^{-1}(Lk, L\dot{\xi})\chi'(Lk,\tau).$$

Using (4.13), (4.18), (4.20) and the formula

$$\hat{H}_{LK}^2 = AH_k^2A,$$

which follows from (4.15), (4.14), we obtain that

$$\zeta'(Lk,\tau) = \hat{R}(L, m\dot{\xi})\zeta(k,\tau). \tag{4.21}$$

In view of (4.21) we now see that the Lorentz covariance of ψ_0 is not broken when we assume that ζ does not depend on k. In Section 5 we show that ζ can be regarded as an instant rest-frame quantity.

Thus, the full Ansatz for ψ_0 has the form

$$\psi_0(\tau,\vec{z}) = N\sum_{\alpha=1,2}\int\frac{d^3k}{(2\pi)^3}\,\frac{m}{k_0}\,M_{\alpha\beta}(k,\dot{\xi})\zeta_\beta(\tau)U^{(\alpha)}(k)$$

$$\exp\left[-\frac{i}{\hbar}\left(k_\mu z^\mu+\frac{(k\dot{\xi})(\xi\dot{\xi})}{\dot{\xi}^2}\right)\right]f(\vec{p}^{\,2}/a_0^2), \tag{4.22}$$

where N is a normalization factor, $M(k,\dot{\xi})$ is given by (4.20) and $\vec{p}^{\,2}$ is given by (4.11) Utilizing a transformation to the instant rest-frame described in the next Section it is easy to prove that N does not depend on ξ, $\dot{\xi}$. The spinor $\zeta(\tau)$ transforms by the Wigner rotation $\hat{R}(L, m\dot{\xi})$ according to (4.21). We shall assume that ζ is normalised to 1,

$$\zeta_\alpha^*\zeta_\alpha = 1. \tag{4.23}$$

The wave function ψ_0 given by (4.22) is, in general, rather complicated function of x^μ — recall that τ and $\vec{z}$ are related to x^μ by (2.1), (2.2).

The classical variables contained in ψ_0 are ξ_μ, $\dot{\xi}_\mu$, ζ_α. The last variable describes spin degrees of freedom. All those classical variables are assumed to be gauge invariant.

At first sight it might be unclear whether ψ_0 given by (4.22) really is a well-localised wave packet. For this reason we would like to note that for fixed τ, in the instantaneous rest-frame, i.e. for $\dot{\xi}/\sqrt{\dot{\xi}^2} = (1,0,0,0)$, the integral in (4.22) has the form

$$\int d^3\vec{k}\,\exp\left(\frac{i}{\hbar}\,\vec{k}\vec{z}_R\right)\tilde{f}(\vec{k}), \tag{4.24}$$

where $\tilde{f}(\vec{k})$ is sharply peaked at $\vec{k}=0$, and $\vec{z}_R$ is equal to $\vec{z}$ transformed to the rest frame. From (4.24) it follows that the rest-frame transform of ψ_0 is indeed well-localised at $\vec{z}_R = 0$.

The full expression for the Lorentz- and gauge-covariant wave packet is given by (4.3), where ψ_0 is given by (4.22).

Using the formulae of Section 5 it is easy to see that for small a_0/m

$$\psi(\tau, \vec{z}) = N \exp\left[-\frac{i}{h} eA_0\right] \sum_\alpha \zeta_\alpha(\tau) U^{(\alpha)}(m\dot{\xi})$$

$$\times \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \exp\left[-\frac{i}{h}\left(k_\mu z^\mu + \frac{(k_\mu \dot{\xi}^\mu)(\xi^\varrho \dot{\xi}_\varrho)}{\dot{\xi}^2}\right)\right] f\left(\frac{\vec{p}^2}{a_0^2}\right) + \mathcal{O}\left(\frac{a_0}{m}\right),$$

where $\mathcal{O}\left(\frac{a_0}{m}\right)$ denotes terms of the order $\frac{a_0}{m}$ or higher. Thus, for the wave packet (4.22) the spin degrees of freedom decouple from the other degrees of freedom in the $a_0/m \to 0$ limit.

Using formulae of the next Section it is easy to check that for the wave packet (4.22)

$$(\psi | z^i \psi) = 0, \tag{4.25}$$

$$(\psi | z^i z^k \psi) = \left(\frac{h}{a}\right)^2 d_0(\delta_{ik} + \dot{\xi}^i \dot{\xi}^k) + \mathcal{O}\left(\frac{a_0^2}{m^2}\right), \tag{4.26}$$

where d_0 is a positive, dimensionless constant. From the formulae (4.25) it follows that

$$(\psi | x^\mu \psi) = \xi^\mu(\tau), \tag{4.27}$$

and from (4.26) it follows that the mean deviation from $\xi^\mu(\tau)$ is of the order $\hbar/a_0$. Thus the wave packet (4.3) is localised at $\vec{\xi}(\tau)$ with a dispersion of the order $\hbar/a_0$.

5. *Transformation to the instant rest-frame*

The Lorentz transformation to the rest-frame is given by $H_{\dot{\xi}}^{-1}$, where $H_{\dot{\xi}}$ is the boost (4.9) for $q \equiv \bar{q} = m\dot{\xi}/\sqrt{\dot{\xi}^2}$. The rest frame quantities will be marked by the subscript R on either the r.h.s. or the l.h.s. of them.

From now on we choose the parameter τ such that $\dot{\xi}^2 = 1$. Thus, now τ is the proper time for the world-line $\xi^\mu(\tau)$.

We have

$$\dot{\xi}^\mu = (H_{\dot{\xi}})^\mu{}_\nu \delta_0^\nu, \tag{5.1}$$

and $\dot{\xi}_R = (\delta_0^\nu)$. Similarly,

$$z^\mu = (H_{\dot{\xi}})^\mu{}_\nu z_R^\nu,$$

i.e.

$$z_R^0 = 0, \quad z^i = H^i{}_k z_R^k, \tag{5.2}$$

where

$$H^i{}_k = \delta^i{}_k + \frac{\dot{\xi}^i\dot{\xi}^k}{1+\dot{\xi}_0}. \tag{5.3}$$

Furthermore,

$$\frac{\partial}{\partial z^i} = (\bar{H}^{-1})^k{}_i \frac{\partial}{\partial z^k_{\mathrm{R}}} \tag{5.4}$$

where $\bar{H} = [H^i{}_k]$ is a 3×3 matrix, and

$$(\bar{H}^{-1})^k{}_i = \delta^k{}_i - \frac{\dot{\xi}^k\dot{\xi}^i}{\dot{\xi}_0(1+\dot{\xi}_0)}. \tag{5.5}$$

Of course,

$$\tau_{\mathrm{R}} = \tau.$$

For the γ^μ matrices and the related matrices like $S^{\mu\nu}$, we may use the relation (2.35), i.e.,

$$S^{-1}(H_{\dot{\xi}})\gamma^\mu S(H_{\dot{\xi}}) = (H_{\dot{\xi}})^\mu{}_\nu\gamma^\nu. \tag{5.6}$$

As an example, let us transform the wave packet (4.22) to the instant rest frame. Substituting into (4.21) $L = H_{\dot{\xi}}$ we obtain

$$\zeta_{\mathrm{R}}(\tau) = \zeta(\tau), \tag{5.7}$$

i.e., the passage to the rest frame does not change the spinor ζ. Next, using the formula (4.12) we can write

$$_{\mathrm{R}}\chi_\alpha(k_{\mathrm{R}},\tau)U^{(\alpha)}(k_{\mathrm{R}}) = S^{-1}(H_{\dot{\xi}})U^{(\beta)}(k)\chi_\beta(k,\tau).$$

On the other hand, from (4.19) it follows that

$$_{\mathrm{R}}\chi(k_{\mathrm{R}},\tau) = M(k_{\mathrm{R}},\dot{\xi}_{\mathrm{R}})\zeta(\tau) = \sqrt{\frac{_{\mathrm{R}}k_0+m}{2m}}\,\zeta(\tau).$$

Thus,

$$\chi_\beta(k,\tau)U^{(\beta)}(k) = \sqrt{\frac{_{\mathrm{R}}k_0+m}{2m}}\,S(H_{\dot{\xi}})\zeta_\alpha(\tau)U^{(\alpha)}(k_{\mathrm{R}}). \tag{5.8}$$

Furthermore,

$$k_\mu z^\mu = {}_{\mathrm{R}}k_\mu z^\mu_{\mathrm{R}} = -{}_{\mathrm{R}}\vec{k}\vec{z}_{\mathrm{R}}, \qquad \frac{(k\dot{\xi})(\xi\dot{\xi})}{\dot{\xi}^2} = k^0_{\mathrm{R}}\xi^0_{\mathrm{R}},$$

and p, present in (4.22) and defined as $H^{-1}_{\dot{q}}k$, is just k_{R}. Thus, because the integration measure in (4.22) is Lorentz invariant, we finally can write

$$\psi_0(\tau,\vec{z}) = S(H_{\dot{\xi}})_{\mathrm{R}}\psi_0(\tau,\vec{z}_{\mathrm{R}}), \tag{5.9}$$

where

$$ {}_{\mathrm{R}}\psi_0(\tau, \vec{z}) = N \sum_{\alpha} \int \frac{d^3k}{(2\pi)^2} \frac{m}{k_0} \sqrt{\frac{k_0+m}{2m}} \zeta_\alpha(\tau) U^{(\alpha)}(k) \cdot \exp\left[-\frac{i}{\hbar} k^0 \xi^0_{\mathrm{R}}\right] $$

$$ \times \exp\left(\frac{i}{\hbar} \vec{k}\vec{z}_{\mathrm{R}}\right) f(\vec{k}^2/a_0^2). \tag{5.10} $$

In the formula (5.10) we have skipped the index R in k_{R}, k^0_{R}.

The phase factor (4.4), i.e. $\exp\left(-\frac{ie}{\hbar} \Lambda_0\right)$, is form-invariant with respect to Lorentz transformations, thus it has the same form (4.4) also in the rest frame.

When passing to the instant rest-frame we have to remember that the corresponding Lorentz transformations are τ-dependent. Therefore, special care is required when dealing with a quantity which contains τ-derivatives. For example,

$$ (\dot{\vec{v}})_{\mathrm{R}} \neq 0, \qquad \frac{d}{d\tau}(\xi_{\mathrm{R}}) \neq (\dot{\xi})_{\mathrm{R}}. $$

The bilinear form (2.30) acquires a simpler form when the wave function $\psi(\tau, \vec{z})$ is expressed by its rest-frame counterpart $\psi_{\mathrm{R}}(\tau, \vec{z}_{\mathrm{R}})$. The relevant formula is

$$ \psi(\tau, \vec{z}) = S(H_{\dot{\xi}})\psi_{\mathrm{R}}(\tau, \vec{z}_{\mathrm{R}}). \tag{5.11} $$

From (5.2) it follows that

$$ d^3\vec{z} = \dot{\xi}_0 d^3\vec{z}_{\mathrm{R}}. \tag{5.12} $$

Using (5.11), (5.12), and (2.35) we obtain that

$$ (\psi|\varphi) = \int d^3\vec{z}_{\mathrm{R}} \psi^\dagger_{\mathrm{R}}(\tau, \vec{z}_{\mathrm{R}}) \varphi_{\mathrm{R}}(\tau, \vec{z}_{\mathrm{R}}). \tag{5.13} $$

Thus, the bilinear form (2.30) can be regarded as the ordinary scalar product of Dirac bispinors in the instant rest-frame.

The form of the r.h.s. of the formula (5.13) implies that the τ-evolution of the rest--frame wave function ψ_{R} is governed by a Hamiltonian which is a Hermitean operator. Indeed, using (5.11), (5.4) and the formula

$$ \left.\frac{\partial}{\partial\tau}\right|_{\vec{z}} = \left.\frac{\partial}{\partial\tau}\right|_{\vec{z}_{\mathrm{R}}} - (\bar{H}^{-1})^p{}_k \dot{H}^k{}_s z^s_{\mathrm{R}} \frac{\partial}{\partial z^p_{\mathrm{R}}}, \tag{5.14} $$

we obtain from (2.39) the following form of the Dirac equation transformed to the rest--frame:

$$ i\hbar \frac{\partial}{\partial\tau} \psi_{\mathrm{R}}(\tau, \vec{z}_{\mathrm{R}}) = H_{\mathrm{R}} \psi(\tau, \vec{z}_{\mathrm{R}}), \tag{5.15} $$

where the rest-frame Hamiltonian H_R has the form (we assume that $\dot{\xi}^2 = 1$)

$$H_R = eB_0^R + m\gamma^0 - \alpha^p \pi_p^R + m\left(\ddot{\xi}^k - \frac{\ddot{\xi}_0 \dot{\xi}^k}{1+\dot{\xi}_0}\right) z_R^k \gamma^0 - \frac{\hbar}{2}\varepsilon_{srp}\frac{\dot{\xi}^s \ddot{\xi}^r}{1+\dot{\xi}_0}\begin{pmatrix}\sigma_p & 0 \\ 0 & \sigma_p\end{pmatrix}$$
$$+\left(\frac{\dot{\xi}^r \ddot{\xi}^p - \ddot{\xi}^r \dot{\xi}^p}{1+\dot{\xi}_0} - \left(\ddot{\xi}^r - \frac{\ddot{\xi}_0 \dot{\xi}_r}{1+\dot{\xi}_0}\right)\alpha^p\right)\tfrac{1}{2}\{z_R^r, \pi_p^R\}_+. \tag{5.16}$$

In this formula $\pi_p^R \equiv i\hbar \dfrac{\partial}{\partial z_R^p} - eB_p^R$, and B_0^R, B_p^R represent B_0, B_p transformed to the rest-frame accordingly to the formulae

$$B_0^R = B_0 + \left.\frac{\partial z^p}{\partial \tau}\right|_{\vec{z}_R} B_p, \qquad B_p^R = \frac{\partial z^k}{\partial z_R^p} B_k,$$

which in turn follow from the fact that $B_\alpha = \dfrac{\partial x^\mu}{\partial s^\alpha} A_\mu$, i.e.

$$B_\alpha^R = \frac{\partial s^\beta}{\partial s^\alpha} B_\beta,$$

see (2.23). Also, we have used the formula

$$S^{-1}\frac{\partial S}{\partial \tau} = -\frac{i}{2(1+\dot{\xi}_0)}\varepsilon_{rst}\dot{\xi}^r \ddot{\xi}^s \begin{pmatrix}\sigma_t & 0 \\ 0 & \sigma_t\end{pmatrix} + \tfrac{1}{2}\left(\ddot{\xi}^s - \frac{\ddot{\xi}_0 \dot{\xi}^s}{1+\dot{\xi}_0}\right)\alpha^s$$

for $S \equiv S(H_{\dot{\xi}})$. This formula follows from (4.16), (4.15), (4.9).

It is clear that H_R is a Hermitean operator with respect to the rest-frame scalar product (5.13). The sum of the first three terms on the r.h.s. of (5.16) has the same form as the usual Hamiltonian for the Dirac equation (2.22) written in Cartesian coordinates. The other terms on the r.h.s. are due to the fact that the instant rest-frame is τ-dependent, in general. These terms vanish when the acceleration $\ddot{v}$ vanishes. The fifth term on the r.h.s. of (5.16) gives the Thomas precession of the rest-frame spin.

Now, let us consider how the rest-frame quantities transform under a Lorentzian change of the coordinates x^μ in Minkowski space-time. Using the formulae

$$z'^\mu = L^\mu{}_\nu z^\nu, \qquad z'^k = (H_{L\dot{\xi}})^k{}_s z_R'^s, \qquad z^k = (H_{\dot{\xi}})^k{}_s z_R^s,$$

it is easy to see that

$$\vec{z}_R' = R(L, \dot{\xi})\vec{z}_R, \tag{5.17}$$

where $R(L, \dot{\xi})$ is the Wigner rotation introduced in Section 4. Similarly, the formulae (2.34), (5.11) and

$$\psi'(x') = S(H_{L\dot{\xi}})\psi_R'(\tau, z_R')$$

give

$$\psi'_R(\tau, \vec{z}'_R) = S(R(L, \dot{\xi}))\psi_R(\tau, \vec{z}_R). \tag{5.18}$$

Thus, the rest-frame quantities merely transform by Wigner rotations.

Particularly interesting is the transformation law (5.18) for the rest-frame wave function, because it does not mix the two upper components of the Dirac bispinor with the two lower components of it. This implies that in the Hamiltonian (5.16) the sum of the terms containing the α^p matrices is Lorentz invariant. Therefore, the standard Foldy-Wouthuysen transformation [20] applied to the rest-frame Dirac Hamiltonian (5.16) also is Lorentz invariant. This is in sharp contrast to the case of the usual Dirac Hamiltonian in which the Foldy-Wouthuysen transformation severely complicates the problem of Lorentz invariance.

The advantage of the Foldy-Wouthuysen representation for the Dirac Hamiltonian consists of the fact that in this representation the Hamiltonian is expressed by operators which have a straightforward physical interpretation from the classical point of view. For instance, it is well-known, that the $\vec{x}$ variables present in the original form of the Dirac Hamiltonian are not what one would like to identify with the position operator, while the $\vec{x}$ variables present in the Foldy-Wouthuysen counterpart of this Hamiltonian have all the desired properties (except for a very complicated transformation law under Lorentz transformations). In our formalism the disadvantage of the Foldy-Wouthuysen representation consisting of the loss of explicit Lorentz invariance disappears. Another problem, namely that of the very existence of the Foldy-Wouthuysen representation, remains on the same level as in the standard approach to the Dirac equation.

The Foldy-Wouthuysen transformation has the form

$$\psi_{F-W} = \exp(iG)\psi_R,$$

$$H_{F-W} = \exp(iG)H_R \exp(iG) - i\hbar \exp(iG)\frac{\partial}{\partial\tau}\exp(-iG), \tag{5.19}$$

where G is Hermitean and is chosen in such a way that H_{F-W} does not contain the α matrices. In the following we shall perform an approximate Foldy-Wouthuysen transformation which leads to H_{F-W} which is free of the α matrices up to terms proportional to m^{-2}.

When estimating the power of m^{-1} in terms contributing to H_{F-W} we shall take into account the fact that also $\ddot{\xi}^i$ and $\dot{\xi}_0$ contain m^{-1} as a factor. This follows from the fact that $\xi^\mu(\tau)$ is a physical trajectory, i.e., it is a line which is followed by a wave packet which evolves according to the Dirac equation (2.22). For large m (large in comparison with the average momentum of the wave packet) the Dirac equation (2.22) can be reduced to the Pauli equation for 2-component spinors, [18]. This nonrelativistic Schrödinger type equation predicts that the wave packet will follow the trajectory which obeys Newton's equation with the Lorentz force, $m\ddot{\vec{\xi}} = e(\vec{E}+\vec{v}\times\vec{B})$. Thus we expect that $\ddot{\xi}^\nu$ vanishes like m^{-1} for large m. The Pauli equation can be regarded as the first in a whole sequence of approximations to the Dirac equation (2.22). The n-th order approximation is given by the equation

$$i\hbar \frac{\partial}{\partial\tau} \psi_{F-W} = H^{(n)}_{F-W}\psi_{F-W}, \tag{5.20}$$

where $H^{(n)}_{F-W}$ is free of odd matrices α^i, γ^5 up to terms proportional to m^{-n}.

In each order the classical trajectory has to be determined anew, because terms by which $H^{(n)}$ differs from $H^{(n-1)}$ will influence the evolution of the wave packet and therefore they can change also the trajectory $\xi^\mu(\tau)$ followed by the wave packet. $H^{(n)}$ differs from $H^{(n-1)}$ by terms proportional to m^{-n}. Hence we expect that the correction to the equation for the classical trajectory is of order m^{-n} too. The following calculations support this expectation. Thus we expect that $\xi^\mu(\tau)$ obeys a classical equation of the form

$$\ddot{\xi}^\nu = \sum_{k=1}^{n} m^{-k} F^{(k)\nu}, \tag{5.21}$$

where $F^{(k)\nu}$ does not contain (a power of) m as a factor.

6. *A derivation of classical equations of motion*

Several times we have made the qualitative statement that the classical trajectory $\xi^\mu(\tau)$ is defined as a line along which a wave packet moves which obeys the Dirac equation (2.22). It is clear that any precise form of this definition of the classical trajectory must contain a certain degree of arbitrariness, because the wave packet has a finite extension. Therefore there are many lines one can say about that the wave packet moves along then. Different definitions will yield different classical trajectories $\xi^\mu(\tau)$ in general. However, for each τ the separation between the trajectories will not exceed the width of the wave packet. Therefore, as long as this width is small on a classical scale, the definition is rather arbitrary. One possible definition of the classical trajectory has already been presented in formula (3.5). The operator x^μ present in (3.5) is the operator of the coordinate x^μ present in the original Dirac equation (2.22). On the basis of available knowledge about the Dirac equation we expect that x^μ contains a Zitterbewegung part, and therefore in principle $\xi^\mu(\tau)$ could contain it too. In order to avoid this rapid oscillation one could utilize an averaging. For example one could try to define $\xi^\mu(\tau)$ by the formula

$$\xi^\mu(\tau) = \frac{1}{2T} \int_{\tau-T}^{\tau+T} d\bar{\tau} (\psi|x^\mu\psi), \qquad T = \frac{\hbar}{mc^2}.$$

Another possibility is to use the formula (3.5) in the Foldy-Wouthuysen representation. This is equivalent to the replacement of the x^μ in (3.5) by another operator, the so-called mean position operator x^μ_{F-W} [20].

In the present paper we shall use the last possibility. That is, in each order of the expansion in powers of m^{-1} we shall require that for all τ

$$\xi^\mu(\tau) = (\psi|x^\mu_{F-W}\psi). \tag{6.1}$$

Because $x^{\mu}_{F-W} = \xi^{\mu}+z^{\mu}_{F-W}$ and $(\psi \mid P_{F-W}\psi) = (\psi_{F-W} \mid P\psi_{F-W})$ for any observable P, the requirement (6.1) is equivalent to

$$(\psi^{(n)}_{F-W}|z^i_R\psi^{(n)}_{F-W}) = 0. \tag{6.2}$$

In the following we shall show that the condition (6.2) leads to the Newton equation of the type (5.21) with $F^{(1)\nu}$ being the Lorentz force.

We would like to include the contribution coming from the anomalous magnetic moment. Therefore we shall include the ΔH term, formula (2.42), in the Hamiltonian. The rest frame counterpart of ΔH is

$$\Delta H_R = \tfrac{1}{2}(g_0-2)\mu_B\left[1+\left(\ddot{\xi}^k - \frac{\ddot{\xi}_0\dot{\xi}^k}{1+\dot{\xi}_0}\right)z^k_R\right] \times\left[\tfrac{1}{2}\varepsilon_{ikr}\gamma^0\begin{pmatrix}\sigma_r & 0\\ 0 & \sigma_r\end{pmatrix}\tilde{F}_{ik} - i\gamma^0\alpha^i\tilde{F}_{0i}\right] \tag{6.3}$$

where $\mu_B = \dfrac{e\hbar}{2m}$ and

$$\tilde{F}_{\lambda\varrho} \equiv (H_{\dot{\xi}})^{\mu}{}_{\lambda}(H_{\dot{\xi}})^{\nu}{}_{\varrho}F_{\mu\nu}. \tag{6.4}$$

Observe that $\tilde{F}_{\lambda\varrho} \neq f^R_{\lambda\varrho}$, where

$$f^R_{0p} \equiv \frac{\partial}{\partial\tau}B^R_p - \frac{\partial}{\partial z^p}B^R_0, \qquad f^R_{ps} \equiv \frac{\partial}{\partial z^p_R}B^R_s - \frac{\partial}{\partial z^s_R}B^R_p \tag{6.5}$$

(the fields $f^R_{\lambda\varrho}$ will appear again in the following). Thus the Hamiltonian to which we shall apply the transformation (5.19) is equal to $H_R+\Delta H_R$, where H_R and ΔH_R are given by (5.16) and (6.3) respectively.

The computation of the approximate Hamiltonians H^R_{F-W} is not significantly different from computations presented in [20], except that we have to remember about (5.21). Performing consecutively the two transformations of the form (5.19), the first one with

$$G_1 = \frac{i}{2m}\gamma^0\alpha^p\pi^R_p, \tag{6.6}$$

and the second one with

$$G_2 = \frac{e\hbar}{(2m)^2}(f^R_{0p}+\tfrac{1}{2}(g_0-2)\tilde{F}_{0p})\alpha^p \tag{6.7}$$

we obtain

$$H^{(1)}_{F-W} = eB^R_0+m\gamma^0\left(1+\left(\ddot{\xi}^k-\frac{\ddot{\xi}_0\dot{\xi}^k}{1+\dot{\xi}_0}\right)z^k_R\right)+\frac{1}{2m}\gamma^0\vec{\pi}^2_R + \frac{e\hbar}{2m}\varepsilon_{psr}\gamma^0\Sigma^r f^R_{ps} - \hbar\varepsilon_{srp}\frac{\dot{\xi}^s\ddot{\xi}^r}{1+\dot{\xi}_0}\Sigma^p + \frac{\dot{\xi}^r\ddot{\xi}^p-\ddot{\xi}^r\dot{\xi}^p}{1+\dot{\xi}_0}\tfrac{1}{2}\{z^r_R, \pi^R_p\} + +\tfrac{1}{2}(g_0-2)\mu_B\varepsilon_{ikr}\gamma^0\Sigma^r\tilde{F}_{ik}+\mathcal{O}\left(\frac{1}{m^2}\right), \tag{6.8}$$

which is free of the odd matrices to the order $\frac{1}{m}$. In the formula (6.8) $\{,\}_+$ denotes the anti-commutator, and

$$\Sigma^p \equiv \frac{1}{2}\begin{pmatrix}\sigma_p & 0\\ 0 & \sigma_p\end{pmatrix}.$$

Notice that the term $\gamma^0 \frac{1}{2m} \vec{\pi}_{\mathrm{R}}^2$ is Lorentz invariant because $\vec{\pi}_{\mathrm{R}}$ transforms by the Wigner rotation. It is easy to check that the equation

$$i\hbar \frac{\partial}{\partial\tau} \psi^{(1)}_{\mathrm{F-W}} = H^{(1)}_{\mathrm{F-W}}\psi^{(1)}_{\mathrm{F-W}} \tag{6.9}$$

is Lorentz covariant. When checking this one has to remember that

$$\left.\frac{\partial}{\partial\tau}\right|_{z_{\mathrm{R}}'} \psi' = \left.\frac{\partial}{\partial\tau}\right|_{z_{\mathrm{R}}} \psi_{\mathrm{R}} - (R^{-1}(L,\dot{\xi})\dot{R}(L,\dot{\xi}))_{kp} z^p_{\mathrm{R}} \frac{\partial}{\partial z^k_{\mathrm{R}}} \psi',$$

$$B_0'^{\mathrm{R}} = B_0^{\mathrm{R}} - (R^{-1}(L,\dot{\xi})\dot{R}(L,\dot{\xi}))_{kp} z^p_{\mathrm{R}} B^{\mathrm{R}}_k,$$

where $\dot{R} \equiv \frac{d}{d\tau} R$, and ψ'_{R} is related to ψ_{R} by (5.18). f^{R}_{ps} transforms by

$$f'^{\mathrm{R}}_{ps} = R_{pk} R_{st} f^{\mathrm{R}}_{kt}.$$

It is also useful to remember that

$$\ddot{\xi}^k - \frac{\ddot{\xi}_0 \dot{\xi}^k}{1+\dot{\xi}_0}$$

is just the spatial part of the acceleration $\ddot{\xi}^\nu$ transformed to the rest-frame, hence

$$(L\ddot{\xi})^k - \frac{(L\ddot{\xi})_0 (L\dot{\xi})^k}{1+(L\dot{\xi})_0} = R_{kp}(L\dot{\xi})\left(\ddot{\xi}^p - \frac{\ddot{\xi}_0\dot{\xi}^p}{1+\dot{\xi}_0}\right).$$

Now we shall obtain Newton's equation (5.21) in the order $n = 1$. It follows from the requirement (6.2) which is assumed to be satisfied for all τ. In fact, it will turn out that the requirement (6.2) is compatible with the equation (6.9) for the wave packet only when $\xi^\mu(\tau)$ obeys an equation of the form (5.21).

The problem of consistency becomes apparent when we notice that the requirement (6.2) implies that

$$\frac{d^k}{d\tau^k} (\psi^{(1)}_{\mathrm{F-W}} | z^i_{\mathrm{R}} \psi^{(1)}_{\mathrm{F-W}}) = 0 \tag{6.10}$$

for any $k = 1, 2, \ldots$, and for all τ. For any observable P we have the relation (following from (6.9))

$$\frac{d}{d\tau}(\psi^{(1)}_{\mathrm{F-W}}|P\psi^{1}_{\mathrm{F-W}}) = \left(\psi^{(1)}_{\mathrm{F-W}}\middle|\left(\frac{i}{\hbar}[H^{(1)}_{\mathrm{F-W}}, P]+\frac{\partial P}{\partial \tau}\right)\psi^{(1)}_{\mathrm{F-W}}\right).$$

Taking $P = z^i$ and using (6.10) we find that (6.2) and (6.9) imply that

$$\left(\psi^{(1)}_{\mathrm{F-W}}\middle|\frac{i}{\hbar}[H^{(1)}_{\mathrm{F-W}}z^i_{\mathrm{R}}]\psi^{(1)}_{\mathrm{F-W}}\right) = 0. \tag{6.11}$$

Taking $k = 2, 3, \ldots$ in (6.10) we find yet other conditions of the type (6.11). On the whole, we find an infinite set of consistency conditions which have to be satisfied if (6.2) is not to contradict (6.9).

We shall analyse the consistency conditions in the approximation consisting of neglecting all terms which are not linear in the external field strength $F_{\mu\nu}$. The acceleration $\ddot{\xi}^{\nu}$ will be regarded as proportional to $F_{\mu\nu}$, in accordance with the intuition gained from the Pauli equation. Moreover, because in $H^{(1)}_{\mathrm{F-W}}$ we have neglected the terms of order m^{-2} and higher the consistency conditions should be considered also with this accuracy.

It is useful to write the Heisenberg equations of motion for the operators z^i, π_k, Σ^i in the Heisenberg picture. It is easy to show that they have the following form

$$\frac{dz^i_{\mathrm{R}}}{dr} = -\frac{\gamma^0}{m}\pi^{\mathrm{R}}_i + \frac{\ddot{\xi}^r\dot{\xi}^i-\dot{\xi}^r\ddot{\xi}^i}{1+\dot{\xi}_0}z^r_{\mathrm{R}}, \tag{6.12}$$

$$\frac{d\pi^{\mathrm{R}}_i}{d\tau} = ef^{\mathrm{R}}_{i0}+m\gamma^0\left(\ddot{\xi}^i-\frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0}\right)+\frac{\dot{\xi}^i\ddot{\xi}^p-\ddot{\xi}^i\dot{\xi}^p}{1+\dot{\xi}_0}\pi^{\mathrm{R}}_p$$

$$+\frac{e}{2m}\gamma^0(\pi^{\mathrm{R}}_p f^{\mathrm{R}}_{pi}+f^{\mathrm{R}}_{pi}\pi^{\mathrm{R}}_p)+\frac{e\hbar}{2m}\varepsilon_{prs}\gamma^0\Sigma^r f^{\mathrm{R}}_{ps,i}$$

$$+\tfrac{1}{2}\mu_{\mathrm{B}}(g_0-2)\varepsilon_{pkr}\gamma^0\Sigma^r\tilde{F}_{pk,i}, \tag{6.13}$$

$$\frac{d\Sigma_i}{d\tau} = \varepsilon_{srp}\varepsilon_{pik}\frac{\dot{\xi}^s\ddot{\xi}^r}{1+\dot{\xi}_0}\Sigma^k-\frac{e}{2m}\varepsilon_{psr}\varepsilon_{rit}\gamma^0\Sigma^t f^{\mathrm{R}}_{ps}$$

$$-\frac{g_0-2}{2\hbar}\mu_{\mathrm{B}}\varepsilon_{pkr}\varepsilon_{rit}\gamma^0\Sigma^t\tilde{F}_{pk}. \tag{6.14}$$

In (6.12)–(6.14) we have neglected terms of the order m^{-2} or higher, and terms nonlinear in $F_{\mu\nu}$.

As the final preparatory step let us specify the assumptions about the wave functions $\psi^{(n)}_{\mathrm{F-W}}$. First of all, we assume that only the two upper components of $\psi^{(n)}_{\mathrm{F-W}}$ do not vanish. We may assume this because $H_{\mathrm{F-W}}$ does not contain the odd matrices. Because of this assumption we can in fact abandon the γ^0 matrix in the Heisenberg equations of motion

(6.12)–(6.14) because now $(\psi \mid \gamma^0 P\psi) = (\psi \mid P\psi)$ for any P. Of course, we also assume that $\psi^{(n)}_{\mathrm{F-W}}$ is a wave packet concentrated around $\vec{z}_{\mathrm{R}} = 0$.

Now we are ready to consider the consistency conditions. (6.2) and (6.10) applied to the covariant expectation value of the equation (6.12) give

$$\frac{1}{m}(\psi^{(1)}_{\mathrm{F-W}} \mid \pi^{\mathrm{R}}_i \psi^{(1)}_{\mathrm{F-W}}) = 0. \tag{6.15}$$

This implies that $(\psi^{(1)}_{\mathrm{F-W}} \mid \pi^{\mathrm{R}}_i \psi^{(1)}_{\mathrm{F-W}})$ is zero in the order m^0. It does not exclude the possibility that $\gamma^0\pi_i$ has nonvanishing covariant expectation value of the order m^{-1}. Now let us consider (6.10) for $k = 2$. Differentiating both sides of (6.12) with respect to τ, using (6.13) and neglecting terms of the order m^{-2} we obtain

$$\frac{d^2 z^i_{\mathrm{R}}}{d\tau^2} = -\frac{\gamma^0}{m}\left(ef^{\mathrm{R}}_{i0} + m\gamma^0\left(\ddot{\xi}^i - \frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0}\right)\right) + \frac{d}{d\tau}\left(\frac{\dot{\xi}^r\ddot{\xi}^i - \dot{\xi}^r\ddot{\xi}^i}{1+\dot{\xi}_0} z^r_{\mathrm{R}}\right). \tag{6.16}$$

Taking the covariant expectation value of the Eq. (6.16) and using (6.2), (6.10) we obtain

$$m\left(\ddot{\xi}^i - \frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0}\right) = -e(\psi^{(1)}_{\mathrm{F-W}} \mid f^{\mathrm{R}}_{i0}\psi^{(1)}_{\mathrm{F-W}}). \tag{6.17}$$

The consistency conditions (6.10) for $k > 2$ follow from (6.16) by differentiations with respect to τ. It is easy to see that (6.17) guarantees that all of them are fulfilled.

The equation (6.17) is Newton's equation (in the instant rest-frame form) for the trajectory. From the definition (6.5) it follows that

$$f^{\mathrm{R}}_{i0} = -F_{\mu\nu}\dot{\xi}^{\mu}(H_{\dot{\xi}})^{\nu}{}_i + (\text{terms of the type } \ddot{\xi}^{\nu}F_{\varrho\sigma}). \tag{6.18}$$

The terms between brackets in (6.18) are nonlinear with respect to $F_{\mu\nu}$, therefore we neglect them. It is easy to check that

$$\ddot{\xi}^i - \frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0} = -\ddot{\xi}_{\nu}(H_{\dot{\xi}})^{\nu}{}_i. \tag{6.19}$$

Then (6.17) is equivalent to the equation (recall that $\dot{\xi}^2 = 1$)

$$m\ddot{\xi}_{\nu} = e\dot{\xi}^{\mu}(\psi^{(1)}_{\mathrm{F-W}} \mid F_{\nu\mu}\psi^{(1)}_{\mathrm{F-W}}). \tag{6.20}$$

For constant $F_{\mu\nu}$ this equation reduces to Newton's equation with the familiar Lorentz force. For nonconstant $F_{\mu\nu}$ we can expand around $\vec{z} = 0$,

$$F_{\mu\lambda}(\tau, \vec{z}) = F_{\mu\nu}(\tau, 0) + z^p_{\mathrm{R}}\frac{\partial}{\partial z^p_{\mathrm{R}}}F_{\mu\nu}\Big|_{\vec{z}_{\mathrm{R}}=0} + \tfrac{1}{2}z^p_{\mathrm{R}}z^s_{\mathrm{R}}\frac{\partial^2}{\partial z^p_{\mathrm{R}}\partial z^s_{\mathrm{R}}}F_{\mu\nu}\Big|_{\vec{z}_{\mathrm{R}}=0}. \tag{6.21}$$

Because of (6.2) the second term on the r.h.s. of (6.21) does not contribute to the expectation value present on the r.h.s. of (6.20). The third term gives the contribution

$$\frac{e}{2}\dot{\xi}^{\mu}\left.\frac{\partial^2 F_{\mu\nu}}{\partial z^p_{\mathrm{R}}\partial z^s_{\mathrm{R}}}\right|_{\vec{z}_{\mathrm{R}}=0}\sigma^{\mathrm{R}}_{ps}(\tau), \tag{6.22}$$

where $\sigma^{\mathrm{R}}_{ps}(\tau) \equiv (\psi^{(1)}_{\mathrm{F-W}} \mid z^p_{\mathrm{R}} z^s_{\mathrm{R}} \psi^{(1)}_{\mathrm{F-W}})$ is the z-dispersion for the wave packet in the instant rest-frame (in the Foldy-Wouthuysen representation). It is easy to compute that

$$\left.\frac{\partial^2 F_{\mu\nu}}{\partial z^p_{\mathrm{R}} \partial z^s_{\mathrm{R}}}\right|_{\vec{z}_{\mathrm{R}}=0} = (H_{\dot{\xi}})^{\lambda}{}_{s} (H_{\dot{\xi}})^{\varrho}{}_{p} F_{\mu\nu,\lambda\varrho}(\zeta^{\mu}(\tau)).$$

It is clear that the expression (6.22) is a Lorentz scalar (σ^{R}_{ps} transforms by the Wigner rotations $R(L, \dot{\xi})$). The fact that the successive terms in the expansion (6.21) give Lorentz covariant contributions to the r.h.s. of the Eq. (6.20) is due to the use of the covariant expectation values.

Now we shall consider the next approximation, i.e. we shall include terms of the order m^{-2}. Again we shall keep only terms which are linear with respect to the external field $F_{\mu\nu}$. The reasonings are strictly analogous to the ones just presented in the order m^{-1}.

Calculating the terms of the order m^{-2} in (6.8) and removing the odd matrices by the transformation (5.19) with $G = G_3$, where G_3 is suitably chosen (G_3 is proportional to m^{-3}), we obtain the Hamiltonian $H^{(2)}_{\mathrm{F-W}}$ which is free of the odd matrices up to the order m^{-2}. It has the following form

$$\begin{aligned} H^{(2)}_{\mathrm{F-W}} = H^{(1)}_{\mathrm{F-W}} &- \frac{e\hbar}{2m^2} \varepsilon_{spr} \Sigma^r \tfrac{1}{2} \{\pi^{\mathrm{R}}_s, f^{\mathrm{R}}_{0p}\}_+ - \frac{e\hbar^2}{8m^2} f^{\mathrm{R}}_{0p,p} \\ &+ \frac{1}{2m}\left(\ddot{\xi}^r - \frac{\ddot{\xi}_0 \dot{\xi}^r}{1+\dot{\xi}_0}\right) \gamma^0 (\pi^{\mathrm{R}}_p z^r_{\mathrm{R}} \pi^{\mathrm{R}}_p - \hbar \varepsilon_{rpk} \Sigma^k \pi^{\mathrm{R}}_p - e\hbar \varepsilon_{ipk} \Sigma^k z^r_{\mathrm{R}} f^{\mathrm{R}}_{pt}) \\ &+ \frac{1}{4m}(g_0 - 2)\mu_{\mathrm{B}}(-\hbar \tilde{F}_{0i,i} + 2\varepsilon_{ipr} \Sigma^r \{\tilde{F}_{0i}, \pi_p\}_+). \end{aligned} \tag{6.23}$$

The corresponding Heisenberg equations of motion for z^i_{R}, π^i_{R}, Σ^i have the following form

$$\begin{aligned} \frac{dz^i_{\mathrm{R}}}{d\tau} &= -\frac{\gamma^0}{m} \pi^{\mathrm{R}}_i + \frac{e\hbar}{2m^2} \varepsilon_{ipr} \Sigma^r f^{\mathrm{R}}_{0p} - \frac{\dot{\xi}^r \ddot{\xi}^i - \ddot{\xi}^r \dot{\xi}^i}{1+\dot{\xi}_0} z^r_{\mathrm{R}} \\ &+ \frac{1}{2m}\gamma^0 \left(\ddot{\xi}^r - \frac{\ddot{\xi}_0 \dot{\xi}^r}{1+\dot{\xi}_0}\right)(\hbar \varepsilon_{rik} \Sigma^k - \{\pi^{\mathrm{R}}_i, z^r_{\mathrm{R}}\}_+) - \frac{1}{m} \mu_{\mathrm{B}}(g_0 - 2)\varepsilon_{kir} \Sigma^r \tilde{F}_{0k}, \end{aligned} \tag{6.24}$$

$$\begin{aligned} \frac{d\pi^{\mathrm{R}}_i}{d\tau} &= e f^{\mathrm{R}}_{i0} + \frac{e\hbar}{2m} \varepsilon_{prs} \gamma^0 \Sigma^r f^{\mathrm{R}}_{ps,i} + \frac{e}{2m} \gamma^0 \{\pi^{\mathrm{R}}_k, f^{\mathrm{R}}_{ki}\}_+ \\ &- \frac{e\hbar}{4m^2} \varepsilon_{spr} \Sigma^r \{\pi^{\mathrm{R}}_s, f^{\mathrm{R}}_{0p,i}\} - \frac{e\hbar^2}{8m^2} f^{\mathrm{R}}_{0p,pi} + m\gamma^0 \left(\ddot{\xi}^i - \frac{\ddot{\xi}_0 \dot{\xi}^i}{1+\dot{\xi}_0}\right) + \frac{\dot{\xi}^i \ddot{\xi}^p - \ddot{\xi}^i \dot{\xi}^p}{1+\dot{\xi}_0} \pi^{\mathrm{R}}_p \\ &+ \frac{1}{2m}\left(\ddot{\xi}^i - \frac{\ddot{\xi}_0 \dot{\xi}^i}{1+\dot{\xi}_0}\right) \gamma^0 \vec{\pi}^2_{\mathrm{R}} + \tfrac{1}{2} \mu_{\mathrm{B}}(g_0 - 2)\varepsilon_{pkr} \gamma^0 \Sigma^r \tilde{F}_{pk,i} \\ &+ \frac{g_0 - 2}{4m} \mu_{\mathrm{B}}(-\hbar \tilde{F}_{0k,ki} + 2\varepsilon_{kpr} \Sigma^r \{\tilde{F}_{0k,i}, \pi^{\mathrm{R}}_p\}_+), \end{aligned} \tag{6.25}$$

$$\frac{d\Sigma^i}{d\tau} = \varepsilon_{stp}\varepsilon_{pik}\frac{\dot{\xi}^s\ddot{\xi}^r}{1+\dot{\xi}_0}\Sigma^k - \frac{e}{2m}\varepsilon_{psr}\varepsilon_{rit}\gamma^0\Sigma^t f^{\mathrm{R}}_{ps}$$

$$+\frac{e}{4m^2}\varepsilon_{rik}\varepsilon_{spr}\Sigma^k\{\pi^{\mathrm{R}}_s, f^{\mathrm{R}}_{0p}\}_+ + \frac{1}{2m}\left(\ddot{\xi}^r - \frac{\ddot{\xi}_0\dot{\xi}^r}{1+\dot{\xi}_0}\right)\gamma^0\Sigma^s\varepsilon_{kis}\varepsilon_{rpk}\pi^{\mathrm{R}}_p$$

$$-\frac{g_0-2}{2\hbar}\mu_{\mathrm{B}}\varepsilon_{pkr}\varepsilon_{rit}\gamma^0\Sigma^t\tilde{F}_{pk} - \tfrac{1}{2}(g_0-2)\frac{\mu_{\mathrm{B}}}{\hbar m}\varepsilon_{kpr}\varepsilon_{rit}\Sigma^t\{\tilde{F}_{0k}, \pi^{\mathrm{R}}_p\}_+. \tag{6.26}$$

Now let us again consider the problem of the compatibility of the conditions (6.2), (6.10) with the equation (6.24) and equations following from (6.24) by differentiations with respect to τ. The conditions (6.2), (6.10) applied to both sides of Eq. (6.24) give the relation

$$(|\pi^{\mathrm{R}}_i) = \frac{e\hbar}{2m}\varepsilon_{ipr}(|\Sigma^r f^{\mathrm{R}}_{0p}) + \tfrac{1}{2}\left(\ddot{\xi}^r - \frac{\ddot{\xi}_0\dot{\xi}^r}{1+\dot{\xi}_0}\right)(|\hbar\varepsilon_{rik}\Sigma^k - \{\pi^{\mathrm{R}}_i, z^r_{\mathrm{R}}\}_+) - \mu_{\mathrm{B}}(g_0-2)\varepsilon_{kir}(|\Sigma^r\tilde{F}_{0k}), \tag{6.27}$$

where the covariant expectation values are taken with respect to the wave function $\psi^{(2)}_{\mathrm{F-W}}$ which obeys the equation

$$i\hbar\frac{\partial}{\partial\tau}\psi^{(2)}_{\mathrm{F-W}} = H^{(2)}_{\mathrm{F-W}}\psi^{(2)}_{\mathrm{F-W}}. \tag{6.28}$$

Differentiating both sides of Eq. (6.24) with respect to τ, using (6.25) and noticing that $d\Sigma/d\tau$ is of the order m^{-1} (as it follows from (6.26)) we obtain

$$\frac{d^2z^i_{\mathrm{R}}}{d\tau^2} = -\frac{\gamma^0}{m}ef^{\mathrm{R}}_{i0} - \ddot{\xi}^i + \frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0} - \frac{e\hbar}{2m^2}\varepsilon_{prs}\Sigma^r f^{\mathrm{R}}_{ps,i}$$

$$-\frac{e}{2m^2}\{\pi^{\mathrm{R}}_k, f^{\mathrm{R}}_{ki}\}_+ - \frac{\gamma^0}{m}\frac{\dot{\xi}^i\ddot{\xi}^p - \ddot{\xi}^i\dot{\xi}^p}{1+\dot{\xi}_0}\pi^{\mathrm{R}}_p - \tfrac{1}{2}\mu_{\mathrm{B}}(g_0-2)\varepsilon_{pkr}\frac{1}{m}\Sigma^r\tilde{F}_{pk,i}$$

$$+\frac{e\hbar}{2m^2}\varepsilon_{ipr}\Sigma^r f^{\mathrm{R}}_{0p,0} - \frac{d}{d\tau}\left(\frac{\dot{\xi}^r\ddot{\xi}^i - \ddot{\xi}^r\dot{\xi}^i}{1+\dot{\xi}_0}z^r_{\mathrm{R}}\right)$$

$$+\frac{1}{2m}\gamma^0\frac{d}{d\tau}\left(\ddot{\xi}^r - \frac{\ddot{\xi}_0\dot{\xi}^r}{1+\dot{\xi}_0}\right)(\hbar\varepsilon_{rik}\Sigma^k - \{\pi^{\mathrm{R}}_i, z^r_{\mathrm{R}}\}_+) - \frac{1}{m}\mu_{\mathrm{B}}(g_0-2)\varepsilon_{kir}\Sigma^r\tilde{F}_{0k,0}. \tag{6.29}$$

Here again we have neglected all terms which are proportional to m^{-n}, $n > 3$, and all terms which are nonlinear with respect to $F_{\mu\nu}$.

Equation (6.29) is much more complicated than Eq. (6.16). When taking the covariant expectation value of the r.h.s. of Eq. (6.29) we encounter covariant expectation values of operators which have not been considered yet, like $\Sigma^r f^{\mathrm{R}}_{ps,i}$. Now we must consider them. First of all, we shall replace f^{R} by its Taylor expansion around $\vec{z}_{\mathrm{R}} = 0$, e.g.,

$$f^{\mathrm{R}}_{ps,i}(\vec{z}_{\mathrm{R}}, \tau) = f^{\mathrm{R}}_{ps,i}(0,\tau) + z^k_{\mathrm{R}}f^{\mathrm{R}}_{ps,ik}(0,\tau) + \ldots \tag{6.30}$$

Because the wave function is, by assumption, a well-localised wave packet, we can cut off this expansion and use only the first few terms. In this manner the problem is reduced to finding the covariant expectation values of operators of the type $\Sigma^r z^k_{\mathrm{R}}$, $\Sigma^r z^k_{\mathrm{R}} z^s_{\mathrm{R}}$, etc. In the case of Eq. (6.29), such operators appear with the second or higher derivative of the external field (at $\vec{z}_{\mathrm{R}} = 0$) as a factor. In this paper we shall neglect such terms, i.e. we will keep only $F_{\mu\nu}$ and $F_{\mu\nu,\varrho}$ (i.e. we consider only weakly varying external fields). Therefore, in this approximation Eq. (6.29) does not require to consider the operators $\Sigma^r z^k_{\mathrm{R}}$, $\Sigma^r z^k_{\mathrm{R}} z^s_{\mathrm{R}}$, etc. However, such operators have to be considered when analysing Eq. (6.26) for the spin. Another operator present on the r.h.s. of Eq. (6.29) is $\{\pi^{\mathrm{R}}_k, z^i_{\mathrm{R}}\}_+$. It appears twice: in the fourth term (after expanding f^{R}_{ki} around $\vec{z}_{\mathrm{R}} = 0$) and in the term which precedes the last term. Furthermore, Eq. (6.26) for the rest-frame spin, in addition to the above listed operators, contains also the operators

$$\Sigma^k \{\pi^{\mathrm{R}}_s, z^p_{\mathrm{R}}\}_+ \quad \text{and} \quad \Sigma^s \pi^{\mathrm{R}}_p .$$

Eq. (6.25) contains yet another operator, namely $\vec{\pi}^2_{\mathrm{R}}$.

The covariant expectation values of those operators in general cannot be reduced to the covariant expectation values of the basic operators z^i_{R}, π^{R}_k, Σ^s. Therefore, in principle they have to be considered as independent classical variables. Let us introduce for them the following notations

$$(|\Sigma^r z^k_{\mathrm{R}}) \equiv Z^{\mathrm{R}}_{rk}, \quad (|\Sigma^r \pi^{\mathrm{R}}_k) \equiv P^{\mathrm{R}}_{rk},$$
$$(|\{\pi^{\mathrm{R}}_k, z^s_{\mathrm{R}}\}_+) \equiv \hbar C^{\mathrm{R}}_{ks}, \quad (|\Sigma^r \{\pi^{\mathrm{R}}_k, z^s_{\mathrm{R}}\}_+) \equiv \hbar D^{\mathrm{R}}_{rks}. \tag{6.31}$$

A complete set of classical equations of motion has to include also equations for these variables in addition to the expected equations for the trajectory $\xi^\mu(\tau)$ and for a classical spin. Those additional equations of motion can be obtained by taking the covariant expectation values of the Heisenberg equations of motion for the corresponding operators. Such Heisenberg equations of motion are easy to obtain with the help of Eqs. (6.24)–(6.26). For example,

$$\frac{d}{d\tau}(\Sigma^s \pi^{\mathrm{R}}_p) = \frac{d\Sigma^s}{d\tau}\pi^{\mathrm{R}}_p + \Sigma^s \frac{d\pi^{\mathrm{R}}_p}{d\tau},$$

where $d\Sigma/d\tau$, $d\pi/d\tau$ are given by (6.26), (6.35) respectively. It is easy to see that these equations imply other new, independent classical variables, e.g. $(|\pi^{\mathrm{R}}_i \pi^{\mathrm{R}}_k)$. In this manner we are led to consider an infinite set of classical equations of motion which probably is just equivalent to the initial wave equation (5.20). The very essence of the idea of the classical approximation is that that infinite set of equations of motion can be approximately replaced by a set consisting of a few equations of a type occurring in classical mechanics. The use of the proper-time coordinate τ and of the covariant expectation values greatly facilitates preserving the Lorentz covariance in the process of approximating.

A detailed mathematical analysis of such an approximation, which would employ a precise, quantitative criterion for the quality of the approximation, is far beyond the scope of this paper. Such an analysis should answer many crucial questions such as, for example, for a given external field $F_{\mu\nu}$ what is the least number of classical variables one has to use in order to have a classical approximation with an error not exceeding an a priori fixed bound? In the present paper we shall be satisfied with an analysis of the consistency conditions. As we have seen, in the order m^{-1} such analysis leads to Eq. (6.20) for the trajectory $\xi^\mu(\tau)$. We think that it is a rather interesting feature of our approach to the classical limit of the Dirac equation that the equation for the classical trajectory $\xi^\mu(\tau)$ arises in this rather unusual way.

Now let us analyze the consistency condition following from (6.29). Taking the covariant expectation values of both sides of Eq. (6.29), utilizing the Taylor expansions of the type (6.30) and neglecting all terms with the second or higher derivatives of $F_{\mu\nu}$, using the fact that $(|\pi_i^{\mathrm{R}})$ is proportional to the external field $F_{\mu\nu}$ (as it follows from (6.27)), and finally utilising the notation (6.31) we obtain the following equation

$$-\ddot{\xi}^i+\frac{\ddot{\xi}_0\dot{\xi}^i}{1+\dot{\xi}_0}-\frac{e}{m}f_{i0}^{\mathrm{R}}-\frac{e\hbar}{2m^2}\varepsilon_{prs}f_{ps,i}^{\mathrm{R}}S_{\mathrm{R}}^r$$

$$-\frac{e\hbar}{2m^2}f_{ki,s}^{\mathrm{R}}C_{sk}^{\mathrm{R}}+\frac{e\hbar}{2m^2}f_{r0,0}^{\mathrm{R}}C_{ri}^{\mathrm{R}}$$

$$-\tfrac{1}{2}(g_0-2)\mu_{\mathrm{B}}\frac{1}{m}\varepsilon_{pkr}S_{\mathrm{R}}^r\tilde{F}_{pk,i}-(g_0-2)\mu_{\mathrm{B}}\frac{1}{m}\varepsilon_{kir}S_{\mathrm{R}}^r\tilde{F}_{0k,0}=0, \tag{6.32}$$

where

$$S_{\mathrm{R}}^k\equiv(\psi_{\mathrm{F-W}}^{(2)}|\Sigma^r\psi_{\mathrm{F-W}}^{(2)}) \tag{6.33}$$

will be called the rest-frame classical spin of the particle. In Eq. (6.32) all fields f^{R} and $\tilde{F}$, as well as their derivatives, are taken at the point $\vec{z}_{\mathrm{R}}=0$.

Eq. (6.32) is Newton's equation for the classical trajectory in the order m^{-2}. Expressing $f_{\alpha\beta}^{\mathrm{R}}$ and $\tilde{F}_{\alpha\beta}$ by $F_{\mu\nu}$, and dropping all the terms which are not linear with respect to $F_{\mu\nu}$, and finally passing from the rest-frame to the lab-frame description, we obtain from (6.32) the following equation ($\varepsilon_{0123}=+1$)

$$m\ddot{\xi}_\nu=eF_{\nu\mu}\dot{\xi}^\mu+\frac{e\hbar}{2m}\varepsilon_{\lambda\mu\sigma\alpha}W^\sigma\dot{\xi}^\lambda F^{\mu\nu}{}_{,\beta}(\delta_\nu^\beta-\dot{\xi}_\nu\dot{\xi}^\beta)$$

$$+\frac{e\hbar}{2m}C^{\varrho\mu}(\delta_\nu^\sigma-\dot{\xi}_\nu\dot{\xi}^\sigma)F_{\mu\sigma,\varrho}+\frac{e\hbar}{2m}C_{\nu\sigma}\dot{\xi}^\varrho\dot{\xi}_\mu F^{\mu\sigma}{}_{,\varrho}$$

$$+\tfrac{1}{2}(g_0-2)\mu_{\mathrm{B}}\varepsilon_{\sigma\mu\alpha\lambda}\dot{\xi}^\sigma W^\lambda F^{\mu\alpha}{}_{,\beta}(\delta_\nu^\beta-\dot{\xi}_\nu\dot{\xi}^\beta)+(g_0-2)\mu_{\mathrm{B}}\varepsilon^\alpha{}_{\lambda\sigma\nu}\dot{\xi}^\varrho\dot{\xi}^\mu\dot{\xi}^\sigma F_{\mu\alpha,\varrho}W^\lambda, \tag{6.34}$$

where

$$C^{\varrho\mu}\equiv(H_{\dot{\xi}})^\varrho{}_r(H_{\dot{\xi}})^\mu{}_kC_{rk}^{\mathrm{R}}, \tag{6.35}$$

and

$$W^{\lambda} \equiv (H_{\xi})^{\lambda}{}_{r} S^{r}_{R} \tag{6.36}$$

is by definition the lab-frame classical spin of the particle. Strictly speaking, (6.34) follows from (6.32) only for $\nu = 1, 2, 3$. Eq. (6.35) with $\nu = 0$ follows from these three equations because $\ddot{\xi}_0 = \xi_0^{-1}\dot{\xi}^i\ddot{\xi}^i$ (recall that $\dot{\xi}^2 = 1$).

It follows from the definition (6.31) that under Lorentz transformations C^{R} transforms by the Wigner rotations. Then, it is easy to check that $C^{\mu\nu}$, as defined by (6.35) is a Lorentz tensor. Similarly, W^{λ} is a Lorentz 4-vector. Thus, Eq. (6.34) is Lorentz covariant.

A comparison of Eq. (6.34) with classical equations which have been presented in the literature will be given in the next Section.

Differentiating both sides of (6.29) with respect to τ and applying the conditions (6.2), (6.10) we obtain further consistency conditions. However, it is not difficult to check that within the adopted approximations, i.e. linear with respect to $F_{\mu\nu}$ and up to m^{-2} in powers of m^{-1}, the equation (6.32) is sufficient in order to guarantee that those further consistency conditions are fulfilled. This is essentially due to the facts that from Eqs. (6.24)–(6.26) it follows that

$$\frac{d\Sigma}{d\tau} \sim \frac{1}{m}, \quad \frac{d}{d\tau}\pi^{R}_{k} \sim F_{\mu\nu}, \quad \frac{dz^{k}_{R}}{d\tau} \sim \frac{1}{m}.$$

Thus Eq. (6.32) is the necessary and sufficient condition in order to satisfy all the consistency conditions following from the assumption (6.2) in the order m^{-2}.

Eq. (6.34) is not self-contained even in the considered case of weakly varying external fields. It has to be associated with an equation for the lab-frame classical spin W^{λ}, and for the tensor $C^{\mu\varrho}$. It is an interesting fact that Eq. (6.20) obtained in the lowest order (m^{-1}) and in the case of weakly varying external fields is self-contained.

The equation of motion for the classical spin S^{k}_{R} follows from Eq. (6.26) by taking the covariant expectation value of both sides of it. Utilising the Taylor expansions around $\vec{z}_R = 0$ and employing the notations (6.31), (6.33) we obtain in the case of weakly varying external fields the following equation

$$\frac{dS^{i}_{R}}{d\tau} = \varepsilon_{srp}\varepsilon_{ikp}\Bigg[\frac{\dot{\xi}^{s}\ddot{\xi}^{r}}{1+\dot{\xi}_0} S^{k}_{R} + \frac{e}{2m}(f^{R}_{rs}S^{k}_{R} + f_{rs,l}Z^{R}_{kl})$$
$$+ \frac{e}{4m^2}(f^{R}_{0r}P^{R}_{ks} + \hbar\, f^{R}_{0r,t}D^{R}_{kst}) - \frac{\hbar}{2m}\left(\ddot{\xi}^{r} - \frac{\ddot{\xi}_0\dot{\xi}^{r}}{1+\dot{\xi}_0}\right)P^{R}_{ks}$$
$$-\tfrac{1}{2}(g_0-2)\frac{\mu_B}{\hbar}(S^{k}_{R}\tilde{F}_{sr} + \tilde{F}_{sr,l}Z^{R}_{kl}) + \tfrac{1}{2}(g_0-2)\frac{\mu_B}{\hbar m}(\tilde{F}_{0r}P^{R}_{ks} + \hbar\tilde{F}_{0r,l}D_{ksl})\Bigg], \tag{6.37}$$

where the fields $f_{\alpha\beta}$, $\tilde{F}_{\alpha\beta}$ and their derivatives are taken at $\vec{z}_R = 0$. Let us introduce the lab-frame counterparts of the quantities (6.31),

$$Z^{\nu\varrho} \equiv H^{\nu}{}_{s}H^{\varrho}{}_{k}Z^{R}_{sk},$$

$$P^{\nu\lambda} \equiv H^{\nu}{}_{s} H^{\lambda}{}_{k} P^{\mathrm{R}}_{sk},$$

$$D^{\nu\lambda\sigma} \equiv H^{\nu}{}_{k} H^{\lambda}{}_{s} H^{\sigma}{}_{t} D^{\mathrm{R}}_{kst}, \tag{6.38}$$

where $H \equiv H_{\dot{\xi}}$ is the boost (4.9). It is easy to see that all these quantities are perpendicular to $\dot{\xi}^{\lambda}$ (in the four-dimensional sense) with respect to each index, for instance

$$\dot{\xi}_{\nu} Z^{\nu\varrho} = \dot{\xi}_{\varrho} Z^{\nu\varrho} = 0. \tag{6.39}$$

Using the definition (6.36) and performing straightforward algebraic manipulations we obtain from (6.37) the following equation for the lab-frame classical spin W^{λ}

$$\begin{aligned}\frac{dW^{\lambda}}{d\tau} = & -\dot{\xi}^{\lambda}\ddot{\xi}_{\mu}W^{\mu} + \frac{e}{m}(\eta^{\lambda\mu} - \dot{\xi}^{\lambda}\dot{\xi}^{\mu})W^{\nu}F_{\mu\nu} + \frac{e}{m}(\eta^{\lambda\mu} - \dot{\xi}^{\lambda}\dot{\xi}^{\mu})Z^{\sigma\varrho}F_{\mu\sigma,\varrho} \\ & + \frac{e}{4m^2}(P^{\nu\lambda}\dot{\xi}^{\mu}F_{\mu\nu} + P^{\nu}{}_{\nu}\dot{\xi}^{\mu}F_{\mu}{}^{\lambda} + \hbar D^{\nu\lambda\sigma}F_{\mu\nu,\sigma}\dot{\xi}^{\mu} \\ & - \hbar D^{\nu}{}_{\nu}{}^{\sigma}\dot{\xi}^{\mu}F_{\mu}{}^{\lambda}{}_{,\sigma}) + \frac{\hbar}{2m}(\ddot{\xi}^{\lambda}P^{\nu}{}_{\nu} + \ddot{\xi}_{\nu}P^{\nu\lambda}) \\ & + (g_0 - 2)\frac{\mu_{\mathrm{B}}}{\hbar}(\eta^{\lambda\mu} - \dot{\xi}^{\lambda}\dot{\xi}^{\mu})(W^{\sigma}F_{\mu\sigma} + Z^{\nu\varrho}F_{\mu\nu,\varrho}) \\ & + \tfrac{1}{2}(g_0 - 2)\frac{\mu_{\mathrm{B}}}{\hbar m}(P^{\varrho}{}_{\varrho}\dot{\xi}^{\mu}F_{\mu}{}^{\lambda} + P^{\nu\lambda}\dot{\xi}^{\mu}F_{\mu\nu} - \hbar D^{\sigma}{}_{\sigma}{}^{\varrho}\dot{\xi}^{\mu}F_{\mu}{}^{\lambda}{}_{,\varrho} + \hbar D^{\nu\lambda\varrho}\dot{\xi}^{\mu}F_{\mu\nu,\varrho}),\end{aligned} \tag{6.40}$$

where $\eta^{\lambda\mu}$ is the Minkowski metric. It is understood that the acceleration $\ddot{\xi}_{\mu}$ is eliminated from the r.h.s. of Eq. (6.40) with the help of Eq. (6.34).

The set of the two equations (6.34) and (6.40) is not selfcontained. It has to be completed with equations for the quantities $C^{\mu\varrho}$, $Z^{\nu\varrho}$, $P^{\nu\lambda}$, $D^{\nu\lambda\sigma}$. As we have already mentioned this would lead to an infinite set of classical equations of motion. In such circumstances it is normal practice to cut off that infinite sequence of equations by making some extra assumptions. In our case such assumptions should be regarded as restrictions on the form of the wave packet. Therefore it should be checked whether the assumptions are compatible with the fact that the wave packet obeys the Dirac equation. A detailed discussion of possibilities for such assumptions is postponed till later. In the present paper we would like only to give an example of a plausible assumption of this type, without actually attempting to answer the question of its compatibility with the Dirac equation. This assumption is motivated by a consideration of wave packets in the absence of an external field, i.e. $F_{\mu\nu} = 0$. It is easy to see that in this case the wave equation (6.11) has wave-packet type solutions such that the spin decouples from $\vec{z}_{\mathrm{R}}$, i.e.

$$(|\Sigma^{r}Q) = S^{r}_{\mathrm{R}}(|Q) \tag{6.41}$$

for any observable Q which does not contain the spin operators Σ^r. For such a wave packet

$$Z^{\mathrm{R}}_{rk} = P^{\mathrm{R}}_{rk} = 0, \qquad D^{\mathrm{R}}_{rsk} = S^r_{\mathrm{R}} C^{\mathrm{R}}_{sk}. \tag{6.42}$$

From (6.42) it follows that

$$Z^{\nu\varrho} = P^{\nu\lambda} = 0, \qquad D^{\nu\lambda\sigma} = W^\nu C^{\lambda\sigma}. \tag{6.43}$$

Assuming also a spherical symmetry of the wave packet in the rest-frame we have

$$C^{\mathrm{R}}_{sk} = c_0 \delta_{sk}, \tag{6.44}$$

$$\text{i.e. } C^{\mu\varrho} = -c_0(\eta^{\mu\varrho} - \dot{\xi}^\mu \dot{\xi}^\varrho), \tag{6.45}$$

where δ_{sk} is Kronecker delta, and c_0 is a dimensionless constant. When switching on the external field we do not expect that the formulae (6.42), (6.44) are still valid, in general. However, we expect that the corrections are regular functionals of the external field $F_{\mu\nu}$, at least for some sufficiently small $F_{\mu\nu}$. Therefore the corrections would give contributions to the right hand sides of Eqs. (6.34), (6.37) which are nonlinear with respect to $F_{\mu\nu}$. Thus in the linear approximation adopted in this Section we can effectively use just (6.42), (6.44), or equivalently (6.43), (6.45). Let us write equations (6.34), (6.40) with the assumptions (6.43), (6.45) taken into account and with the substitution of (6.34) into (6.40):

$$\ddot{\xi}^\mu = \frac{e}{m} F^\mu{}_\nu \dot{\xi}^\nu - (2 - \tfrac{1}{2} g_0)\mu_{\mathrm{B}} \frac{1}{m} \varepsilon_{\lambda\nu\alpha\sigma} W^\sigma \dot{\xi}^\lambda F^{\nu\alpha,\beta}(\delta^\mu_\beta - \dot{\xi}_\beta \dot{\xi}^\mu)$$

$$- c_0 \mu_{\mathrm{B}} \frac{1}{m} (\eta^{\mu\sigma} - \dot{\xi}^\mu \dot{\xi}^\sigma) F^\varrho{}_{\sigma,\varrho} + (g_0 - 2)\mu_{\mathrm{B}} \frac{1}{m} \varepsilon_{\alpha\sigma\lambda}{}^\mu \dot{\xi}^\lambda \dot{\xi}^\nu \dot{\xi}^\varrho W^\sigma F_\nu{}^\alpha{}_{,\varrho}, \tag{6.46}$$

where

$$\mu_{\mathrm{B}} = \frac{e\hbar}{2m},$$

$$\frac{dW^\lambda}{d\tau} = \frac{eg_0}{2m} F^\lambda{}_\nu W^\nu - (g_0 - 2)\frac{e}{2m} \dot{\xi}^\lambda \dot{\xi}^\mu W^\nu F_{\mu\nu}$$

$$- c_0 \frac{e\hbar}{4m^2}(g_0 - 1)\left[(\eta^{\lambda\sigma} - \dot{\xi}^\lambda \dot{\xi}^\sigma)\dot{\xi}^\mu W^\nu F_{\mu\nu,\sigma} - \dot{\xi}^\mu W^\sigma F_\mu{}^\lambda{}_{,\sigma}\right]$$

$$+ c_0 \frac{e\hbar}{2m^2} \dot{\xi}^\lambda W^\sigma F^\varrho{}_{\sigma,\varrho} + (2 - \tfrac{1}{2} g_0)\frac{e\hbar}{2m^2} \varepsilon_{\beta\nu\alpha\sigma} \dot{\xi}^\lambda \dot{\xi}^\beta W^\sigma W_\gamma F^{\nu\alpha,\gamma}. \tag{6.47}$$

This set of equations is self-contained. However, the price for this is that this set follows from the Dirac equation only when the assumptions (6.43), (6.45) are not in contradiction with the fact that the wave packet obeys the Dirac equation (2.22) (with the anomalous magnetic moment term included).

Eqs. (6.46), (6.47), as well as Eqs. (6.34), (6.40) are compatible with the two conditions

$$\dot{\xi}^2 = 1, \quad \dot{\xi}^\mu W_\mu = 0. \tag{6.48}$$

Eq. (6.47) is compatible also with the condition

$$W_\mu W^\mu = \text{const}. \tag{6.49}$$

Eq. (6.40) is not compatible with (6.49). We comment upon this fact in the next Section.

7. *Discussion*

In this Section we would like to discuss two topics. First, we would like to present some general remarks about the classical equations of motion obtained in the preceding Section. Next we shall discuss limitations of the proper-time formulation of the Dirac equation coming from the fact that the τ, $\vec{z}$ coordinates are local.

A. First let us note that in the lowest order approximation, i.e. when keeping only the terms of the order m^{-1}, Eqs. (6.46), (6.47) reduce to the well-known B-M-T equations [11], even for nonconstant external fields $F_{\mu\nu}$. Looking at the more general equation (6.40) we see that the B-M-T equation for the classical spin W^λ in the lab-frame follows from (6.40) in the order m^{-1} and in the linear approximation if $Z^{\sigma\varrho}$ is of the order m^{-1} or higher, or if $Z^{\sigma\varrho}$ is proportional to $F_{\mu\nu}$. These two conditions for $Z^{\sigma\varrho}$ are non-trivial in the sense we cannot exclude the possibility that there exist wave-packets for which $Z^{\sigma\varrho}$ does not meet any of these conditions. Therefore, the B-M-T equation for W^λ is not universal in the context of the classical limit of the Dirac equation even in the order m^{-1}. It is valid only for a subclass of wave packets. On the other hand, Newton's equation in the order m^{-1}, i.e.

$$\ddot{\xi}^\mu = \frac{e}{m} F^\mu{}_\nu \dot{\xi}^\nu,$$

turns out to be universal in that sense.

Let us also note that in the case of constant external fields, $\partial_\mu F_{\varrho\sigma} = 0$, the general equations (6.34), (6.40) valid up to the order m^{-2}, reduce again to the B-M-T equations. In this case particular assumptions about the wave packet are not necessary. This is due to the fact that all the terms in Eqs. (6.34), (6.40) which contain $Z^{\sigma\varrho}$, $P^{\nu\lambda}$, $D^{\nu\lambda\varrho}$, $C^{\mu\sigma}$ or are of the order m^{-2}, are proportional to the derivatives of the external field.

Thus the B-M-T equations appear in the classical limit of the Dirac equation as equations which are universal for all wave packets in the case of a constant external field ($\partial_\mu F_{\varrho\sigma} = 0$), at least up to the order m^{-2} considered in this paper. Moreover, these equations appear also in the case of non-constant external fields in the order m^{-1}. However, in this last case the equations are not universal.

The terms in Eqs. (6.46), (6.47) proportional to $F_{\mu\nu,\varrho}$ differ from such terms present in other equations of motion for classical spinning particles which have been presented in the literature, see e.g. [4–10]. The general structure of these terms is identical, but coefficients and several other details are different. This is not a surprise because in most cases the equations given in the literature have been derived from entirely different starting points like for instance, an a priori assumed classical Lagrangian. In the case of derivations which, like our derivation, start from the Dirac equation (e.g. the derivation presented in [6]), the differences are due to the fact that there is no unique definition of a classical trajectory on the ground of quantum mechanics. For instance, there is freedom in choosing the position operator, as well as in defining the classical variables. Different definitions of the classical trajectory obviously lead to different classical equations of motion, in general. The classical equation obtained in the previous Section should be considered as relevant only within the framework developed in the present paper.

We would like to underline two particular differences between the classical equations of motion obtained in the present paper and equations presented in the literature. Both differences are due to the fact that in our approach classical variables are defined as the covariant expectation values of the corresponding operators. Therefore it is natural that the classical equations which we obtain might contain some parameters which are directly related to the underlying wave packet. The c_0 coefficient present in Eqs. (6.46), (6.47) is an example of such a parameter. Its value has to be calculated with the help of (6.44) (for wave packets which actually obey the assumption (6.44)). The classical equations presented in the literature do not contain such parameters.

The other difference which we would like to underline is related to the condition (6.49). This condition is equivalent to $\vec{S}_R^2 = \text{const.}$, which means that the classical spin has constant length. This condition is adopted in most derivations of classical equations of motion for spinning particles which have been presented in the literature. However, within the framework used in the present paper the condition (6.49) in general is not justified. In order to see this, let us compute $\vec{S}_R^2$ using the definition (6.33). We shall assume that $\psi^{(2)}_{F-W}$ is a normalised wave packet, i.e.

$$(\psi^{(2)}_{F-W}|\psi^{(2)}_{F-W}) \equiv \int d^3\vec{z}_R \psi^{\dagger(2)}_{F-W}(\tau, \vec{z}_R)\psi^{(2)}_{F-W}(\tau, \vec{z}_R) = 1. \tag{7.1}$$

For S_R^k we have the following explicit formula

$$S_R^k = \tfrac{1}{2}\int d^3\vec{z}_R \psi^{\dagger(2)}_{F-W}(\tau, \vec{z}_R)\sigma_k\psi^{(2)}_{F-W}(\tau, \vec{z}_R) \tag{7.2}$$

(by the assumption $\psi^{(2)}_{F-W}$ has 2 nonvanishing components). Using Schwarz's inequality it is easy to prove that from (7.1), (7.2) it follows that

$$(\vec{S}_R)^2 \leqslant \tfrac{1}{4}\,. \tag{7.3}$$

Thus in general we should not expect that $\vec{S}_R^2 = -W_\lambda W^\lambda$ is constant in τ.

The equality in (7.3) occurs only if $\psi^{(2)}_{F-W}(\tau, \vec{z})$ has the form

$$\psi^{(2)}_{F-W}(\tau, \vec{z}_R) = \begin{bmatrix} e^1(\tau) \\ e^2(\tau) \\ 0 \\ 0 \end{bmatrix} \varphi(\tau, \vec{z}_R), \tag{7.4}$$

where $\varphi(\tau, \vec{z}_R)$ is a c-number valued function. One can say that in (7.4) the spin degrees of freedom factorize from the translational degrees of freedom. Thus in the particular case of wave packets of the form (7.4) one should expect that $\vec{S}_R = -W_\lambda W^\lambda = \frac{1}{4} = \text{const.}$ It is easy to see that (7.4) implies (6.41) and, subsequently (6.42), (6.43). Thus we would expect that in the case when the assumption (6.41) is valid, the classical equation of motion (6.40) is compatible with the condition (6.49). It is easy to check that indeed this is the case.

Finally, let us recall that our classical equations of motion have been obtained in the approximation linear with respect to $F_{\mu\nu}$, and that we have neglected all higher than the first derivatives of $F_{\mu\nu}$. We have also neglected all the terms which are proportional to m^{-n}, $n > 2$.

B. Now let us turn to the second topic planned for this Section. The region of correctness of the coordinates τ, $\vec{z}$ is determined by the condition of non-singularity of the metric tensors (2.9), (2.11). This is equivalent to nonvanishing of the determinant g of the metric $g_{\alpha\beta}$ and of its inverse. Using the formula (2.10) and adopting the condition $\dot{\xi}^2 = 1$ we obtain the following restriction (for the $\vec{z}_R$ coordinates defined by (5.2))

$$1 + \left(\ddot{\xi}^r - \frac{\ddot{\xi}_0 \dot{\xi}^r}{1+\dot{\xi}_0}\right) z_R^r > 0. \tag{7.5}$$

Using Eqs. (6.17), (6.18) and neglecting the terms nonlinear with respect to $F_{\mu\nu}$ we obtain from (7.5) the following condition

$$1 + \frac{e}{m} \vec{E}^R \vec{z}_R > 0 \tag{7.6}$$

where $\vec{E}^R$ is the electric field in the instant rest-frame. The condition (7.6) is certainly satisfied if

$$\frac{e}{m} |\vec{E}^R|\,|\vec{z}_R| < 1. \tag{7.7}$$

For a well-known reason the wave packet should not be smaller than the Compton wave length $\lambda_C = \dfrac{\hbar}{m}$. In order to obtain a rough estimate we shall assume that this applies just to the $\vec{z}_R$ coordinates, i.e. that the maximal $|\vec{z}_R|$ allowed by (7.7) should be greater than λ_C. This gives a bound for the electric field

$$|\vec{E}^R| < |\vec{E}^R_C| \sim \frac{m^2 c^3}{e\hbar}, \tag{7.8}$$

where we have reintroduced the velocity of light c. The definition of $|\vec{E}^R_C|$ can be written in the form

$$e|\vec{E}^R_C|\lambda_C \sim mc^2. \tag{7.9}$$

From (7.9) we see that the allowed values of $|\vec{E}^R|$ are rather high — the work performed by the field $\vec{E}^R_C$ on the microscopic distance λ_C is equal to the rest energy of the particle.

The electric field on a Bohr orbit in a hydrogen atom is of the order $eE \sim 10^{-7}\, m^2c^3\hbar^{-1}$, where m is the mass of the electron. When passing from the lab-frame to the rest-frame the components of electric and magnetic field transverse to the velocity increase by the factor γ, while the parallel components do not change. Thus the relation (7.9) can be written in the form

$$e\gamma_C |F_{\mu\nu}^{\mathrm{Lab}}| \sim mc^2,$$

where $\gamma = (1-\vec{v}^2)^{-\frac{1}{2}}$. Even for a very fast particle, e.g. $\gamma \sim 10^5$, the allowed fields (in the lab-frame) are rather strong.

The condition (7.7) has also to be satisfied for the sake of the validity of the linear approximation. For instance, looking at the formula (6.3) for the contribution of the anomalous magnetic moment we see that the magnitude of the terms nonlinear in $F_{\mu\nu}$ relative to the linear terms is characterised just by the value of the l.h.s. of the inequality (7.7).

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REFERENCES

[1] S. I. Rubinov, J. B. Keller, *Phys. Rev.* **131**, 2789 (1963); K. Rafanelli, R. Schiller, *Phys. Rev.* **B135**, 279 (1964); R. Schiller, *Phys. Rev.* **128**, 1402 (1962).
[2] K. Huang, *Am. J. Phys.* **20**, 479 (1952).
[3] D. M. Fradkin, R. H. Good, *Rev. Mod. Phys.* **33**, 343 (1961).
[4] M. C. Corben, *Classical and Quantum Theories of Spinning Particles*, Holden-Day, Inc., 1968.
[5] I. M. Ternov, W. A. Bordovicin, *Vestnik Moscow Univ. Ser. Phys. and Astron.* **21**, 8 (1980).
[6] L. G. Suttorp, S. R. de Groot, *Nuovo Cimento* **65A**, 245 (1970).
[7] H. van Dam, Th. W. Ruijgrok, *Physica* **104A**, 281 (1980); S. Ragusa, *Phys. Rev.* **D26**, 1979 (1981); A. P. Balachandran, G. Marmo, B.-S. Skagerstam, A. Stern, *Gauge Symmetries and Fibre Bundles*, Lecture Notes in Physics, Vol. 188, Springer, Berlin 1983.
[8] A. Barducci, R. Casalbuoni, L. Lusanna, *Nucl. Phys.* **B124**, 521 (1977); J. A. de Azcárraga, J. Lukierski, *Phys. Lett.* **113B**, 170 (1982); J. Gomis, M. Novel, K. Rafanelli, *Phys. Rev.* **D34**, 1072 (1986).
[9] W. Drechsler, *Nuovo Cimento* **50A**, 405 (1979); U. Heinz, *Phys. Lett.* **144B**, 228 (1984).
[10] H. Arodź, K. Golec, *Phys. Rev.* **D33**, 534 (1986).
[11] V. Bargmann, L. Michel, V. L. Telegdi, *Phys. Rev. Lett.* **2**, 435 (1959).
[12] P. Ehrenfest, *Z. Phys.* **45**, 455 (1927).
[13] J. L. Synge, *Relativity: The General Theory*. Chapt. II, § 10, North-Holl. Publ. Company, Amsterdam 1960.
[14] The problem is explicitly mentioned in, e.g. J. Hilgevoord, S. A. Wouthuysen, *Nucl. Phys.* **40**, 1 (1963).
[15] For a short review of this topic see, e.g. S. R. de Groot, L. C. Suttorp, *Foundations of Electrodynamics*, Chapt. VIII. North-Holl. Publ. Comp., Amsterdam 1972.
[16] G. N. Fleming, *J. Math. Phys.* **7**, 1959 (1966).
[17] R. G. Littlejohn, *Phys. Rep.* **138**, 194 (1986).
[18] C. Itzykson, J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill Book Company 1980.
[19] N. N. Bogoljubov, A. A. Logunov, I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin, Inc., 1975.
[20] L. L. Foldy, S. A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).