

ERWIN SCHRÖDINGER AND THE CREATION OF WAVE MECHANICS*

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On the occasion of the 100th birthday of Erwin Schrödinger, an outline of his life and work is presented, together with an analysis of the steps that led him to wave mechanics.

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Introduction

Often one thinks of Einstein's relation $E = mc^2$ — where E and m denote the energy and mass of a system and c the velocity of light in vacuo — as *the* physical equation of the twentieth century. Nevertheless, this relation is certainly exceeded in importance by another equation, whose first form was written down in early 1926: the wave equation of Schrödinger. The Schrödinger equation describes quantitatively not only the phenomena of atomic, molecular and solid state physics; it also governs inorganic *and* organic chemistry up to biological structures. Modifications and generalisations of the original equation account for the behaviour of the smallest submicroscopic particles, of the fundamental constituents of matter, the present elementary particles and their prospective subparticles. Finally the atomic theory represented by Schrödinger's equation, i.e., wave mechanics, has brought along an epistemological extension of our concepts of microphysics. In this lecture I want to sketch the most important steps that have led to the Schrödinger equation and wave mechanics. Although this scheme is usually regarded as constituting the more *anschauliche* version of modern atomic theory, it was created *after* the more abstract formalism of matrix mechanics existed already. Although it further originated from *one* author — based on a concept of another one — this author went in rather twisted paths to arrive at wave mechanics, which I shall try to outline below.

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1. Vita of Erwin Schrödinger (1887–1961)

This brings me to the main actor of the story, to Erwin Schrödinger, whose hundredth anniversary the scientific world celebrates on 12 August of this year. The important biographical data of Schrödinger may be recalled briefly.

Schrödinger was born in Vienna on 12 August 1887, about 43 years after his fellow-citizen Ludwig Boltzmann and 13 years before Wolfgang Pauli, another compatriot. Rudolf Schrödinger, the father, had married the daughter of Alexander Bauer, his university teacher in chemistry, and ran a linoleum business (*Wachstuchmanufaktur*). His grandmother came from England, where the young Schrödinger spent several summer vacations and learned to speak English early in his life. After private tutoring at home, Erwin attended the *Öffentliches Akademisches Gymnasium* in Vienna, a humanistic gymnasium, teaching especially Latin and Greek. Schrödinger was an excellent pupil who liked mathematics and physics, but also Greek and German literature — he was an addict of the *Burgtheater* and would later write poems (both in German and English). In 1906, just after Boltzmann committed suicide, he entered the University of Vienna to study physics, mathematics, chemistry and meteorology. He did his doctoral dissertation at the *II. Physikalisches Institut*, headed by Franz Exner, the successor of Joseph Loschmidt.

Upon having performed his one-year military duty, Schrödinger joined Exner's institute as laboratory assistant in the fall of 1911. He became a *Privatdozent* in early 1914 and started to lecture there, then World War I broke out. He joined the army as artillery officer, serving at the Austrian-Italian front near Trieste. During the last years of the war, he was posted back, teaching meteorology at an officers school in Wiener Neustadt. In spring of 1920 he left Austria, becoming in fast sequence assistant to Max Wien in Jena, *Extraordinarius* of theoretical physics in Stuttgart and *Ordinarius* in Breslau/Wrocław (1921). At the end of 1921 he moved to the University of Zurich, and then in fall of 1927 to Berlin to succeed Max Planck. In 1933 he voluntarily resigned from the Berlin chair; the next years he spent at Oxford (1933–1936), and as professor at the University of Graz (1936–1938). Discharged after the annexation of Austria by the German *Reich*, Schrödinger spent a year in Belgium and was called then by Eamon de Valera, President of Eire and mathematician, to the directorship of the School of Physics at the newly founded Institute for Advanced Studies, Dublin. Retiring in 1956, he finally returned to the University of Vienna on a special chair created for him. Schrödinger died on 4 January 1961 in his home-city; he is buried in the Tyrolian village Alpbach.

2. Scientific work of Erwin Schrödinger (1910–1961) [1]

At the time he invented wave mechanics, Schrödinger was 38 years old — being in the middle of his professional life. He had started as an experimentalist — doctoral thesis on a problem of insulation, under the supervision of Egon von Schweidler — but soon also worked on theoretical physics, namely on the kinetic theory of magnetism, of dielectrics and of point-lattices (1911–1914). Especially several papers on the theory of Laue diagrams (1913–1914) and the dynamics of point lattices contributed valuable results establishing

the atomistic structure of solids. Attention might also be drawn to a theoretical paper of late 1912 on the origin of the “penetrating radiation”, which played some rôle in confirming the discovery of cosmic radiation by his senior Viennese colleague Victor Franz Hess. In the following years, interrupted by war, Schrödinger discussed — mostly in theory — several problems of statistical mechanics; e.g., his two important papers of 1918/1919 served, together with simultaneous experiments by Elisabeth Bormann, as the decisive proof of Schweidler’s fluctuations, i.e., of the statistical nature of the radioactive decay processes. Having studied Einstein’s theory of general relativity since 1916, Schrödinger wrote two notes on the theory (published 1918). He further applied the concepts of Riemannian geometry in a series of memoirs on color theory, perhaps the deepest investigation of this subject (1920).

Quantum theory entered Schrödinger’s work first through studies of the literature and review articles on the specific heats (1914–1919). Later, between 1921 and 1925, he wrote some isolated papers on atomic constitution (within the Bohr-Sommerfeld theory), and he also concerned himself with the problem of the proper (quantum) statistics of microscopic particles in a more systematic way. In the middle of the twenties, Schrödinger’s reputation rested mainly on his papers on statistical mechanics and color theory, in addition on a few learned review articles on these and similar topics. After his invention of wave mechanics, which established for him a world-wide fame and earned him the 1933 Nobel prize for physics (shared with Paul Dirac), he worked on extensions of wave mechanics and Dirac’s electron theory (since 1928), on generalized field theories, meson fields and unified field theories (beginning on the late thirties) and, of course, further on statistical problems. He also wrote essays on philosophy and theory of cognition and on biology — his book *What is Life?* of 1944 stimulated James Watson to study biology, thus starting him on the way to discover the structure of the substance of which genes are composed (Schrödinger was especially interested in the nature of inheritance).

3. Atomic and quantum theory (1900–1925)

In order to understand the rôle of the discovery of wave mechanics, we have to recall the status of atomic theory up to 1925. In this century, progress in that field was closely tied to the progress in quantum theory, which emerged in 1900 with Max Planck’s derivation of the black body radiation law and was decisively promoted by Albert Einstein from 1905 on: he introduced the concept of light quanta (1905), solved the riddle of specific heat of solids (1906) and demonstrated the dual (quantum-wave) nature of radiation (1909). In 1910 Arthur Haas proposed the first model of atomic structure involving the quantum of action; in 1913 Niels Bohr explained atomic and molecular constitution plus the radiation emitted by them on the basis of a quantum-theoretical nucleus-electron model; two years later Arnold Sommerfeld considerably extended Bohr’s theory.

The main fact is that since 1913 there existed a concept of atomic constitution, in which the atom appeared as a planetary system of electrons orbiting around the central nucleus, with the discrete orbits being determined by the Bohr-Sommerfeld quantum conditions. Certain detailed properties, such as selection rules and intensity of spectral lines,

could be obtained with the help of the correspondence principle (Bohr, 1918), whose application also allowed a quite detailed explanation of the organisation of the chemical elements in the periodic system (Bohr, 1921–1922). Still the whole model seemed to fail already in the case of the simplest more-than-two body systems (He atom, H_2^+ molecule ion, H_2 molecule, crossed field problem). Hence the so-called old quantum theory had to be replaced by a radically new scheme. Some elements of this new scheme showed up in the dispersion-theoretical approach (Ladenburg, 1921; Kramers, Born, Heisenberg, 1924), the hypothesis of matter waves (Louis de Broglie, 1923), a new statistics for the light quantum (Bose, 1924) plus its application to the theory of ideal gases (Einstein, 1924–1925), and a rearrangement of quantum numbers in the atom (Stoner, 1924) leading to the exclusion principle (Pauli 1925). Finally, Werner Heisenberg formulated in early summer of 1925 the foundations of quantum mechanics, which were developed during the next half year by Max Born, Dirac, Heisenberg and Pascual Jordan into systematical mathematical theories (matrix mechanics, q -number algebra).

Up to 1925 Schrödinger participated in this development only from time to time. So he accepted in 1913 the theory of the specific heats of solids as proposed the previous year by Born and Theodore von Kármán; in 1919 he suggested an experiment to decide about the nature of light (directed light quantum or wave). For the proper description of the energy states of non-hydrogen-like atoms, he invented the idea of diving orbits — an excentric external electron penetrates into the inner shell of electrons (1921). In 1922 he discovered a “notable property” of the Bohr-Sommerfeld electron orbits: they seemed to follow from a special application of the “distance (phase) factor” introduced by Hermann Weyl in his extension of general relativity and electrodynamics (based on affine geometry). During the next three years Schrödinger took part in the discussion of the Bohr-Kramers-Slater theory of radiation (1924, agreeing with the possibility of a only statistical conservation of energy and momentum in atomic processes) and contributed one paper on hydrogen-like atoms (1925).

4. Statistical mechanics, especially the statistics of identical particles (1915–1925)

Problems of statistical theory always interested Schrödinger, who considered himself a student and heir of Ludwig Boltzmann, Fritz Hasenöhr and Marian von Smoluchowski. He composed the first four papers dealing with various applications of classical statistical methods during World War I (on Brownian motion of the particles in the Millikan-Ehrenhaft experiment, 1915; ordering of chance sequences, 1918; theory of an experiment providing the first proof of radioactive fluctuations, 1918–1919). In the same period, probably after the death of Smoluchowski in 1917, he wrote extensive notes analysing in detail the late papers of this author (published since 1912) on fluctuation phenomena. Throughout his life Schrödinger’s views oscillated between two extremes: the laws of nature operated in a way that was either strictly causal or strictly accidental. The latter view had been expressed by his teacher Franz Exner in his inaugural address of *Rector* of the Vienna University in 1908 [2] and repeated in some detail in his book *Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften*, which appeared in print in 1919. Schrödinger

expounded the statistical credo “that chance is the common root of all rigid conformity to law” in his Zurich inaugural address of 1922 (published 1929), and again in the inaugural address which he presented seven years later to the Prussian Academy of Sciences [3].

In the first half of the 1920’s Schrödinger’s concern with statistical problems deepened. Besides publishing several papers on different topics, such as the Gibbs paradox (1921), the specific heats of solids and the hydrogen molecule (1924), gas degeneracy (1922, 1924) and equilibrium between light and acoustic rays (1924), he composed in this period several extended manuscripts containing outlines and parts of the text of books or review articles on “molecular statistics” and “quantum statistics”.

More than on any other problem Schrödinger’s attention focussed on the particular question as to which statistical method of counting had to be applied to an assembly of identical particles. This question coincided, since Ludwig Boltzmann’s proposal of 1872, with the statistical entropy definition for an ideal gas of N identical molecules (or even electrons), Boltzmann’s entropy principle (in the form given to it by Max Planck),

$$\text{entropy } S = k \log W \quad (1)$$

— with the probability W being obtained (by maximation) from the number of ways P to distribute N particles among different states 1, 2, 3, ...,

$$P = \frac{N!}{n_1!n_2!n_3!\dots} \quad (2)$$

It entered crucially Planck’s derivation of the law of black-body radiation in 1900. Since 1916 Planck worked on a quantum theory of the ideal gas, in which the expression obtained for the probability had to be divided by $N!$. He argued, in support of this division, that identical particles (such as the molecules of a pure gas) could not be distinguished physically [4]. In contrast to Planck, Paul Ehrenfest rejected the division by $N!$; he even declared Planck’s Eq. (1) to be an artificial restriction of Boltzmann’s original proposal (and the whole of Planck’s procedure in obtaining the black-body law) as arbitrary [5].

In 1924 Satyendra Nath Bose suggested a new derivation of Planck’s law, based upon a new counting method of light quanta [6]; Albert Einstein immediately extended Bose’s method to develop a quantum theory of the ideal gas [7]. Schrödinger studied the literature on the ideal gas problem in detail, especially the papers of Planck and Einstein. From July to December 1925 he submitted three papers, in which he attempted to derive essentially Einstein’s gas theory. While he first tried to introduce suitable counting methods [8], or to treat the gas as a big quantum system [9], he finally turned — on a hint in Einstein’s second paper — to consider the ideal gas as a system of standing matter wave à la de Broglie [10]. At that time he had already entered the route to wave mechanics.

5. Origin of wave mechanics (middle to late 1925)

Since fall of 1924, Schrödinger — who felt quite isolated in Zurich — got into an increasingly lively scientific exchange with Albert Einstein and Max Planck from Berlin and Arnold Sommerfeld and Willy Wien from Munich, all of whom he had met at the

Innsbruck *Naturforscherversammlung* (September 1924), resulting in a frequent correspondence as well as in a growing number of publications on various problems of atomic and statistical theory. Following a strenuous summer semester 1925, he used his vacations in the Swiss mountains (Arosa) partly to compose an essay based on earlier notes on psychological and philosophical questions (published later as "*Suche nach dem Weg*" in *Meine Weltansicht* [11]). Then he prepared his lecture course on the theory of atomic and molecular spectra (to be given in the winter semester), for which he heavily used the last (1924) edition of Sommerfeld's *Atombau und Spektrallinien*. Soon after the return to Zurich, before the beginning of the semester, he obtained a copy of Louis de Broglie's doctoral dissertation [12].

Einstein had referred to this thesis in his second paper on gas statistics. In particular, he had quoted there de Broglie's equations for the phase velocity u of a particle (mass m , velocity v) and its frequency ν ,

$$u = \frac{c^2}{v}, \quad (3)$$

and

$$\nu = \frac{mc^2/h}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4)$$

and had pointed to de Broglie's interesting interpretation of the Bohr-Sommerfeld quantum conditions from standing electron phase waves in Bohr orbits [13]. Schrödinger found that this interpretation was related to his own discussion of electron orbits in atoms of 1922 ([14], see above), and he immediately tried to extend de Broglie's geometrical phase-wave construction (case of circular orbits) to general elliptic orbits and to the Stark and Zeeman effect cases; however, he did not get very far with it. In the discussion of Schrödinger's Zurich colloquium talk on de Broglie's theory — probably given in late November 1925 — Debye asked for the wave equation of the phase waves. Such an equation had not been written down by de Broglie, who considered the phase or matter wave just as an auxiliary effect accompanying the particle motion.

Schrödinger quickly developed the first example of a wave equation for the case of the electron wave in the hydrogen atom in a manuscript entitled "*H-Atom. Eigenschwingungen*" (H-atom. Eigenvibrations). In this manuscript of three pages he started from an extension of de Broglie's equation (4), i.e.

$$h\nu = \frac{mc^2}{\sqrt{1 - \beta^2}} - \frac{e^2}{r} \quad (5)$$

and combined it with Eq. (3), or,

$$u = \frac{h\nu}{mv\sqrt{1 - \beta^2}}, \quad (6)$$

to eliminate $\beta (= v/c)$. Then he inserted the expression for u into the standard wave equation,

$$\Delta\psi = -\frac{4\pi^2\nu^2}{u^2}\psi, \quad (7)$$

to obtain the *relativistic* equation

$$\Delta\psi + \frac{4\pi^2m^2c^2}{h^2} \left[\left(\frac{h\nu}{mc^2} + \frac{e^2}{mc^2r} \right)^2 - 1 \right] \psi = 0, \quad (8)$$

which should describe the behaviour of the matter wave associated with the electron in the hydrogen atom.

The solution sketched on the few sheets contained the two steps:

- (i) Introduction of spherical harmonics to separate the angular dependence (in ϕ and ϑ).
- (ii) The resulting radial equation,

$$\frac{d^2\phi_r}{dr^2} + \frac{2}{r} \frac{d\psi_r}{dr} + \left[-A + \frac{2B}{r} - \frac{C}{r^2} \right] \psi_r = 0, \quad (9)$$

possessed a unique finite solution only if

$$\frac{B}{\sqrt{A}} - \sqrt{C + \frac{1}{4}} + \frac{1}{2} = \text{integer}, \quad (10)$$

as Schrödinger learned from Ludwig Schlesinger's book on integral equations [15]. Now C could be written as $(k + \frac{1}{2})^2$, with k an integer. On the other hand, Eq. (10) resembled closely an expression written down by Sommerfeld in 1916 to describe the relativistic fine structure of hydrogen [16], i.e.,

$$\frac{B}{\sqrt{A}} - \sqrt{k^2 + \frac{1}{4}} + \frac{1}{2} = \text{integer}. \quad (11)$$

As a consequence, the new result of Schrödinger in which k^2 was replaced by $(k + \frac{1}{2})^2$ would not describe the known data, and Schrödinger felt that his first approach to use an atomic wave equation had failed.

Fortunately at about the same time — i.e., in the middle of December 1925 — he scored a success in applying the phase wave idea to the ideal gas [10]; hence he soon returned to the wave equation of the hydrogen atom.

6. Nonrelativistic Schrödinger equation for the hydrogen problem (December 1925–January 1926)

From May 1925 Willy Wien and Erwin Schrödinger exchanged many letters on various questions. The correspondence started with the request of Wien to Schrödinger to write an article on physiological optics for the *Handbuch der Experimentalphysik*, edited by Wien and F. Harms. Although Schrödinger finally did not fulfill Wien's request, their correspond-

ence provided ample opportunity to discuss many important problems then existing, e.g., the repetition of the Michelson-Morley experiment (on the velocity of the ether — the American Dayton Miller seemed to have discovered a positive effect) and the nature of radiation. In his letter of 24 December 1924, Wien expressed a particular wish: “Since long I have not obtained from Zurich anything for the *Annalen der Physik* [of which he was also an editor]. I would be happy if I could soon publish again several Swiss papers”.

The letter did not reach Schrödinger immediately, because he had gone for Christmas vacation to Arosa. From there he reported to Wien on 27 December: “At the moment I am plagued by a new atomic theory ... I believe that I can construct a vibrating system — in relatively natural manner, not by *ad hoc* assumptions — which yields the hydrogen-term frequencies as eigenfrequencies”. After his return to Zurich, and reading Wien’s letter of 24 December, he promised to submit the paper in progress to the *Annalen*. He worked out the details of the solution of the *nonrelativistic* hydrogen atom and finished the paper, “*Quantisierung als Eigenwertproblem. (Erste Mitteilung)*”, such that it was received by Wien on 27 January 1926 [17].

The path to the first communication on the eigenvalue problem may be reconstructed in detail from a notebook of Schrödinger — entitled “*Eigenwertproblem des Atoms. I*” (Eigenvalue problem of the atom I) — which he began to write in Arosa and continued after his return (around 7 January 1926) to Zurich. In Arosa he achieved the following goals:

- (a) establishing the wave equation for the nonrelativistic hydrogen atom;
- (b) equation for the Stark effect and preliminary treatment;
- (c) detailed solution of the hydrogen equation with the help of Schlesinger’s book: discrete energy solutions creating the correct Balmer spectrum; continuous solutions; structure of the radial eigenfunctions in the case of discrete energy solutions.

After the Christmas vacation Schrödinger worked out essentially mathematical details in Zurich (partly assisted by Hermann Weyl, the mathematician at the ETH); he added also the derivation of the wave equation from a variation principle, probably taken over from his student-time notes of the Vienna theoretical seminar. (A notebook of 1909/10 is kept in the Schrödinger *Nachlass* — it also contains the separation of electromagnetic radiation problems with the help of spherical coordinates, as used by Schrödinger in solving both the relativistic and nonrelativistic hydrogen equations.)

The first communication starts from the Hamilton-Jacobi equation of classical dynamics,

$$H\left(q, \frac{\partial S}{\partial q}\right) = E, \quad (12)$$

with H denoting the Hamiltonian as a function of the position coordinates q and the momentum coordinates, written as the partial derivatives of the action function S . Schrödinger then suggested replacing the classical action function in Eq. (12) via the transformation

$$S = K \log \psi, \quad (13)$$

with K being a constant (having the dimension of an action) and ψ a function of the position variables. He thus arrived at the equation

$$H\left(q, \frac{K}{\psi} \frac{\partial \psi}{\partial q}\right) = E, \quad (14)$$

which in the case of the nonrelativistic hydrogen system — one electron (mass m , charge $-e$) moving in a Coulomb potential of a proton (charge $+e$), described by the classical Hamilton-Jacobi equation

$$-\frac{e^2}{r} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] = E \quad (15)$$

— yielded the equation

$$\left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 - \frac{2m}{K^2} \left(E + \frac{e^2}{r} \right) \psi \right] = 0. \quad (16)$$

Schrödinger did not use Eq. (16) as the equation determining the unknown function ψ , but rather inserted the square bracket expression on its left-hand side into the variation problem

$$\delta J = \delta \int dx dy dz [\quad] = 0. \quad (17)$$

The Euler-Lagrange equation of this variation problem finally represented the Schrödinger equation for the wave function ψ of non-relativistic hydrogen atom, i.e.,

$$\Delta \psi + \frac{2m}{K^2} \left(E + \frac{e^2}{r} \right) \psi = 0, \quad (18)$$

provided the constant K took on the value (h Planck's constant)

$$K = \frac{h}{2\pi}. \quad (19)$$

The solutions of the second order differential equation (18), which were everywhere finite and unique, separated into two classes:

(a) for negative "eigenvalues" E only discrete solutions existed, with E assuming the values

$$E = - \frac{2\pi^2 m e^4}{h^2 n^2}, \quad n = 1, 2, 3, \dots; \quad (20)$$

(b) all positive values $E > 0$ allowed a consistent solution of Eq. (18).

7. *Foundation of wave mechanics by the mechanical-optical analogy*
(January-February 1926)

Schrödinger did not sit back after achieving these results. He wanted in particular to test his new undulatory description of atomic systems in other examples, knowing that his transformation trick (13) did not work in general. Rather he had to dive into a deeper formulation of the whole approach, which he developed in the second communication submitted only four weeks after the first one [18].

The deeper foundation was provided by what he called “mechanical-optical analogy”, which again started from the Hamilton-Jacobi equation — used, however, in a form slightly different from Eq. (12), i.e.,

$$2\mathcal{T}\left(q, \frac{\partial S}{\partial q}\right) = 2(E - V). \quad (21)$$

\mathcal{T} denotes the kinetic energy of the system, given as a function of position (q) and momentum variables $\left(\frac{\partial S}{\partial q}\right)$, E and V are its total and potential energies, respectively. Schrödinger then considered the dynamical system within the mechanics of Heinrich Hertz. The latter had proposed, before the turn of the century, to abolish the concept of force altogether and to replace it by constraint conditions, which make systems (particles) move in general on curved orbits in a many-dimensional Riemannian space [19]. The line element in the space — whose dimensions coincided with the number of degrees of freedom of the system under investigation — may be written in terms of the position variables and their time-derivatives as

$$ds^2 = \sum_{i,k=1}^4 dq_i dq_k = 2\mathcal{T}(q_k, \dot{q}_k) dt^2. \quad (22)$$

Schrödinger now assumed a relation to exist between the trajectory of a particle in dynamics and the propagation of wave fronts in classical optics. If the wave fronts are given in the equation

$$S = \text{const.}, \quad (23)$$

with a gradient

$$|\text{grad } S| = \sqrt{2(E - V)}, \quad (24)$$

then they propagate with a phase velocity u ,

$$u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E - V)}}, \quad (25)$$

and an automatic identity between Fermat’s optical and Maupertuis’ mechanical principles can be established, since obviously

$$O = \delta \int_{P_1}^{P_2} \frac{ds}{u} = \delta \int_{P_1}^{P_2} \frac{ds \sqrt{2(E - V)}}{E} = \delta \int_{\text{(Maupertuis)}} \frac{2\mathcal{T}}{E} dt. \quad (26)$$

(Fermat)

However, the wave equation describing the motion of the wave front, i.e.,

$$\operatorname{div} \operatorname{grad} \psi + \frac{1}{u^2} \ddot{\psi} = 0, \quad (27)$$

in the Riemannian space à la Hertz also turns out to be identical with the Schrödinger equation, provided the time-dependence of the wave function has the simple form (with period ν^{-1})

$$\psi = \exp(2\pi i \nu t) \psi_q. \quad (28)$$

That is, from the “mechanical-optical analogy” thus the generalized wave-mechanical equation can be “derived”, namely

$$\operatorname{div} \operatorname{grad} \psi_q + \frac{8\pi^2}{h^2} (E - V) \psi_q = 0. \quad (29)$$

Schrödinger applied Eq. (29) directly to various special problems, such as the harmonic oscillator, the rotator, or a model for diatomic molecules, with the differential operator $\operatorname{div} \operatorname{grad}$ being defined in the appropriate Riemannian space, whose line-element has been attained from the (classical) expression for the kinetic energy of the system according to Eq. (22). (See Section 3 of paper [18].)

8. A further “derivation” of the general wave equation and the time-dependent Schrödinger equation

The above outlined foundation of the wave equation (from Hertz’s mechanics and Schrödinger’s mechanical-optical analogy) may look a little far-fetched, and different suggestions have been given in the literature how one could “derive” the wave equation in a simpler way [20]. However, one should notice two facts in this context. On the one hand, a simple mechanical-ray-optical analogy does not cover the desired result — the wavelength of electrons in atoms is comparable with the atomic dimensions. On the other hand, Schrödinger had available the entire formalism used in Sections 1 and 2 of his second communication already since 8 years and had written it down in detailed notebooks, entitled “*Hertz’sche Mechanik und Einstein’sche Gravitationstheorie*” and “*Tensoranalytische Mechanik I, II, III*” (all composed around 1918 when he showed particular interest in Einstein’s general relativity theory). Now in early 1926 he had just to copy the relevant considerations and formulae from his old notebooks (which have been fortunately kept in his *Nachlass*).

Still he provided another less learned approach to obtain the appropriate Schrödinger equation for a given mechanical system. In his next paper “*Über das Verhältnis der Heisenberg-Born-Jordanschen Quantenmechanik zu der meinen*” (On the relation of the Heisenberg-Born-Jordan quantum mechanics to my own), which was received by the *Annalen* on 17 March 1926 [21], he introduced the following rule translating a mechanical system into one of undulatory or wave mechanics: Be the system under consideration given by its

classical Hamiltonian, expressed in terms of position (q) and momentum variables (p), $H(p, q)$; then it will be described in undulatory mechanics by the wave equation

$$H\left(q, K' \frac{\partial}{\partial q}\right) \psi = E\psi, \quad (30)$$

with

$$K' = \frac{h}{2\pi i}. \quad (31)$$

That is, the operator H acting on the wave function ψ is obtained from $H(q, p)$ by replacing each position variable q_k by the operator q_k and each momentum variable p_k by K' times the differential operator $\partial/\partial q_k$.

Evidently Eq. (30) is not the most general wave equation, as it does not depend on time and hence will describe only periodic systems (see Eq. (28)) [22]. In the fourth communication on the quantization problem, however, Schrödinger further extended his approach to cover also systems having general time dependence; thus he arrived at the time-dependent Schrödinger equation [23], i.e.,

$$\Delta\psi - \frac{8\pi}{h^2} V\psi \mp \frac{4\pi i}{h} \frac{\partial\psi}{\partial t} = 0. \quad (32)$$

While this equation allowed him to treat in particular the dispersion of light by atoms, he had to use a different wave equation for describing relativistic systems (Section 7 of the fourth communication). This equation agreed in the special case with the relativistic hydrogen equation (8), from which he had started the whole wave mechanics in late 1925; but he now wrote it down in the time-dependent form including, besides the electric potential V , also the magnetic vector potential \vec{A} , hence

$$\begin{aligned} \Delta\psi - \frac{1}{c^2} \frac{\partial^2\psi}{\partial t^2} \mp \frac{4\pi ie}{hc} \left(\frac{V}{c} \frac{\partial\psi}{\partial t} + \vec{A} \cdot \text{grad } \psi \right) \\ + \frac{4\pi^2 e^2}{h^2 c^2} \left(V^2 - \vec{A}^2 - \frac{m^2 c^4}{e^2} \right) \psi = 0. \end{aligned} \quad (33)$$

Equation (33) is normally named after Oskar Klein and Walter Gordon, who published it a little earlier and afterwards, respectively [24, 25]. But simultaneously many other authors hit on it, including Vladimir Fock, Théodore de Donder and Frans van den Dungen, Louis de Broglie, and Janós Kudar. The so-called Klein-Gordon equation with the “many fathers” constituted the first relativistic equation describing particles with spin zero and thus opened an area in the future of subnuclear physics.

Schrödinger’s memoirs, in which he founded, applied and generalized the wave mechanical description of atoms, soon extended the range of application of quantum theory. Decades later another great theoretician, Max Born, who could well compete with Schrödinger in contributing to atomic physics, judged competently: “What more magnificent exists in theoretical physics than his [Schrödinger’s] first six papers on wave mechanics” [26].

The story has been treated in detail in J. Mehra and H. Rechenberg: *The Historical Development of Quantum Theory*, Volume 5: *Erwin Schrödinger and the Rise of Wave Mechanics*, Springer-Verlag: New York, Berlin, Heidelberg, London and Tokyo, 1987.

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