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ON HARMONIC SPINORS*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

We study the question to what extend classical Hodge–deRham theory for harmonic differential forms carries over to harmonic spinors. Despite some special phenomena in very low dimensions and despite the Atiyah–Singer index theorem which provides a link between harmonic spinors and the topology of the underlying manifold it turns out that in many dimensions harmonic spinors are not topologically obstructed. In this respect harmonic spinors behave very differently from harmonic differential forms. We also discuss parallel spinors and Killing spinors.

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1. Introduction

Let M be a closed n-dimensional differential manifold. Here "closed" means compact, connected, and without boundary. Let $\Omega^p(M)$ denote the space of smooth p-forms on M, $p = 0, \dots, n$. Then the *exterior derivative* d maps $\Omega^p(M)$ into $\Omega^{p+1}(M)$. If M is equipped with a (positive definite) Riemannian metric g, then we can form its L^2-adjoint $\delta_g : \Omega^{p+1}(M) \to \Omega^p(M)$, characterised by $(d\omega, \eta)_{L^2(M,g)} = (\omega, \delta_g \eta)_{L^2(M,g)}$, $\omega \in \Omega^p(M)$, $\eta \in \Omega^{p+1}(M)$. The subindex g indicates that δ_g depends on the Riemannian metric while d does not.

The *Laplace–Beltrami operator* $\Delta_g = (d + \delta_g)^2 = d\delta_g + \delta_g d$ respects the degree p of forms. It is an elliptic second order differential operator with nonnegative discrete spectrum. The eigenvalues of Δ_g depend on the Riemannian metric g in a complicated manner, except for the eigenvalue 0. Namely, one has

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Hodge–deRham Theory:

$$b^p(M) = \dim\left(\ker(\Delta_g|_{\Omega^p(M)})\right)$$

is a topological invariant, the p^{th} Betti number.

This is an amazing fact. Harmonic differential forms (solutions of $\Delta_g\omega=0$) can be counted topologically!

There are other important fields on a manifold like spinors which describe fermions in quantum mechanics. Is there a similar strong link between harmonic spinors and the topology of the underlying manifold?

To be more specific, let $\mathfrak{s}$ be a *spin structure* on M and let $\Sigma M \to M$ be the corresponding complex *spinor bundle*. See *e.g.* [22] for a definition of these concepts. At each point $p \in M$ there is the *Clifford multiplication map* from the tangent space to linear maps of $\Sigma_p M$ to itself

$$\gamma : T_pM \to \mathrm{End}(\Sigma_p M)$$

satisfying the relation

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X,Y)\cdot \mathrm{Id}$$

for all $X, Y \in T_pM$. The Levi–Civita connection ∇ induces canonical connection on spinors, again denoted ∇. The *Dirac operator* is defined by

$$D = D_{g,\mathfrak{S}} = \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j}\,,$$

where $e_1, \dots, e_n$ is a local orthonormal basis of TM.

Example. Let $M = \mathbb{R}^2$ with the Euclidean metric g. Then spinors are simply maps $\mathbb{R}^2 \to \mathbb{C}^2$ and Clifford multiplication is given by the Pauli matrices

$$\gamma(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \gamma(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Dirac operator is

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\partial}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \cdot \frac{\partial}{\partial y}.$$

We call a spinor φ *harmonic* if it satisfies the Dirac equation without potential,

$$D\varphi = 0.$$

To see the analogy to harmonic differential forms note that $D\varphi = 0$ is equivalent to $D^2\varphi = 0$. Namely, $D\varphi = 0$ obviously implies $D^2\varphi = 0$. Conversely, if $D^2\varphi = 0$ take the L^2-product with φ and compute

$$0 = (D^2\varphi, \varphi)_{L^2} = (D\varphi, D\varphi)_{L^2}.$$

This partial integration is possible because D is formally self-adjoint and M is closed. One concludes $D\varphi = 0$.

Now the operator D^2 is an elliptic second order differential operator completely analogous to $\Delta = (d+\delta)^2$.

Question. *Is there topological information contained in*

$$h(M, g, \mathfrak{S}) := \dim (D_{g,\mathfrak{S}}),$$

does $h(M, g, \mathfrak{S})$ really depend on g and/or $\mathfrak{S}$?

It is not hard to see [19] that h is a conformal invariant, *i.e.* if $g' = u \cdot g$ for some positive function u, then

$$h(M, g, \mathfrak{S}) = h(M, g', \mathfrak{S}).$$

2. Parallel spinors

Before we study harmonic spinors let us first look at a stronger equation. A spinor is called *parallel* if for all $X \in TM$

$$\nabla_X \varphi = 0.$$

This is a linear overdetermined elliptic equation. Of course, every parallel spinor is harmonic. In the "general case" this equation has no nonzero solutions. Existence of parallel fields can always be characterized by a reduction of the holonomy group $\mathrm{Hol}\,(M, g)$. A generic Riemannian spin manifold has holonomy group $\mathrm{Hol}\,(M, g) = \mathrm{SO}\,(n)$ and no parallel spinors. Going through the list of possible holonomy groups of (irreducible, simply connected) Riemannian spin manifolds one obtains the following table [19, 33]:

TABLE I

$n = \dim (M)$	$\mathrm{Hol}\,(M, g)$	geometric condition	N
$2m,\quad m \geq 2$	$\mathrm{SU}\,(m)$	Kähler, Ricci-flat	2
$4m,\quad m \geq 2$	$\mathrm{Sp}\,(m)$	hyperkähler	$m+1$
8	$\mathrm{Spin}\,(7)$		1
7	G_2		2

Here N denotes the dimension of the space of parallel spinors. For example, every $(4m)$-dimensional hyperkähler manifold has $m+1$ linearly independent parallel spinors.

The irreducibility condition is technical. It means that a general simply connected Riemannian manifold is a product of irreducible ones and the various combinations of the holonomy groups in Table I can occur. For the nonsimply connected case see [34].

3. Killing spinors

The concept of parallel spinors generalizes as follows. Let $\alpha \in \mathbb{C}$. A spinor φ is called a *Killing spinor* with Killing constant α if for all $X \in TM$

$$\nabla_X \varphi = \alpha \cdot \gamma(X)\varphi.$$

Again this is an overdetermined elliptic equation. Killing spinors were first introduced by Penrose [32] in General Relativity. They are important in Supergravity theories [9]. If $\alpha = 0$, then we are back to parallel spinors. It can be shown that on a closed manifold M only real α can occur. If $\alpha \in \mathbb{R} - \{0\}$ then by appropriately rescaling the metric we can assume $\alpha = \pm\frac{1}{2}$. Let $N^{\pm}$ denote the dimension of the space of Killing spinors with Killing constant $\pm\frac{1}{2}$.

The essential idea to characterize manifolds with Killing spinors geometrically and to compute $N^{\pm}$ is the following.

Let CM be the cone over M, *i.e.* $CM = M \times \mathbb{R}_+$ with the Riemannian metric $g_{CM} = dt^2 + t^2 g$, $t \in \mathbb{R}_+$. The crucial observation is that Killing spinors on M correspond to parallel spinors on CM. Applying the results from the previous section to CM and translating the geometric conditions on CM into conditions on M yields the following possibilities for simply connected M with Killing spinors [3]:

TABLE II

$n = \dim(M)$	$\mathrm{Hol}(CM, g_{CM})$	geometry of M	(N^+, N^-)
arbitrary	$\{1\}$	S^n	$(2^{[n/2]}, 2^{[n/2]})$
$2m-1$	$\mathrm{SU}(m)$	Einstein–Sasaki	$(1,1)$ or $(2,0)$
$4m-1$	$\mathrm{Sp}(m)$	3-Sasaki	$(m+1, 0)$
7	$\mathrm{Spin}(7)$	nearly parallel vector cross product	$(1,0)$
6	G_2	nearly Kähler, nonkähler	$(1,1)$

In particular, if the dimension n is even but $n \neq 6$, then the standard sphere S^n is the only simply connected manifold with Killing spinors. In dimension $n = 4$ and $n = 8$ this had been known before, see [8] and [18]. For other previously known special cases of the classification, obtained by different methods, see [11–13, 17].

4. The Atiyah–Singer index theorem

We have seen that parallel spinors or, more generally, Killing spinors exist only on very special Riemannian manifolds. A slight perturbation of the metric will immediately destroy their existence. This is not too surprising since they are defined by overdetermined equations. Let us therefore return to harmonic spinors.

If $n = \dim(M)$ is even, then the spinor bundle splits into the so-called half-spinor (or Weyl spinor) bundles, $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. The Dirac operator maps positive half-spinors into negative ones and vice versa. Thus

$$h(M, g, \mathfrak{S}) = h^+(M, g, \mathfrak{S}) + h^-(M, g, \mathfrak{S}),$$

where $h^\pm(M, g, \mathfrak{S})$ is the dimension of the space of harmonic positive/negative half-spinors.

Atiyah–Singer Index Theorem [1].
The number

$$h^+(M, g, \mathfrak{S}) - h^-(M, g, \mathfrak{S}) = \hat{A}(M)$$

is a topological invariant (independent of g and $\mathfrak{S}$), the $\hat{A}$-genus of M.

Corollary. $h(M, g, \mathfrak{S}) \geq |\hat{A}(M)|$, *i.e. there is a topological lower bound for* $h(M, g, \mathfrak{S})$.

This gives a nontrivial estimate only if n is divisible by 4 since otherwise $\hat{A}(M) = 0$. In certain dimensions there is a refinement of the index theorem [2] using Milnor's α-genus:

$$\begin{aligned} &\text{If } n \equiv 1 \bmod 8: \quad h(M, g, \mathfrak{S}) \equiv \alpha(M, \mathfrak{S}) \bmod 2\,, \\ &\text{If } n \equiv 2 \bmod 8: \quad h^+(M, g, \mathfrak{S}) \equiv \alpha(M, \mathfrak{S}) \bmod 2\,. \end{aligned}$$

The index theorem in its various versions establishes a certain link between $h(M, g, \mathfrak{S})$ and the topology of the underlying manifold. The analog for differential forms is the formula

$$\sum_{p=0}^{n} (-1)^p \dim(\ker \Delta_g|_{\Omega^p(M)}) = \sum_{p=0}^{n} (-1)^p b^p(M) = \chi(M)$$

for the Euler–Poincaré characteristic of M. But we know that more is true, the individual $b^p(M)$ are topological.
What can we say about $h(M, g, \mathfrak{S}) = h^+(M, g, \mathfrak{S}) + h^-(M, g, \mathfrak{S})$?

5. The 1-dimensional case

To start let us look at the simplest case. There is only one closed 1-dimensional manifold, the 1-sphere S^1. Its metric is determined by its length $L > 0$, $S^1 = \mathbb{R}/L \cdot \mathbb{Z}$. There are two spin structures on S^1, $\mathfrak{S}_1$ and $\mathfrak{S}_2$. For $\mathfrak{S}_1$ spinors correspond to periodic functions $\varphi : \mathbb{R} \to \mathbb{C}$, $\varphi(t+L) = \varphi(t)$. For $\mathfrak{S}_2$ they correspond to antiperiodic functions $\varphi : \mathbb{R} \to \mathbb{C}$, $\varphi(t+L) = -\varphi(t)$. In both cases the Dirac operator is $D = i\frac{d}{dt}$. Obviously, the kernel of D consists of constant functions which are admissible only for $\mathfrak{S}_1$. Thus

$$\begin{aligned} h(S^1, g, \mathfrak{S}_1) &= 1, \\ h(S^1, g, \mathfrak{S}_2) &= 0. \end{aligned}$$

This example already shows that h depends in general on the spin structure.

6. Surfaces

Let us turn to two dimensions. The situation depends a lot on the genus of the surface.

Case 1: genus = 0.

Theorem [4]. *Let M be a closed surface of genus 0. Then all eigenvalues λ of the Dirac operator on M satisfy*

$$\lambda^2 \geq \frac{4\pi}{\mathrm{area}(M, g)}.$$

In particular, $h(M, g, \mathfrak{S}) = 0$, so again h does not depend on the metric. This also follows from conformal invariance of h and the fact that all metrics on S^2 are conformally equivalent.

Case 2: genus = 1.

The 2-torus has four different spin structures, one of which, $\mathfrak{S}_{\mathrm{tr}}$, is in a sense trivial (biinvariant). Every metric on a 2-torus is conformally equivalent to a flat metric, hence it suffices to consider flat metrics. For flat metrics the eigenvalues of the Dirac operator are easily computed [10]. It follows from this that for *any* metric

$$h\left(T^2, g, \mathfrak{S}\right) = \begin{cases} 2, & \text{if } \mathfrak{S} = \mathfrak{S}_{\mathrm{tr}} \\ 0, & \text{otherwise.} \end{cases}$$

Again, h does not depend on g.

Case 3: genus = 2.

This case turns out to be similar to the torus case, h depends on the spin structure but not on the metric [19].

Case 4: genus > 2.

It turns out that in general h *does* depend on both, the spin structure and the metric [19]. Despite the promising first results in dimension one and for genus ≤ 2 we get disappointed at this stage. It turns out that harmonic spinors are less closely related to the topology of the manifold than harmonic differential forms.

7. Berger spheres

To get an idea of what to expect in higher dimensions let us look at a family of Riemannian metrics on spheres constructed as follows.

Let $S^1 \to S^{2m+1} \to \mathbb{C}P^m$ be the *Hopf fibration.* Here $\mathbb{C}P^m$ denotes complex projective space. Let g be the standard metric on S^{2m+1} of constant curvature 1. At each point $p \in S^{2m+1}$ the tangent space T_pS^{2m+1} splits into the 1-dimensional vertical subspace V_p tangent to the fiber S^1 and its orthogonal complement H_p, the "horizontal" subspace. Hence we have $T_pS^{2m+1} = H_p \oplus V_p$. With respect to this splitting we can define for each $T > 0$ a new metric by $g_T := g|_H \oplus T \cdot g|_V$. In other words, we keep the splitting $TS^{2m+1} = H \oplus V$ orthogonal, on H the metric remains unchanged whereas it becomes rescaled by T along the fibers. This way we obtain a one-parameter family of metrics g_T on S^{2m+1}, called *Berger metrics.*

It is possible to explicitely compute the spectrum of the Dirac operator on S^{2m+1} for each g_T. The reason for this is the fact that all metrics g_T are homogeneous for the unitary group when we write

$$S^{2m+1} = U(m+1)/U(m).$$

Therefore one can apply methods from harmonic analysis using representation theory for the unitary groups to compute the spectrum, see [19] for $m = 1$ and [5] for the general case.

It turns out that if m is odd, *i.e.* if $n = 2m + 1 \equiv 3 \bmod 4$, then $h(S^{2m+1}, g_T, \mathfrak{S})$ is zero for general T but for special choices $h(S^{2m+1}, g_T, \mathfrak{S}) > 0$. Actually, $h(S^{2m+1}, g_T, \mathfrak{S})$ is unbounded for fixed m and for $T \in (0, \infty)$. One of these special values of T is $2(m + 1)$. More precisely, the Dirac operator on (S^{2m+1}, g_T), m odd, has an eigenvalue

$$\lambda(T) = (-1)^{\frac{m-1}{2}} \cdot \left(\tfrac{T}{2} - (m+1)\right)$$

with multiplicity $\begin{pmatrix} m+1 \\ (m+1)/2 \end{pmatrix}$. This eigenvalue vanishes for $T = 2(m+1)$ and in a neighborhood of $2(m + 1)$ no other eigenvalue vanishes.

8. The conjecture

The example of Berger spheres indicates that one can produce harmonic spinors by special choices of the metric and it gives rise to

Conjecture. *Let $(M, \mathfrak{S})$ be a spin manifold of dimension $n \geq 3$. Then there exists a Riemannian metric g on M such that*

$$h(M, g, \mathfrak{S}) > 0.$$

In other words, we believe that in contrast to harmonic differential forms harmonic spinors are not topologically obstructed for dimension $n \geq 3$.

Using the Atiyah–Singer index theorem for families and the theory of exotic spheres Hitchin [19] has shown that the conjecture is true for $n \equiv 0, 1, 7 \bmod 8$. We give a completely different argument for dimension $n \equiv 3 \bmod 4$. First we prove the following

Gluing Theorem [6].
Let $(M_1, g_1, \mathfrak{S}_1)$ and $(M_2, g_2, \mathfrak{S}_2)$ be closed Riemannian spin manifolds of dimension $n \geq 3$. Let $\varepsilon > 0$ and $\Lambda > 0$. Then there exists a Riemannian metric g on the connected sum of M_1 and M_2 such that the Dirac spectrum of $(M_1 \# M_2, g, \mathfrak{S}_1 \# \mathfrak{S}_2)$ is (Λ, ε)-close to the disjoint union of the Dirac spectra of $(D_i, g_i, \mathfrak{S}_i)$.

Here (Λ, ε)*-close* means that if $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$ are the eigenvalues of D_g in the range $(-\Lambda, \Lambda)$ (repeated according to their multiplicity) and $\mu_1 \leq \ldots \leq \mu_l$ are the eigenvalues of D_{g_1} and D_{g_2} in the same range, then $k = l$ and $|\mu_j - \lambda_j| < \varepsilon$.

Now let $(M, \mathfrak{S}_M)$ be a closed spin manifold of dimension $n \equiv 3 \bmod 4$. Pick any Riemannian metric g on M. If $h(M, g, \mathfrak{S}_M) > 0$ we are done, otherwise we can assume (after possibly rescaling g) that all Dirac eigenvalues λ of $(M, g, \mathfrak{S}_M)$ satisfy $|\lambda| \geq 1$. We apply the Gluing Theorem to $(M_1, g_1, \mathfrak{S}_1) = (M, g, \mathfrak{S}_M)$ and $(M_2, g_2, \mathfrak{S}_2) = (S^n, g_T, \mathfrak{S}_{S^n})$ with ε very small and $\Lambda = 1$. Then there exist metrics $\tilde{g}_T$ on $M \# S^n = M$ with Dirac eigenvalues $\lambda(T)$ such that $\left|\lambda(T) - (-1)^{\frac{m-1}{2}} \left(\frac{T}{2} - (m+1)\right)\right| < \varepsilon$. Since λ depends continuously on T it must necessarily vanish for some $\tilde{T}$ close to $2(m+1)$.

Thus $\tilde{g}_{\tilde{T}}$ is the desired metric on $(M \# S^n, \mathfrak{S}_M \# \mathfrak{S}_{S^n}) = (M, \mathfrak{S}_M)$ with

$$h(M, \tilde{g}_{\tilde{T}}, \mathfrak{S}_M) > 0.$$

Summarizing, we now know that the conjecture holds in dimension $n \equiv 0, 1, 3, 7 \bmod 8$, $n \geq 3$, it is not true for $n = 1$ or $n = 2$, and it is still open for $n \equiv 2, 4, 5, 6 \bmod 8$, $n \geq 4$.

9. Further aspects of harmonic spinors

Generic metrics. Our conjecture, which we have seen to be true in many dimensions, tells us that for specific choices of the Riemannian metric there are nontrivial harmonic spinors. On the other hand, all examples which one can explicitly compute, like the Berger metrics on odd-dimensional spheres, indicate that for *generic metrics* the number of linearly independent harmonic spinors is minimal in the sense that there are not more than there must be by the index theorems. This has recently proven to be true in dimensions ≤ 4 at least, see [24].

Positive Scalar Curvature. The scalar curvature function of a Riemannian manifold is a very weak geometric invariant. It is known that every function f on an n-dimensional closed manifold, $n \geq 3$, which is negative somewhere, is the scalar curvature function for some Riemannian metric on M [20, 21]. In other words, if the scalar curvature s is negative somewhere, then it contains no topological information at all.

But from Lichnerowicz's formula [23]

$$D^2 = \nabla^*\nabla + \frac{s}{4}$$

it follows that if the scalar curvature is positive, then D^2 is a strictly positive operator. Hence, $h(M, g, S) = 0$. In particular, if n is divisible by 4, then $\hat{A}(M) = 0$. We see that nonvanishing of the $\hat{A}$-genus is a topological obstruction against existence of a metric of positive scalar curvature. A similar remark holds for the α-genus in general dimensions.

Combining surgery results obtained independently by Gromov/Lawson and Schoen/Yau with homotopy theoretic work by Stolz one obtains the remarkable fact that for simply connected manifolds this is the only obstruction, see [15, 27, 29, 30].

The nonsimply connected case is still a topic of active research [14]. The corresponding conjecture is known as *Gromov–Lawson–Rosenberg conjecture*. See [25] or [31] for a survey, see also [16] for the noncompact case. Very recently, the Gromov–Lawson–Rosenberg conjecture in its original (unstable) form has been shown to fail in dimension 5,6, and 7 [26].

In the case of zero scalar curvature, $s \equiv 0$, there can be nontrivial harmonic spinors, $h(M, g, S)$ can be positive. But then, again by Lichnerowicz's formula $D^2 = \nabla^*\nabla$, every harmonic spinor must be parallel and we are back to Section 2.

Nodal Sets. Let φ be a harmonic spinor or, more generally, a solution of $(D + h)\varphi = 0$ where h denotes any potential. The set of zeros $N_\varphi = \{x \,|\, \varphi(x) = 0\}$ is called its *nodal set*. This terminology stems from the analogous classical theory for Laplace operators describing vibrating membranes. For example, if φ is an eigenspinor on a 3-dimensional manifold, $(D-\lambda)\varphi = 0, \lambda \in \mathbb{R}$, then

$$\Phi(x,t) = e^{i\lambda t} \cdot \varphi(x)$$

describes a fermion in "pure state". The nodal set N_φ is precisely the set of points where the probability density $|\Phi|^2$ vanishes, *i.e.* the locus at which the probability to find the particle is zero.

How big can such an "exclusive" set be? The answer is the following [7]: *If φ solves $(D+h)\varphi = 0$ on an n-dimensional manifold, then*

$$dim(N_\varphi) \leq n-2.$$

Here dim denotes Hausdorff dimension. In particular, the exclusive set for a fermion is at most 1-dimensional.

Seiberg–Witten Theory. In order to explain confinement of electric charge in $N = 2$ supersymmetric gauge theory Seiberg and Witten [28] introduced equations which led recently to spectacular results in differential topology of 4-manifolds. It seems that most theorems proved by Donaldson's instanton theory can also be proved using Seiberg–Witten theory, only in a simpler way. Moreover, there have been new important applications. The Seiberg–Witten equations couple the harmonic spinor equation to an equation for a $U(1)$-potential.

REFERENCES

[1] M.F. Atiyah, I.M. Singer, *Ann. Math.* **87**, 546 (1968).
[2] M.F. Atiyah, I.M. Singer, *Ann. Math.* **93**, 139 (1971).
[3] C. Bär, *Commun. Math. Phys.* **154**, 509 (1993).
[4] C. Bär, *Math. Ann.* **293**, 39 (1992).
[5] C. Bär, GAFA **6**, 899 (1996).
[6] C. Bär, *Math. Ann.* **309**, 225 (1997).
[7] C. Bär, *Commun. Math. Phys.* **188**, 709 (1997).
[8] H. Baum, T. Friedrich, R. Grunewald, I. Kath, *Twistor and Killing Spinors on Riemannian Manifolds*, *Teubner-Texte zur Mathematik*, Teubner 1991.
[9] M.J. Duff, B.E.W. Nilsson, C.N. Pope, *Phys. Rep.* **130**, 1 (1986).
[10] T. Friedrich, *Collect. Math.* **48**, 57 (1984).
[11] T. Friedrich, I. Kath, *J. Diff. Geom.* **29**, 263 (1989).
[12] T. Friedrich, I. Kath, *C.R. Acad. Sci. Paris* **307**, 967 (1988).
[13] T. Friedrich, I. Kath, *Commun. Math. Phys.* **133**, 543 (1990).
[14] M. Gromov, H.B. Lawson, *Ann. Math.* **111**, 209 (1980).

[15] M. Gromov, H.B. Lawson, *Ann. Math.* **111**, 423 (1980).

[16] M. Gromov, H.B. Lawson, *Publ. Math. I.H.E.S.* **58**, 295 (1983).

[17] R. Grunewald, *Ann. Glob. Anal. Geom.* **8**, 43 (1990).

[18] O. Hijazi, *C.R. Acad. Sci. Paris* **303**, 417 (1986).

[19] N. Hitchin, *Adv. Math.* **14**, 1 (1974).

[20] J.L. Kazdan, F.W. Warner, *Ann. Math.* **101**, 317 (1975).

[21] J.L. Kazdan, F.W. Warner, *J. Diff. Geom.* **10**, 113 (1975).

[22] H.B. Lawson, M.L. Michelsohn, *Spin Geometry*, Princet. Math. Series, Princeton 1989.

[23] A. Lichnerowicz, *C.R. Acad. Sci. Paris* **257**, 7 (1963).

[24] S. Maier, Generic metrics and connections on spin- and spin^c-manifolds, preprint 1996.

[25] J. Rosenberg, S. Stolz, *Springer Math. Sci. Res. Inst. Publ.* **27**, 241 (1994).

[26] T. Schick, A counterexample to the (unstable) Gromov–Lawson–Rosenberg conjecture, preprint 1997.

[27] R. Schoen, S.T. Yau, *Manuscr. Math.* **28**, 159 (1979).

[28] N. Seiberg, E. Witten, *Nucl. Phys.* **B431**, 484 (1994).

[29] S. Stolz, *Bull. Am. Math. Soc.* **23**, 427 (1990).

[30] S. Stolz, *Ann. Math.* **136**, 511 (1992).

[31] S. Stolz, Positive scalar curvature metrics — existence and classification questions, Proc. Intern. Congr. of Mathematicians, Zürich Vol. 1, 625 (1994).

[32] M. Walker, R. Penrose, *Commun. Math. Phys.* **18**, 265 (1970).

[33] M. Wang, *Ann. Glob. Anal. Geom.* **7**, 59 (1989).

[34] M. Wang, *Ann. Glob. Anal. Geom.* **13**, 31 (1995).