

DISTRIBUTED-ORDER FRACTIONAL KINETICS *

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Fractional diffusion equations are widely used to describe anomalous diffusion processes where the characteristic displacement scales as a power of time. For processes lacking such scaling the corresponding description may be given by distributed-order equations. In the present paper we consider different forms of distributed-order fractional kinetic equations and investigate the effects described by different classes of such equations. In particular, the equations describing accelerating and decelerating subdiffusion, as well as those describing accelerating and decelerating superdiffusion are presented.

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1. Introduction

The diffusion equation proposed by Adolf Fick almost 150 years ago is a partial differential equation of parabolic type, with the first temporal derivative on its l.h.s. and the second spatial derivative on its r.h.s. The corresponding equation can be put down both for the particles' concentration

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in a diffusing cloud and for a probability of a single particle's position:

$$\frac{\partial}{\partial t}p(x, t) = K \frac{\partial^2}{\partial x^2}p(x, t). \quad (1.1)$$

The mean squared particle's displacement from its initial position given by the solution of this equation grows linearly in time: $\langle x^2(t) \rangle = 2Kt$. This scaling behavior follows immediately from the structure of Fick's equation, being second order in spatial coordinate and first order in time: Changing the spatial scale by a factor of 2 corresponds to changing the time scale by a factor of 4.

In complex systems this kind of behavior is often violated, being replaced by an anomalous diffusion relationship,

$$\langle x^2(t) \rangle \propto t^\beta, \quad (1.2)$$

with $\beta \neq 1$, or (in absence of the second moment) by related forms where the lower, not necessarily integer, moment of the distribution scales as a function of time. The continuous description of anomalous diffusion is given by generalizations of Fick's scheme based on fractional derivatives [1]. The fractional generalizations of the diffusion equation may have either a corresponding fractional derivative instead of the whole-number one, or an additional fractional derivative on the "wrong" side of the equation. In what follows we refer to these two possibilities as to "normal" and "modified" forms of a fractional diffusion equation. An example of the "normal" form is an equation for superdiffusion with the Riesz fractional spatial derivative instead of the second derivative on the r.h.s. [2]. The situation with an additional derivative on the "wrong" side is exemplified by the standard fractional diffusion equation for subdiffusion [3]. We note that the two forms, *i.e.* the form with a Caputo derivative on the l.h.s. for subdiffusive processes, and a form with an additional spatial derivative on the l.h.s. for a superdiffusive process, are equivalent to the commonly used ones. This equivalence will be discussed in detail in Section 2 of the present work, introducing the corresponding fractional operators.

Many physical processes, however, lack power-law scaling, Eq. (1.2), over the whole time-domain. These processes can not be characterized by a single scaling exponent β . Examples of processes lacking scaling include several cases of decelerating subdiffusion (*e.g.* the Sinai diffusion [4]) and decelerating superdiffusion (as exemplified by truncated Levy-flights [5]). Such processes can be described by derivatives of *distributed* order [6, 7], introduced by Caputo [8]. The equivalence between the different forms of fractional diffusion equations is lost in this case: Different forms of the distributed-order fractional equations describe different situations. Thus,

the equations with the distributed-order derivatives on the “proper” side describe processes getting more anomalous in course of the time (accelerating superdiffusion and decelerating subdiffusion) [9], while the equations with additional distributed-order derivative on the “wrong” side describe the situations getting less anomalous (decelerating superdiffusion and accelerating subdiffusion). As an example a special model with two fractional derivatives of different orders is used throughout the work.

In Sections 2 and 3 we discuss the four forms of fractional diffusion equations for the processes showing scaling behavior. These forms correspond to possible permutations of fractional temporal/spatial derivative on the “proper”/“wrong” side of the equation. Three of the four forms discussed below are well-known. The fourth one, to our best knowledge, was not previously discussed. We first turn to the forms pertinent to temporal fractional equations, *i.e.* to subdiffusion, and then to the superdiffusive case. Sections 4 and 5 are devoted to the distributed-order generalizations of the corresponding equations. The results are summarized in Section 6.

2. Two forms of time fractional diffusion equations

2.1. Riemann–Liouville form

The time fractional diffusion equation (TFDE) in the Riemann–Liouville form (RL-form), which in our terminology corresponds to a “modified” form of the fractional diffusion equation, reads [2]:

$$\frac{\partial}{\partial t} p(x, t) = K_\beta {}_0D_t^{1-\beta} \frac{\partial^2}{\partial x^2} p(x, t), \tag{2.1}$$

$$p(x, t = 0) = \delta(x), \quad 0 < \beta \leq 1.$$

Here K_β is a positive constant, $[K_\beta] = \text{cm}^2/\text{sec}^\beta$, and ${}_0D_t^\mu$ is the Riemann–Liouville fractional derivative on the right semi-axis, which, for a “sufficiently well-behaved” function $\phi(t)$, is defined as follows:

$${}_0D_t^\mu \phi = \frac{d}{dt} J^{1-\mu} \phi = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t d\tau \frac{\phi(\tau)}{(t-\tau)^\mu}, \quad 0 \leq \mu < 1, \tag{2.2}$$

where $J^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t-\tau)^{\alpha-1} \phi(\tau)$, $t > 0$, $\alpha \in \mathbb{R}^+$ is the Riemann–Liouville fractional integral of the order α . In what follows, we omit the subscript “0”, for brevity.

Applying the Laplace transform,

$$\tilde{\phi}(s) \equiv L \{ \phi(t) \} = \int_0^\infty dt e^{-st} \phi(t),$$

and the Fourier-transform

$$g(k) \equiv \Phi \{g(x)\} = \int_{-\infty}^{\infty} dx e^{ikx} g(x)$$

in succession, and using the Laplace transform of the Riemann–Liouville derivative (2.2),

$$L \{D_t^\mu \phi(t)\} = s^\mu \tilde{\phi}(s) \tag{2.3}$$

we get from Eq. (2.1) the form of the Laplace-transformed characteristic function $f(k, t)$ of the distribution $p(x, t)$:

$$\tilde{f}(k, s) = \frac{s^{\beta-1}}{s^\beta + K_\beta k^2}. \tag{2.4}$$

The corresponding probability density function (PDF) is exactly the one typical for continuous-time random walks with a power-law waiting-time distribution [1, 2].

2.2. Caputo form

The TFDE in the Caputo form (C-form, corresponding to a “normal” form of the fractional diffusion equation) is written as follows:

$$\frac{\partial^\beta}{\partial t^\beta} p(x, t) = K_\beta \frac{\partial^2}{\partial x^2} p(x, t), \tag{2.5}$$

$p(x, 0) = \delta(x)$, where K_β is the same constant as in Eq. (2.1), and the time fractional derivative of order β , $0 < \beta < 1$ is understood in the Caputo sense [11],

$$\frac{\partial^\beta}{\partial t^\beta} \phi \equiv D_{*t}^\beta \phi(t) = J^{1-\beta} \frac{d}{dt} \phi = \frac{1}{\Gamma(1-\beta)} \int_0^t d\tau (t-\tau)^{-\beta} \frac{d}{d\tau} \phi(\tau). \tag{2.6}$$

Here, the sequence of temporal integration and differentiation is reversed with respect to a Riemann–Liouville operator. Recalling the Laplace transform of the Caputo derivative,

$$L \left\{ \frac{d^\mu \phi}{dt^\mu} \right\} = s^\mu \tilde{\phi}(s) - s^{\mu-1} \phi(0), \tag{2.7}$$

$0 < \mu < 1$, and making the Fourier–Laplace transform of Eq. (2.5) we again arrive at Eq. (2.4). Thus, both forms of TFDE are equivalent.

In the literature, the third form of TFDE is mentioned, the one using the Riemann–Liouville derivative [12]:

$$D_t^\beta p(x, t) = K_\beta \frac{\partial^2}{\partial x^2} p(x, t) + \frac{t^{-\beta}}{\Gamma(1 - \beta)} p(x, 0), \tag{2.8}$$

$0 < \beta < 1$. The equivalence of the RL- and this third form can be easily shown by applying $\int_0^t dt' \dots$ to both sides of Eq. (2.1) and using the Riemann–Liouville derivative of a constant, $D_t^\beta 1 = t^{-\beta} / \Gamma(1 - \beta)$. The equivalence of the “normal” (C-) and the third form can also be shown easily, if one uses the relation between the Riemann–Liouville and Caputo derivatives which can be obtained straightforwardly:

$$D_t^\beta \phi(t) = \frac{\partial^\beta}{\partial x^\beta} \phi + \frac{t^{-\beta}}{\Gamma(1 - \beta)} \phi(0) \tag{2.9}$$

with $0 < \beta < 1$. Then, starting, *e.g.*, from the C-form of TFDE, Eq. (2.5), and using Eq. (2.9) we immediately arrive at Eq. (2.8).

In this paper we are interested in the mean squared displacement (MSD) given by

$$\langle x^2(t) \rangle = L^{-1} \left\{ \left(-\frac{\partial^2 \tilde{f}}{\partial k^2} \right)_{k=0} \right\}. \tag{2.10}$$

From Eq. (2.4) we get:

$$\langle x^2(t) \rangle = L^{-1} \left\{ 2K_\beta s^{-\beta-1} \right\} = \frac{2K_\beta t^\beta}{\Gamma(1 + \beta)}. \tag{2.11}$$

3. Two forms of space fractional diffusion equations

3.1. The “normal” form

The “normal” form of space fractional diffusion equation reads as:

$$\frac{\partial p}{\partial t} = K_\alpha \frac{\partial^\alpha p}{\partial |x|^\alpha}, \tag{3.1}$$

$p(x, t = 0) = \delta(x)$, $0 < \alpha \leq 2$, where K_α is a positive constant, $[K_\alpha] = \text{cm}^\alpha / \text{sec}$, and the Riesz fractional derivative $\partial^\alpha / \partial |x|^\alpha$ (we adopt here the notation introduced in [12]) is defined for a “sufficiently well-behaved” function $f(x)$ through the Liouville–Weil derivatives [10]:

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \begin{cases} -\frac{1}{2 \cos(\pi\alpha/2)} [D_+^\alpha + D_-^\alpha] & \alpha \neq 1 \\ -\frac{d}{dx} \hat{H} f(x) & \alpha = 1 \end{cases}, \tag{3.2}$$

where D_{\pm}^{α} are the left- and right-hand side Liouville–Weil derivatives,

$$\begin{aligned}
 D_{+}^{\alpha}\phi &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^x \frac{\phi(\xi)d\xi}{(\xi-x)^{\alpha-1}}, \\
 D_{-}^{\alpha}\phi &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^{\infty} \frac{\phi(\xi)d\xi}{(x-\xi)^{\alpha-1}}
 \end{aligned}
 \tag{3.3}$$

for $0 < \alpha < 2$, $\alpha \neq 1$ (for $\alpha = 1$ $D_{\pm}^1 = \pm d/dx$), and H is the Hilbert transform operator,

$$H\phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\xi)d\xi}{x-\xi}.$$

In Fourier space the operators of fractional derivatives have a simple form:

$$\Phi(D_{\pm}^{\alpha}\phi) = \int_{-\infty}^{\infty} dx \exp(ikx) D_{\pm}^{\alpha}\phi = (\mp ik)^{\alpha} \phi(k),
 \tag{3.4}$$

where

$$(\mp ik)^{\alpha} = |k|^{\alpha} \exp\left(\mp \frac{\alpha\pi i}{2} \text{sign } k\right).$$

Since

$$\Phi(H\phi) = i \text{sign } k \phi(k)
 \tag{3.5}$$

then, with the use of Eqs. (3.2)–(3.5) we get the expression, which is valid for the Fourier transform of the Riesz fractional derivative, for all values of α :

$$\Phi\left(\frac{d^{\alpha}\phi}{d|x|^{\alpha}}\right) = -|k|^{\alpha} \phi(k).
 \tag{3.6}$$

Applying the Fourier-transform to Eq. (3.1), and noting Eq. (3.6), we get the characteristic function for the PDF of Lévy flights,

$$f(k, t) = \exp(-D |k|^{\alpha} t),
 \tag{3.7}$$

see Ref. [2].

3.2. The “modified” form

Let us turn to the fractional equation for superdiffusion with the additional spatial derivative on its l.h.s.

$$\frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \frac{\partial p}{\partial t} = -K_\alpha \frac{\partial^2}{\partial x^2} p, \tag{3.8}$$

where K_α is the same as in Eq. (3.1). Note the minus sign in Eq. (3.8). This sign gets clear when turning to a Fourier representation: applying Fourier transform and using Eq. (3.6), we arrive at the characteristic function (3.7). Thus, both forms of the space fractional diffusion equation, Eqs. (3.1) and (3.8), are equivalent.

Since the mean square displacement diverges for Lévy flights [2], their anomalous nature can be characterized by a typical displacement δx of the diffusing particle,

$$\delta x \propto \langle |x|^q \rangle^{1/q}, \quad q < \alpha, \tag{3.9}$$

which differs, of course, from the MSD discussed in Eq. (2.10). To get the q -th moment we use the following expression [13]:

$$\langle |x|^q \rangle = \frac{2}{\pi} \Gamma(1+q) \sin \frac{\pi q}{2} \int_0^\infty dk (1 - \text{Re} f(k, t)) k^{-q-1}. \tag{3.10}$$

Inserting Eq. (3.7) into Eq. (3.10) and changing to a new variable, $\xi = K_\alpha k^\alpha t$, we obtain

$$\int_0^\infty dk \dots = \frac{(K_\alpha t)^{1/\alpha}}{\alpha} \int_0^\infty d\xi (1 - e^{-\xi}) \xi^{-q/\alpha-1} = \frac{(K_\alpha t)^{q/\alpha}}{q} \Gamma\left(1 - \frac{q}{\alpha}\right), \tag{3.11}$$

and, thus

$$\langle |x|^q \rangle = C(q, \alpha) (K_\alpha t)^{q/\alpha}, \quad q < \alpha. \tag{3.12}$$

Here

$$C(q, \alpha) = \frac{2}{\pi q} \sin\left(\frac{\pi q}{2}\right) \Gamma(1+q) \Gamma\left(1 - \frac{q}{\alpha}\right). \tag{3.13}$$

Note that for $a = q = 2$ Eqs. (3.12), (3.13) give

$$\langle x^2 \rangle = 2Kt.$$

4. Distributed-order time fractional diffusion equations

4.1. Distributed-order time fractional diffusion equation in the RL form

The fractional diffusion equation with a distributed-order Riemann–Liouville derivative reads:

$$\frac{\partial p}{\partial t} = \int_0^1 d\beta w(\beta) K(\beta) D_i^{1-\beta} \frac{\partial^2 p}{\partial x^2}, \quad (4.1)$$

$p(x, 0) = \delta(x)$, where $K(\beta) = K\tau^{1-\beta}$, $[K] = \text{cm}^2/\text{sec}$, $[\tau] = \text{sec}$, $w(\beta)$ is a dimensionless non-negative function, which should fulfill $\int_0^1 d\beta w(\beta) = 1$. If we set $w(\beta) = \delta(\beta - \beta_0)$, $0 < \beta_0 < 1$, then we arrive at time fractional diffusion equation in the RL form, see Eq. (2.1), where $K_\beta = K\tau^{1-\beta_0}$.

We now prove that the solution of Eq. (4.1) is a PDF. The derivation here follows the method used in [14]. Its aim is to show that the random process whose PDF obeys Eq. (4.1) is subordinated to the Wiener process. This means that the trajectory of such a process corresponds to a simple Brownian motion. However, this motion takes place in some intrinsic (“operational”) time, which in turn is a nondecreasing random function of the physical time t [14]. Making a Fourier–Laplace transform of Eq. (4.1) and using Eq. (2.3) we get

$$\tilde{f}(k, s) = \frac{1}{sI_{\text{RL}}(I_{\text{RL}}^{-1} + k^2 K\tau)}, \quad (4.2)$$

where

$$I_{\text{RL}}(s\tau) = \int_0^1 d\beta (s\tau)^{-\beta} w(\beta). \quad (4.3)$$

We rewrite Eq. (4.2) as follows:

$$\tilde{f}(k, s) = \frac{1}{sI_{\text{RL}}} \int_0^\infty du \exp[-u(I_{\text{RL}}^{-1} + k^2 K\tau)] = \int_0^\infty du e^{-uk^2 K\tau} \tilde{G}_{\text{RL}}(u, s), \quad (4.4)$$

where

$$\tilde{G}_{\text{RL}}(u, s) = \frac{1}{sI_{\text{RL}}(s\tau)} \exp\left[-\frac{u}{I_{\text{RL}}(s\tau)}\right] \quad (4.5)$$

is a Laplace transform of a function whose properties will be specified below. Now, with the help of Eqs. (4.4) and (4.5) the PDF $p(x, t)$ can be written

as

$$\begin{aligned}
 p(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \int_{\text{Br}} \frac{ds}{2\pi i} e^{st} \int_0^{\infty} du e^{-uk^2 K\tau} \tilde{G}_{\text{RL}}(u, s) \\
 &= \int_0^{\infty} du \frac{e^{-x^2/4uK\tau}}{\sqrt{4\pi uK\tau}} G_{\text{RL}}(u, t), \tag{4.6}
 \end{aligned}$$

where Br denotes the Bromwich integration contour.

In order to prove the positivity of $p(x, t)$ we demonstrate that the function $G_{\text{RL}}(u, t)$ is the PDF providing the subordination transformation from time scale t to time scale u , that is, $G_{\text{RL}}(u, t)$ is positive and normalized with respect to u for any t . At first we demonstrate normalization,

$$\int_0^{\infty} du G_{\text{RL}}(u, t) = L^{-1} \left\{ \int_0^{\infty} du \frac{e^{-u/I_{\text{RL}}}}{s I_{\text{RL}}} \right\} = L^{-1} \left\{ \frac{1}{s} \right\} = 1. \tag{4.7}$$

Now, to prove the positivity of function $G_{\text{RL}}(u, t)$, according to the Bernstein theorem [15], it is enough to show that $\tilde{G}_{\text{RL}}(u, s)$, Eq. (4.5), is completely monotonic as a function of s on positive real axis, *i.e.*, it is positive and the signs of its derivatives alternate. We do this for the special case when

$$w(\beta) = B_1 \delta(\beta - \beta_1) + B_2 \delta(\beta - \beta_2) \tag{4.8}$$

with $0 < \beta_1 < \beta_2 \leq 1$, $B_1 > 0$, $B_2 > 0$, $B_1 + B_2 = 1$. This choice allows us to show in a simple way the property of the diffusive behavior governed by distributed-order diffusion equations. This is why this case is repeatedly discussed in our article.

Inserting Eq. (4.8) into Eq. (4.3) we have

$$I_{\text{RL}}(s\tau) = b_1 s^{-\beta_1} + b_2 s^{-\beta_2}, \tag{4.9}$$

where $b_1 = B_1/\tau^{\beta_1}$, $b_2 = B_2/\tau^{\beta_2}$. From Eq. (4.5) it follows that $\tilde{G}_{\text{RL}}(u, s)$ is a product of two functions: $\tilde{G}_{\text{RL}}(u, s) = \phi_1(s)\phi_2(u, s)$ with

$$\phi_1 = \frac{1}{s I_{\text{RL}}(s\tau)}, \quad \phi_2 = \exp \left[-\frac{u}{I_{\text{RL}}(s\tau)} \right]. \tag{4.10}$$

We will prove that both functions ϕ_1 and ϕ_2 are completely monotonic, and therefore, as a product, $\tilde{G}_{\text{RL}}(u, s)$ is completely monotonic too.

Let us start from ϕ_1 . It can be rewritten as $\phi_1 = \phi(h(s))$, where $\phi(y) = 1/y$ is a completely monotonic function, and $h(s) = b_1 s^{1-\beta_1} + b_2 s^{1-\beta_2}$ is a

positive function with a completely monotonic derivative (this is evident by direct inspection). Therefore, $\phi_1(s)$ is completely monotonic according to Criterion 2 in [15], Chapter XIII, §4.

Now we turn to ϕ_2 . Again, it can be rewritten as $\phi_2 = \phi(\psi(s))$, where $\phi(y) = \exp(-uy)$ is a completely monotonic function. According to the same Criterion, it is enough to show that the positive function

$$\psi(s) = \frac{1}{I_{RL}(s\tau)} = \frac{1}{b_1s^{-\beta_1} + b_2s^{-\beta_2}} = \frac{s^{\beta_2}}{b_2} \frac{1}{1 + \frac{b_1}{b_2}s^{\beta_2-\beta_1}} \tag{4.11}$$

possesses completely monotonic derivative. By differentiating Eq. (4.11) we get

$$\begin{aligned} \psi'(s) &= \frac{s^{\beta_2-1}}{b_2} \frac{1}{1 + \frac{b_1}{b_2}s^{\beta_2-\beta_1}} - \frac{s^{\beta_2}}{b_2} \frac{\frac{b_1}{b_2}(\beta_2 - \beta_1)s^{\beta_2-\beta_1-1}}{\left(1 + \frac{b_1}{b_2}s^{\beta_2-\beta_1}\right)^2} \\ &= \frac{\beta_2s^{\beta_2-1}}{b_2} \frac{1}{1 + \frac{b_1}{b_2}s^{\beta_2-\beta_1}} \frac{1 + \frac{b_1\beta_1}{b_2\beta_2}s^{\beta_2-\beta_1}}{1 + \frac{b_1}{b_2}s^{\beta_2-\beta_1}}. \end{aligned} \tag{4.12}$$

Denoting $\xi = \beta_2 - \beta_1$, $0 < \xi < 1$, we note that this function is a product of three completely monotonic functions, and thus is itself completely monotonic, too. Indeed, (i) the first function is s^{β_2-1} being a negative power of s ; (ii) the second one is completely monotonic since it has the form $g(h(s))$, with $g(y) = 1/(1 + y)$ being completely monotonic, and $h(\xi) = s^\xi$ is a positive function with a completely monotonic derivative; (iii) the last function has the same form with $g(y) = (1 + cy)/(1 + dy)$, where $0 < c < d$ with $c = b_1\beta_1/b_2\beta_2$, $d = b_1/b_2$. The function $g(y)$ is completely monotonic since its n -th derivative (obtained using the Leibnitz rule) has the form

$$g^{(n)}(y) = \frac{(-1)^{n-1}(c - d)n!d^{n-1}}{(1 + dy)^{n+1}},$$

and the signs of the successive derivatives alternate. Thus, we have proved that $\psi'(s)$, Eq. (4.12), is a completely monotonic function. Therefore the exponential ϕ_2 , Eq. (4.10), is a completely monotonic function, too, and the whole function, $\tilde{G}_{RL}(u, s)$, is completely monotonic as a product of two completely monotonic functions, ϕ_1 and ϕ_2 . Therefore, the function $G_{RL}(u, t)$ is a PDF and, according to Eq. (4.6), the function $p(x, t)$ is a PDF, too. Thus, we have proved that the solution of the distributed order time fractional diffusion equation in the Riemann–Liouville form is a PDF.

We are interested in the MSD which is given by

$$\langle x^2(t) \rangle = L^{-1} \left\{ \left(-\frac{\partial^2 \tilde{f}}{\partial k^2} \right)_{k=0} \right\} = K\tau L^{-1} \left\{ \frac{2I_{RL}(s\tau)}{s} \right\}. \tag{4.13}$$

For the particular case (4.8) we get, by inserting Eq. (4.9) into Eq. (4.13) and making an inverse Laplace transform,

$$\langle x^2(t) \rangle = \frac{2\kappa_1}{\Gamma(1 + \beta_1)} \left(\frac{t}{\tau}\right)^{\beta_1} + \frac{2\kappa_2}{\Gamma(1 + \beta_2)} \left(\frac{t}{\tau}\right)^{\beta_2}, \tag{4.14}$$

where $\kappa_1 = B_1 K \tau$ and $\kappa_2 = B_2 K \tau$. Since $0 < \beta_1 < \beta_2 \leq 1$, at small times the first term in the r.h.s. of Eq. (4.14) prevails, whereas at large times the second one dominates. Thus, the overall behavior corresponds to accelerating subdiffusion.

4.2. Distributed-order time fractional diffusion equation in the C-form

The distributed-order time fractional diffusion equation in the “normal” form can be written as

$$\int_0^1 d\beta \tau^{\beta-1} w(\beta) \frac{\partial^\beta p}{\partial t^\beta} = K \frac{\partial^2 p}{\partial x^2}, \tag{4.15}$$

$p(x, 0) = \delta(x)$, where τ is a positive constant representing some characteristic time of the problem (vide infra), $[\tau] = \text{sec}$, K is the diffusion coefficient, $[K] = \text{cm}^2/\text{sec}$, $w(\beta)$ is a dimensionless non-negative function, and the time fractional derivative of order β , $0 < \beta < 1$ is understood in the Caputo sense, Eq. (2.6). Note the difference between Eq. (4.15) and Eq. (4.1).

If we set $w(\beta) = \delta(\beta - \beta_0)$, $0 < \beta_0 < 1$, we arrive at the “normal” form of a fractional diffusion equation, Eq. (2.5), with $\beta = \beta_0$ and $K_\beta = K \tau^{1-\beta_0}$.

Let us prove that the solution of Eq. (4.15) is a PDF. Applying the Laplace- and Fourier-transforms in succession, we get:

$$\tilde{f}(k, s) = \frac{1}{s} \frac{I_C(s\tau)}{I_C(s\tau) + k^2 K \tau}, \tag{4.16}$$

where

$$I_C(s\tau) = \int_0^1 d\beta (s\tau)^\beta w(\beta). \tag{4.17}$$

We rewrite Eq. (4.16) in the form analogous to Eq. (4.4),

$$\tilde{f}(k, s) = \frac{I_C}{s} \int_0^\infty du e^{-u[I_C + k^2 K \tau]} = \int_0^\infty du e^{-uk^2 K \tau} \tilde{G}_C(u, s), \tag{4.18}$$

where

$$\tilde{G}_C(u, s) = \frac{I_C(s\tau)}{s} e^{-uI_C(s\tau)} \tag{4.19}$$

is the Laplace transform of a function $G_C(u, t)$ whose properties will be specified below. Now, $p(x, t)$ can be written in the form analogous to Eq. (4.6):

$$\begin{aligned} p(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \int_{\text{Br}} \frac{ds}{2\pi i} e^{st} \int_0^{\infty} du e^{-uk^2 K\tau} \tilde{G}_C(u, s) \\ &= \int_0^{\infty} du \frac{e^{-x^2/4uK\tau}}{\sqrt{4\pi uK\tau}} G_C(u, t). \end{aligned} \tag{4.20}$$

Similar to the RL case, the function $G_C(u, t)$ is the PDF providing the subordination transformation, from time scale t to time scale u . Indeed, at first we note that $G_C(u, t)$ is normalized with respect to u for any t . Using Eq. (4.19) we get

$$\int_0^{\infty} du G_C(u, t) = L^{-1} \left\{ \int_0^{\infty} du \frac{I_{\text{RL}}}{s} e^{-u/I_C} \right\} = L^{-1} \left\{ \frac{1}{s} \right\} = 1. \tag{4.21}$$

To prove the positivity of $G_C(u, t)$ it is enough to show that its Laplace transform is completely monotonic on the positive real axis [15]. The last statement is proved by noting that it is a product of two completely monotonic functions, I_C/s and $\exp(-uI_C)$. We again demonstrate this for the particular choice of $w(\beta)$, see Eq. (4.8). For this choice we obtain from Eq. (4.17)

$$I_C(s) = b_1 s^{\beta_1} + b_2 s^{\beta_2}, \tag{4.22}$$

where $b_1 = B_1\tau^{\beta_1}$, $b_2 = B_2\tau^{\beta_2}$. Then, I_C/s is completely monotonic as a sum of the negative powers of s , and I_C itself is a positive function with a completely monotonic derivative. Thus, $\exp(-uI_C)$ is also completely monotonic. Thus, we have proved that it is completely monotonic, and that $G_{\text{RL}}(u, t)$ is a PDF, according to the Bernstein theorem.

We are interested in the behavior of the MSD, *i.e.* in the second moment of the PDF. Using Eq. (4.16), we have

$$\langle x^2(t) \rangle = L^{-1} \left\{ \left(-\frac{\partial^2 \tilde{f}}{\partial k^2} \right)_{k=0} \right\} = 2K\tau L^{-1} \left\{ \frac{1}{sI_C(s\tau)} \right\}. \tag{4.23}$$

Inserting Eq. (4.22) into Eq. (4.23) one obtains:

$$\langle x^2(t) \rangle = 2K\tau L^{-1} \left\{ \frac{1}{s(b_1 s^{\beta_1} + b_2 s^{\beta_2})} \right\} = \frac{2K\tau}{b_2} L^{-1} \left\{ \frac{s^{-\beta_1-1}}{\frac{b_1}{b_2} + s^{\beta_2-\beta_1}} \right\}. \tag{4.24}$$

Recalling the Laplace transform of the generalized Mittag–Leffler function $E_{\mu,\nu}(z)$, $\mu > 0, \nu > 0$, which can be conveniently written as [11]

$$L \{ t^{\nu-1} E_{\mu,\nu}(-\lambda t^\mu) \} = \frac{s^{\mu-\nu}}{s^\mu + \lambda}, \quad \text{Res} > |\lambda|^{1/\mu}, \tag{4.25}$$

we get from Eq. (4.24):

$$\langle x^2(t) \rangle = \frac{2K\tau}{b_2} t^{\beta_2} E_{\beta_2-\beta_1, \beta_2+1} \left(-\frac{b_1}{b_2} t^{\beta_2-\beta_1} \right). \tag{4.26}$$

To obtain asymptotics at small t , we use an expansion, which is, in fact the definition of $E_{\mu,\nu}(z)$, see [16], Ch. XVIII, Eq. (19):

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \tag{4.27}$$

which yields in the main order for the MSD

$$\langle x^2(t) \rangle = \frac{2K\tau}{B_2 \Gamma(\beta_2 + 1)} \left(\frac{t}{\tau} \right)^{\beta_2} \propto t^{\beta_2}. \tag{4.28}$$

For large t we use the following expansion valid on the real negative axis, see [16], Ch. XVIII, Eq. (21):

$$E_{\mu,\nu}(z) = - \sum_{n=0}^N \frac{z^{-n}}{\Gamma(-\mu n + \nu)} + O(|z|^{-1-N}), \quad |z| \rightarrow \infty \tag{4.29}$$

which yields

$$\langle x^2(t) \rangle = \frac{2K\tau}{B_1 \Gamma(\beta_1 + 1)} \left(\frac{t}{\tau} \right)^{\beta_1} \propto t^{\beta_1}. \tag{4.30}$$

Since $\beta_1 < \beta_2$, we have retarded subdiffusion. It is worthwhile to note that the distributed-order equation (4.15) with the Caputo derivative always describes retarding, or slowing-down, sub-diffusive processes. Indeed, it is clearly seen from Eqs. (4.22), (4.23) that, for $w(\beta)$ given by Eq. (4.8), for large s (short times) $I(s\tau)$ behaves as s^{β_2} , so that, due to the Tauberian theorem, the MSD behaves as t^{β_2} , while at long times (small s) $I(s\tau) \propto s^{\beta_1}$ and, respectively, $\langle x^2 \rangle \propto t^{\beta_1}$, $\beta_1 < \beta_2$. Thus, Eq. (4.15) cannot describe the accelerating sub-diffusive process. The same arguments applied to Eq. (4.1) with the Riemann–Liouville derivative demonstrate that Eq. (4.1) cannot describe retarded sub-diffusive processes.

5. Distributed-order space fractional diffusion equation

5.1. The “normal” form

We write the distributed-order space fractional diffusion equation in the “normal” form as

$$\frac{\partial p}{\partial t} = \int_0^2 d\alpha K(\alpha) \frac{\partial^\alpha p}{\partial |x|^\alpha}, \quad p(x, 0) = \delta(x), \quad (5.1)$$

where $K(\alpha)$ is a (dimensional) function of the order of the derivative α , and the Riesz space fractional derivative $\partial^\alpha/\partial|x|^\alpha$ is given by its Fourier transform, see Eq. (3.6). Setting $K(\alpha) = K_{\alpha_0}\delta(\alpha-\alpha_0)$ we arrive at Eq. (3.1). In the general case $K(\alpha)$ can be represented as

$$K(\alpha) = l^{\alpha-2} K w(\alpha), \quad (5.2)$$

where l and K are dimensional positive constants, $[l] = \text{cm}$, $[K] = \text{cm}^2/\text{sec}$ and w is a dimensionless non-negative function of α . The equation for the characteristic function of Eq. (5.1) has the solution

$$f(k, t) = \exp \left\{ -\frac{Kt}{l^2} \int_0^2 d\alpha w(\alpha) (|k|l)^\alpha \right\}. \quad (5.3)$$

Note that the normalization condition,

$$\int_{-\infty}^{\infty} dx p(x, t) = f(k=0, t) = 1 \quad (5.4)$$

is fulfilled.

Let us consider the simple particular case,

$$w(\alpha) = A_1 \delta(\alpha - \alpha_1) + A_2 \delta(\alpha - \alpha_2), \quad (5.5)$$

where $0 < \alpha_1 < \alpha_2 \leq 2$, $A_1 > 0$, $A_2 > 0$. Inserting Eq. (5.5) into Eq. (5.3) we have

$$f(k, t) = \exp \{ -a_1 |k|^{\alpha_1} t - a_2 |k|^{\alpha_2} t \}, \quad (5.6)$$

where $a_1 = A_1 K/l^{2-\alpha_1}$, $a_2 = A_2 K/l^{2-\alpha_2}$. The characteristic function (5.6) is a product of two characteristic functions of Lévy stable PDFs with the Lévy indexes α_1 , α_2 , and the scale parameters a_1^{1/α_1} and a_2^{1/α_2} , respectively.

Therefore, the inverse Fourier transformation of Eq. (5.6) gives the PDF which is a convolution of two stable PDFs,

$$p(x, t) = a_1^{-\frac{1}{\alpha_1}} a_2^{-\frac{1}{\alpha_2}} t^{-\frac{1}{\alpha_1} - \frac{1}{\alpha_2}} \int_{-\infty}^{\infty} dx' L_{\alpha_1,1} \left(\frac{x - x'}{(a_1 t)^{\frac{1}{\alpha_1}}} \right) L_{\alpha_2,1} \left(\frac{x'}{(a_2 t)^{\frac{1}{\alpha_2}}} \right), \tag{5.7}$$

where $L_{\alpha,1}$ is the PDF of a symmetric Lévy stable law given by its characteristic function

$$\hat{L}_{\alpha,1}(k) = \exp(-|k|^\alpha). \tag{5.8}$$

The PDF given by Eq. (5.7) is, obviously, positive, as the convolution of two positive PDFs.

The PDF will be also positive, if the function $A(\alpha)$ is represented as a sum of N delta-functions multiplied by positive constants, N is a positive integer. Moreover, if $A(\alpha)$ is a continuous positive function, then discretizing the integral in Eq. (5.1) by a Riemann sum and passing to the limit we can also conclude on the positivity of the PDF in the general case.

Let us consider the q -th moment, Eq. (3.10), at small and large t . After inserting Eq. (5.6) into Eq. (3.10) we get

$$\langle |x|^q \rangle = \frac{2}{\pi} \Gamma(1 + q) \sin\left(\frac{\pi q}{2}\right) \int_0^\infty dk \left(1 - e^{-t\phi(k)}\right) k^{-q-1}, \tag{5.9}$$

where

$$\phi(k) = a_1 k^{\alpha_1} + a_2 k^{\alpha_2}. \tag{5.10}$$

In order to get the q -th moment at small t , we expand $\exp(-a_1 |k|^{\alpha_1} t)$ in power series with subsequent integration over k . As the result we have the following expansion valid at $q < \alpha_1$:

$$\begin{aligned} \langle |x|^q \rangle &= \frac{2}{\pi q} (a_2 t)^{q/\alpha_2} \sin\left(\frac{\pi q}{2}\right) \Gamma(1 + q) \Gamma\left(1 - \frac{q}{\alpha_2}\right) \\ &\times \left\{ 1 + \frac{1}{\Gamma\left(1 - \frac{q}{\alpha_2}\right)} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\alpha_2 n!} a_1^n a_2^{-\frac{n\alpha_1}{\alpha_2}} \Gamma\left(\frac{n\alpha_1 - q}{\alpha_2}\right) t^{n\left(1 - \frac{\alpha_1}{\alpha_2}\right)} \right\} \end{aligned} \tag{5.11}$$

for $t \rightarrow 0$. We recall that the radius R of convergence of a series $\sum_{n=0}^\infty c_n x^n$ is determined from the equation

$$\frac{1}{R} = \lim_{n \rightarrow \infty} |c_n|^{1/n}.$$

Then, by using the Stirling formula, one can see that in case of Eq. (5.11) $|c_n|^{1/n} \sim n^{\alpha_1/\alpha_2-1} \rightarrow 0$ for $n \rightarrow \infty$, and therefore expansion (5.11) is valid at all t . The leading term of a series (5.11) gives for the characteristic displacement at small t ,

$$\delta x \sim \langle |x|^q \rangle^{1/q} \propto t^{1/\alpha_2}. \tag{5.12}$$

In order to get the q -th moment at large t we integrate by parts the right hand side of Eq. (5.9):

$$\langle |x|^q \rangle = \frac{2}{\pi} \Gamma(1+q) \sin\left(\frac{\pi q}{2}\right) \frac{t}{q} J(t), \tag{5.13}$$

where

$$J(t) = \int_0^\infty dk \zeta(k) e^{-t\phi(k)} \tag{5.14}$$

with

$$\zeta(k) = \frac{\phi'(k)}{k^q} = \frac{a_1 \alpha_1 k^{\alpha_1} + a_2 \alpha_2 k^{\alpha_2}}{k^{q+1}}.$$

Since at small k one has $\phi(k) \approx a_1 k^{\alpha_1}$, $\zeta(k) \approx a_1 \alpha_1 k^{\alpha_1-q-1}$, then at large t we have exactly the case of Laplace asymptotic integral, see [17], Chapter III, Theorem 7.1. For $J(t)$ one immediately obtains $J(t) = a_1^{q/\alpha_1} t^{q/\alpha_1-1} \Gamma(1 - q/\alpha_1)$ and

$$\langle |x|^q \rangle \sim \frac{2}{\pi q} (a_1 t)^{q/\alpha_1} \sin\left(\frac{\pi q}{2}\right) \Gamma(1+q) \Gamma\left(1 - \frac{q}{\alpha_1}\right), \quad q < \alpha_1. \tag{5.15}$$

For the characteristic displacement at large t we get

$$\delta x \sim \langle |x|^q \rangle^{1/q} \propto t^{1/\alpha_1}. \tag{5.16}$$

Therefore, at small times the characteristic displacement grows as t^{1/α_2} , whereas at large times it grows as t^{1/α_1} . Since $\alpha_1 < \alpha_2$, one encounters an accelerated superdiffusion.

5.2. The “modified” form

Let us now consider the conjugated form of the equation, namely one with an additional distributed-order fractional operator in the l.h.s.:

$$\int_0^2 d\alpha w(\alpha) l^{2-\alpha} \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}} \frac{\partial p}{\partial t} = -K \frac{\partial^2 p}{\partial x^2}, \quad p(x, 0) = \delta(x), \tag{5.17}$$

where l , K and w have the same meaning as in Eq. (5.2). Setting $w(\alpha) = \delta(\alpha - \alpha_0)$ we arrive at Eq. (3.8) where $K_\alpha \equiv K_{\alpha_0} = Kl^{\alpha_0-2}$. The equation for the characteristic function of the solution of Eq. (5.17) reads:

$$f(k, t) = \exp \left[-\frac{Kt/l^2}{\int_0^2 d\alpha w(\alpha)(|k|l)^\alpha} \right] \tag{5.18}$$

(compare with Eq. (5.3)). Note that the normalization condition, Eq. (5.4), is fulfilled.

As we did it throughout the present article, let us consider a particular case of

$$w(\alpha) = A_1\delta(\alpha - \alpha_1) + A_2\delta(\alpha - \alpha_2), \tag{5.19}$$

$0 < \alpha_1 < \alpha_2 \leq 2$. From Eq. (5.18) we get for the characteristic function,

$$f(k, t) = \exp \left[-\frac{t}{\frac{a_1}{|k|^{\alpha_1}} + \frac{a_2}{|k|^{\alpha_2}}} \right], \tag{5.20}$$

where $a_{1,2} = A_{1,2}l^{2-\alpha_{1,2}}/K$. The proof of non-negativity of the PDF given by inverse Fourier transform of Eq. (5.20) follows along the same lines as for the accelerating subdiffusion case.

The q -th moment is given by Eq. (5.9), where

$$\phi(k) = \frac{1}{a_1k^{-\alpha_1} + a_2k^{-\alpha_2}}. \tag{5.21}$$

We insert Eq. (5.21) into Eq. (5.9) and pass to a new variable $\xi = tk^{\alpha_1}/\alpha_1$. For the integral over k we get

$$\int_0^\infty dk \dots = \left(\frac{t}{a_1}\right)^{q/\alpha_1} \frac{1}{\alpha_1} \times \int_0^\infty d\xi \xi^{-1-q/\alpha_1} \left\{ 1 - \exp \left[-\frac{\xi}{1 + \frac{a_2}{a_1}\xi^{1-\frac{\alpha_2}{\alpha_1}} \left(\frac{t}{a}\right)^{\frac{\alpha_2}{\alpha_1}-1}} \right] \right\}. \tag{5.22}$$

At small t we can neglect the term with t in the denominator in square brackets. Therefore,

$$\langle |x|^q \rangle = C(q, \alpha_1) \left(\frac{t}{\alpha_1}\right)^{q/\alpha_1}, \quad q < \alpha_1, \tag{5.23}$$

where $C(q, \alpha_1)$ is given by Eq. (3.13), and

$$\delta x \sim \langle |x|^q \rangle^{1/q} \propto t^{1/\alpha_1} \quad (5.24)$$

for $t \rightarrow 0$.

In order to get the q -th moment at large t , we again use the Laplace method. Turn to Eq. (5.13), where $\phi(k)$ is given by Eq. (5.21) and

$$\zeta(k) = \frac{\phi'(k)}{k^q} = \frac{1}{k^{1+q}} \frac{a_1 \alpha_1 k^{-\alpha_1} + a_2 \alpha_2 k^{-\alpha_2}}{(a_1 k^{-\alpha_1} + a_2 k^{-\alpha_2})^2}. \quad (5.25)$$

Since at small k we have $\phi(k) \approx k^{\alpha_2}$, $\zeta(k) \approx (\alpha_2/a_2) k^{\alpha_2-q-1}$, then at large t we again have exactly the case of Laplace asymptotic integral [17]. For $J(t)$ we have immediately $J(t) = a_2^{-q/\alpha_1} t^{q/\alpha_2-1} \Gamma(1 - q/\alpha_2)$, and

$$\langle |x|^q \rangle \approx C(q, \alpha_2) \left(\frac{t}{\alpha_2} \right)^{q/\alpha_1}, \quad q < \alpha_1, \quad (5.26)$$

where $C(q, \alpha)$ is given by Eq. (3.13), and

$$\delta x \sim \langle |x|^q \rangle^{1/q} \propto t^{1/\alpha_2} \quad (5.27)$$

for $t \rightarrow \infty$.

Therefore, at small times the characteristic displacement grows as t^{1/α_1} , whereas at large times it grows as t^{1/α_2} . Since $\alpha_1 < \alpha_2$, we encounter retarding superdiffusion.

6. Conclusions

Distributed-order diffusion equations generalize the approach based on fractional diffusion equations to processes lacking temporal scaling. The typical forms of fractional diffusion equations can be classified with respect to the position of the fractional derivative in time/coordinate instead of or in addition to the first and second derivatives in the classical Fick's form. In the present paper we considered the corresponding forms of distributed-order fractional diffusion equations and elucidate the effects described by different classes of such equations. We show that equations with the distributed-order fractional operator replacing the corresponding whole-number derivative describe processes getting more anomalous in course of the time, *i.e.* the accelerating superdiffusion or retarded subdiffusion. On the opposite, equations with additional fractional operators on the "wrong" side of the diffusion equation describe processes getting less anomalous, *i.e.* retarded superdiffusion and accelerating subdiffusion.

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