ON EIGHT KINDS OF SPINORS

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Dedicated to Andrzej Staruszkiewicz on the occasion of his 65th birthday

The eight inequivalent spinorial double covers of the full Lorentz group $\mathbb{L}$ are described explicitly. Two among them include the antilinear representations of space and time reflections on two-component spinors, discovered in 1976 by Staruszkiewicz. The group of all inequivalent central extensions of $\mathbb{L}$ by $\mathbb{Z}_2$ has 16 elements and contains an eight-element subgroup of 'vectorial' double covers, characterized by the property of being trivial when restricted to the proper Lorentz group.

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1. Introduction

In a remarkable paper *On two kinds of spinors*, Andrzej Staruszkiewicz [1] described two representations in the space of two-component spinors that differ by their behavior under reflections. His approach was motivated by the spinor flags of Roger Penrose, based on the relation between spinors and null bivectors, and led to the conclusion that there is a kind of spinors that cannot be conveniently represented on the Riemann sphere of complex numbers.

Transformation properties of spinors under space and time reflections interested physicists even before the discovery of parity violation in weak interactions. As early as 1937, Racah [2] noticed that the action of space reflections on Dirac spinors could be represented either by the matrices $\pm\gamma_0$ or by $\pm i\gamma_0$. Yang and Tiomno [3] extended Racah's observation and considered possible physical consequences of the distinction between spinors and 'pseudospinors'. Shirokov [4] pointed out that there may be as many as eight double covers of the full Lorentz group $\mathbb{L} = \mathrm{O}_{1,3}$, corresponding to the possible choices of signs $\lambda, \mu, \nu \in \{+, -\}$ in the relations

$$PT = \lambda TP, \quad P^2 = \mu 1, \quad T^2 = \nu 1\,, \tag{1}$$

where P and T are the elements of the group, covering the space and time reflections, respectively. He also considered the possibility of using different covers of the Lorentz group for the description of elementary particles. Dąbrowski [5] extended Shirokov's ideas to the general pseudo-orthogonal group $\mathrm{O}_{m,n}$ and outlined the construction of the double covers. Chamblin [6] determined the topological obstructions, in terms of Stiefel–Whitney classes, to the existence of the corresponding generalized pin structures on manifolds. None of those authors seems to have shown that there are precisely eight double covers of $\mathrm{O}_{m,n}$ for $m, n \geqslant 1$. Moreover, for a long time, the only explicitly known, non-trivial double covers were given by the groups $\mathrm{Pin}_{m,n}$ and $\mathrm{Pin}_{n,m}$. Some time ago, I determined the number of inequivalent central extensions of $\mathrm{O}_{m,n}$ by the two-element group $\mathbb{Z}_2 = \{1, -1\}$. To my surprise, it turned out that $\mathbb{L}$ has as many as 16 such inequivalent extensions [7]. In agreement with Shirokov and Dąbrowski, only eight among them are *spinorial* in the sense that, by restriction to the proper Lorentz group $\mathbb{L}^0 = \mathrm{SO}^0_{1,3}$, the extensions reduce to

$$\mathbb{Z}_2 \to \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\rho} \mathbb{L}^0. \tag{2}$$

The four spinorial extensions characterized by anticommuting P and $T (\lambda = -)$ are *Cliffordian*: the corresponding double covers of $\mathbb{L}$ can be realized as subgroups of the Clifford group, as defined by Chevalley [8], associated with the complexified Minkowski space $\mathbb{C}^4$. Among them are the groups $\mathrm{Pin}_{1,3}$ and $\mathrm{Pin}_{3,1}$. The eight non-spinorial extensions are *vectorial*: by restriction to $\mathbb{L}^0$, they trivialize to

$$\mathbb{Z}_2 \to \mathbb{L}^0 \times \mathbb{Z}_2 \xrightarrow{\mathrm{pr}_1} \mathbb{L}^0\,.$$

In this paper, I undertake the modest task of describing explicitly the eight inequivalent spinorial double covers of $\mathbb{L}$, relating them to the work of Staruszkiewicz, and complementing the remarks by Chamblin and Gibbons [9] on this subject. To make the paper self-contained, in the next section, I recall a few definitions from the theory of groups, their representations and extensions. Section 3 is the main part of the paper. It contains a description of the spinorial double covers of $\mathbb{L}$ and of their spinorial representations. Two among them are of the real type: these are the ones discovered by Staruszkiewicz. The last Section contains remarks on the vectorial double covers and on the structure of the group $\mathrm{Ext}\,\mathbb{L}$.

2. Preliminaries on groups

2.1. Central extensions

Mathematicians define a *central extension* of a (topological) group L by the Abelian group $\mathbb{Z}_2$ to be an exact sequence of (continuous) group homomorphisms,

$$\mathbb{Z}_2 \xrightarrow{i} H \xrightarrow{p} L\,, \tag{3}$$

such that i is injective, p is surjective and $i(\mathbb{Z}_2)$ is contained in the center of the group H. One identifies $\mathbb{Z}_2$ with its image by i so that $\pm 1 \in H$. Another extension

$$\mathbb{Z}_2 \xrightarrow{i'} H' \xrightarrow{p'} L$$

is said to be *equivalent* to (3) whenever there is an isomorphism of groups $j : H \to H'$ such that $j \circ i = i'$ and $p' \circ j = p$. There is always the trivial extension given as the direct product,

$$\mathbb{Z}_2 \to L \times \mathbb{Z}_2 \xrightarrow{\mathrm{pr}_1} L\,.$$

It is often convenient to say that the group H, appearing in (3), is the extension of L by $\mathbb{Z}_2$ or, simply, a *double cover* of L. Note, however, that the groups H and H' may be isomorphic without the extensions being equivalent, so that there is involved here an abuse of the language requiring considerable attention; see Section 2.2 for an example.

The set $\mathrm{Ext}\, L$ of equivalence classes of all extensions of L by $\mathbb{Z}_2$ has the structure of an Abelian group. Namely, the composition of the extensions

$$\mathbb{Z}_2 \xrightarrow{i_\alpha} H_\alpha \xrightarrow{p_\alpha} L, \quad \alpha = 1, 2\,,$$

is an extension (3), defined as follows. Let

$$\tilde{H} = \{(h_1, h_2) \in H_1 \times H_2 \,|\, p_1(h_1) = p_2(h_2)\}\,. \tag{4}$$

The injection $\mathbb{Z}_2 \to \tilde{H}$, given by $\pm 1 \mapsto (\pm 1, \pm 1)$, makes $\mathbb{Z}_2$ into a normal subgroup of $\tilde{H}$; let H be the resulting quotient group: $[(h_1, h_2)] = [(h_1', h_2')] \in H$ whenever either $h_1' = h_1$ and $h_2' = h_2$ or $h_1' = -h_1$ and $h_2' = -h_2$. The map $p : H \to L$ given by $p([(h_1, h_2)]) = p_1(h_1)$ is a surjective homomorphism and its kernel is the subgroup of H generated by the element $[(-1, 1)] \in H$. This extension is denoted here by $H_1 \bullet H_2$. One easily checks that $H_2 \bullet H_1$ is an extension equivalent to $H_1 \bullet H_2$ and that so defined composition of (equivalence classes of) extensions is associative. The trivial extension, denoted in this context by $\boldsymbol{I}$, is the neutral element ($H \bullet \boldsymbol{I} = H$), and $H \bullet H = \boldsymbol{I}$ for every extension H, so that $\mathrm{Ext}\, L$ is indeed an Abelian group with composition of elements given by $\bullet$.

According to [7], the group $\operatorname{Ext} \mathrm{O}_{m,n}$ is isomorphic to the $l(m,n)$-fold direct product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where

$$l(m,0) = 2,\ l(1,1) = 2,\ l(1,n) = 4,\ l(m,n) = 5 \ \text{ for } \ m,n > 1\,.$$

For $n > 1$, the connected groups SO_n and $\mathrm{SO}^0_{1,n}$ have each only one non-trivial double cover given by the connected component of the spin group.

2.2. The eight double covers of $\mathbb{Z}_2 \times \mathbb{Z}_2$

The group $\mathrm{O}_1 = \mathbb{Z}_2$ has two double covers: the trivial one and

$$\mathbb{Z}_4 = \{\pm 1, \pm \mathrm{i}\} \to \mathbb{Z}_2, \quad \text{where} \quad \mathrm{i} = \sqrt{-1}\,.$$

The group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \pi\} \times \{1, \tau\}$ has eight inequivalent double covers that can be described as follows. Consider the group

$$H_{\lambda,\mu,\nu} = \{\pm 1, \pm P, \pm T, \pm PT\}\,,$$

where the element -1 is central, $(-1)^2 = 1$ and, for every triple of signs (λ, μ, ν), the elements P and T are subject to the relations (1). The covering homomorphism is

$$\eta : H_{\lambda,\mu,\nu} \to \{1, \pi\} \times \{1, \tau\}, \quad \eta(-1) = 1,\ \eta(P) = \pi,\ \eta(T) = \tau\,, \tag{5}$$

and the groups are:

$\lambda\ \mu\ \nu$	$H_{\lambda,\mu,\nu}$	
$+ + +$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$= \{1, -1\} \times \{1, P\} \times \{1, T\}$
$+ + -$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$= \{1, P\} \times \{\pm 1, \pm T\}$
$+ - +$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$= \{1, T\} \times \{\pm 1, \pm P\}$
$+ - -$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$= \{1, PT\} \times \{\pm 1, \pm P\}$
$- + +$	D_4	$P^2 = 1,\ T^2 = 1,\ (PT)^2 = -1$
$- + -$	D_4	$P^2 = 1,\ T^2 = -1,\ (PT)^2 = 1$
$- - +$	D_4	$P^2 = -1,\ T^2 = 1,\ (PT)^2 = 1$
$- - -$	Q	the quaternion group

The dihedral group D_4 is the group of all isometries of a square.

2.3. Homomorphisms of semi-direct products

To define the double covers, it is convenient to use the notion of a semi-direct product of groups. Let $k : H \to \operatorname{Aut} G$ be a homomorphism of a group H into the group of automorphisms of the group G. If $h, h' \in H$ and $g, g' \in G$, then the composition law

$$(h, g) \cdot (h', g') = (hh', gk(h)(g'))$$

defines the semi-direct product $H \times_k G$ of the groups H and G with respect to k. The maps $h \mapsto (h, 1)$ and $g \mapsto (1, g)$ make H and G into subgroups of $H \times_k G$. Let L be another group and consider homomorphisms

$$\eta : H \to L \quad \text{and} \quad \rho : G \to L\,.$$

One easily proves the following fact:

Lemma. *If*

$$\eta(h)\rho(g) = \rho(k(h)(g))\eta(h) \quad \textit{for every} \quad g \in G \;\; \textit{and} \;\; h \in H\,, \tag{6}$$

then the map

$$H \times_k G \to L, \quad \textit{given by} \quad (h, g) \mapsto \rho(g)\eta(h)\,,$$

defines a homomorphism from the semi-direct product to L.

2.4. Complex conjugate representations

Given a representation κ of a group H in a complex vector space S,

$$\kappa : H \to \mathrm{GL}(S)\,,$$

one can form the complex conjugate representation,

$$\bar{\kappa} : H \to \mathrm{GL}(\bar{S})\,.$$

The representations κ and $\bar{\kappa}$ are complex-equivalent if there is an isomorphism of complex vector spaces

$$C : S \to \bar{S}$$

such that

$$\overline{\kappa(h)}\, C = C\kappa(h) \quad \text{for every} \quad h \in H\,.$$

The representation κ is said to be of *complex type* if κ and $\bar{\kappa}$ are not complex-equivalent. If the representation κ is irreducible and complex-equivalent to $\bar{\kappa}$, then, by Schur's lemma, C can be normalized so that either *(i)* $\bar{C}C = -\,\mathrm{id}_S$ (*quaternionic type*) or *(ii)* $\bar{C}C = \mathrm{id}_S$ (*real type*).

In the last case, the real vector space

$$\mathrm{Re}\, S = \{\psi \in S \,|\, \bar{\psi} = C\psi\} \tag{7}$$

is of real dimension equal to the complex dimension of S. For every $h \in H$ one then has $\kappa(h)\,\mathrm{Re}\, S \subset \mathrm{Re}\, S$. The representation κ is given by real matrices in $\mathrm{Re}\, S$.

In spinor algebra, the intertwiner C defines the charge conjugate $C^{-1}\bar{\psi}$ of the spinor ψ. The equivalence of the representations κ and $\bar{\kappa}$ is essential for the construction, from complex spinors, of real 'covariant' quantities, such as currents. If κ is of real type, then the elements of $\mathrm{Re}\, S$ are called Majorana spinors.

3. Spinorial double covers of the full Lorentz group

3.1. Preliminaries on the Lorentz group and spinors

The full Lorentz group $\mathbb{L}$, considered as a manifold, has four connected components: *(i)* the proper Lorentz group $\mathbb{L}^0$, *(ii)* a component consisting of products of all proper Lorentz transformations by the space reflection $\pi(t,x,y,z) = (t,-x,-y,-z)$, *(iii)* a component consisting of products of all proper Lorentz transformations by the time reflection $\tau(t,x,y,z) = (-t,x,y,z)$, and *(iv)* a component consisting of products of all proper Lorentz transformations by the total reflection $\pi\tau$.

Let g be a complex 2 by 2 matrix. Recall that

$$\text{if} \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{then} \quad g\varepsilon g^{\mathrm{T}} = \varepsilon \det g\,,$$

where g^{T} denotes the transpose of g. Complex conjugation is denoted with a bar and $g^\dagger = \bar{g}^{\mathrm{T}}$. If $X = (t,x,y,z) \in \mathbb{R}^4$, then the matrix

$$\sigma(X) = \begin{pmatrix} t+z & x-\mathrm{i}y \\ x+\mathrm{i}y & t-z \end{pmatrix}$$

is Hermitian and $\det\sigma(X) = t^2 - x^2 - y^2 - z^2$. Let $g \in \mathrm{SL}_2(\mathbb{C})$; the equation

$$\sigma(\rho(g)X) = g\sigma(X)g^\dagger$$

defines $\rho(g)$ and can be used to justify the exact sequence (2) which establishes $\mathrm{SL}_2(\mathbb{C})$ as the connected spin group $\mathrm{Spin}^0_{1,3}$. The map $g \mapsto \bar{g}$ is an automorphism of $\mathrm{SL}_2(\mathbb{C})$ that is not inner. The equations

$$\sigma(\pi(X)) = \varepsilon\,\overline{\sigma(X)}\,\varepsilon^{-1} \quad \text{and} \quad \sigma(\tau(X)) = -\varepsilon\,\overline{\sigma(X)}\,\varepsilon^{-1} \tag{8}$$

are important for the construction of spinorial double covers and their representations.

3.2. Construction of the spinorial double covers

The eight spinorial double covers of $\mathbb{L}$ are labeled by the triples (λ,μ,ν) of $+$ and $-$ signs, as in (1), and denoted here by $\mathrm{Pin}_{\lambda,\mu,\nu}$, so that there are exact sequences

$$\mathbb{Z}_2 \to \mathrm{Pin}_{\lambda,\mu,\nu} \xrightarrow{\rho_{\lambda\mu\nu}} \mathbb{L}\,, \quad \lambda,\mu,\nu \in \{+,-\}\,.$$

This notation differs mainly graphically from that of references [5,6] and [9]. Note that Dąbrowski's signs are related to mine by $a = -\mu$, $b = \nu$ and

$abc = \lambda$. Every group $\mathrm{Pin}_{\lambda,\mu,\nu}$ has four simply connected components, each doubly covering the corresponding connected component of $\mathbb{L}$. In particular, the reflections π and τ are covered, respectively, by the pairs $(P, -P)$ and $(T, -T)$ of elements of $\mathrm{Pin}_{\lambda,\mu,\nu}$. The element P can be continuously deformed to $-P$ and one can not give preference to one or the other as providing the representation of the action of π on spinors. Similar remarks apply to τ.

Let $H_{\lambda,\mu,\nu}$ be the eight-element group described in Section 2.2. Define the homomorphism

$$k : H_{\lambda,\mu,\nu} \to \mathrm{Aut}\,\mathrm{SL}_2(\mathbb{C})$$

so that

$$k(-1)g = g, \quad k(P)g = k(T)g = \varepsilon\bar{g}\varepsilon^{-1} \tag{9}$$

for every $g \in \mathrm{SL}_2(\mathbb{C})$. Let $I \in \mathrm{SL}_2(\mathbb{C})$ be the unit matrix. The map $-1 \mapsto (-1, -I)$ makes $\mathbb{Z}_2$ into a normal subgroup of $H_{\lambda,\mu,\nu} \times_k \mathrm{SL}_2(\mathbb{C})$. Following Dąbrowski (see p. 11 in [5]), one defines

$$\mathrm{Pin}_{\lambda,\mu,\nu} = (H_{\lambda,\mu,\nu} \times_k \mathrm{SL}_2(\mathbb{C}))/\mathbb{Z}_2$$

so that $[(h, g)] = [(-h, -g)]$ in $\mathrm{Pin}_{\lambda,\mu,\nu}$. The covering homomorphism is given by

$$\rho_{\lambda\mu\nu}([(h, g)]) = \rho(g)\eta(h)\,, \tag{10}$$

where η is as in (5). To check that equation (10) defines a homomorphism, one uses the Lemma of Section 2.3 and (8). The kernel of $\rho_{\lambda\mu\nu}$ is seen to be $\mathbb{Z}_2$ generated by the element $[(1, -I)]$.

3.3. Spinorial representations of the groups $\mathrm{Pin}_{\lambda,\mu,\nu}$

In view of (8), to represent the groups $\mathrm{Pin}_{\lambda,\mu,\nu}$ complex-linearly and faithfully upon restriction to $\mathrm{SL}_2(\mathbb{C})$, it is necessary to double the dimension of the space of (Weyl) spinors. One puts

$$\gamma(X) = \begin{pmatrix} 0 & \sigma(X)\varepsilon \\ -\sigma(X)^{\mathrm{T}}\varepsilon & 0 \end{pmatrix} \in \mathbb{C}(4)$$

so that

$$\gamma(X)^2 = (t^2 - x^2 - y^2 - z^2)\,\mathrm{id}\,, \quad \text{where} \quad \mathrm{id} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in \mathrm{GL}_4(\mathbb{C})\,.$$

Embedding $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{GL}_4(\mathbb{C})$ by putting

$$\kappa(g) = \begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix} \quad \text{for } g \in \mathrm{SL}_2(\mathbb{C})\,,$$

one obtains

$$\gamma(\rho(g)X) = \kappa(g)\gamma(X)\kappa(g)^{-1} .$$

The matrix $\kappa(g)$ acts on the *Dirac spinor*

$$\psi = \begin{pmatrix} u \\ \bar{v} \end{pmatrix} \in \mathbb{C}^4 . \tag{11}$$

The components of its Weyl (chiral) parts $\bar{v}$ and u are traditionally labeled by dotted (Penrose: primed) and undotted (unprimed) indices, respectively.

To complete the *spinorial representation*

$$\kappa_{\lambda\mu\nu} : \mathrm{Pin}_{\lambda,\mu,\nu} \to \mathrm{GL}_4(\mathbb{C}), \quad \kappa_{\lambda\mu\nu}(g) = \kappa(g) \quad \text{for} \quad g \in \mathrm{SL}_2(\mathbb{C}) ,$$

one has to specify the matrices $\kappa_{\lambda\mu\nu}(P)$ and $\kappa_{\lambda\mu\nu}(T) \in \mathrm{GL}_4(\mathbb{C})$, so as to satisfy

$$\kappa_{\lambda\mu\nu}(PT) = \lambda\kappa_{\lambda\mu\nu}(TP), \quad \kappa_{\lambda\mu\nu}(P)^2 = \mu\,\mathrm{id}, \quad \kappa_{\lambda\mu\nu}(T)^2 = \nu\,\mathrm{id} ,$$

and, in view of (6) and (9), the equation

$$\kappa_{\lambda\mu\nu}(P)\kappa(g) = \kappa(\varepsilon\bar{g}\varepsilon^{-1})\kappa_{\lambda\mu\nu}(P) ,$$

and a similar equation with P replaced by T.

Let

$$E_\pm = \begin{pmatrix} 0 & \varepsilon \\ \pm\varepsilon & 0 \end{pmatrix} ,$$

so that $E_\pm^2 = \mp\,\mathrm{id}$. The matrices E_+ and E_- anticommute.

For every pair (μ, ν) of signs, define $p_{\mu\nu}$ and $t_{\mu\nu} \in \{1, \mathrm{i}\}$,

$$\begin{array}{rcccc} \mu\,\nu = & ++ & +- & -+ & -- \\ p_{\mu\nu} = & 1 & 1 & \mathrm{i} & \mathrm{i} \\ t_{\mu\nu} = & \mathrm{i} & 1 & \mathrm{i} & 1 \end{array} \tag{12}$$

and put

$$\kappa_{-\mu\nu}(P) = p_{\mu\nu}E_-, \qquad \kappa_{-\mu\nu}(T) = t_{\mu\nu}E_+ \tag{13}$$

$$\kappa_{+\mu\nu}(P) = p_{\mu\nu}E_-, \qquad \kappa_{+\mu\nu}(T) = -\mathrm{i}t_{\mu\nu}E_-. \tag{14}$$

For the Cliffordian groups ($\lambda = -$), the representation $\rho_{-\mu\nu}$, defining the action of $\mathrm{Pin}_{-,\mu,\nu}$ on vectors, can be described by

$$\gamma(\rho_{-\mu\nu}(a)X) = \kappa_{-\mu\nu}(a)\gamma(X)\kappa_{-\mu\nu}(a)^{-1} \quad \text{for every} \quad a \in \mathrm{Pin}_{-,\mu,\nu} \tag{15}$$

or by the definition (2).

In the non-Cliffordian case ($\lambda = +$), there is no analog of formula (15). The spinorial representations of the groups Pin_{+++} and Pin_{+--} are not faithful.

To determine the type of the spinorial representations of the groups $\mathrm{Pin}_{\lambda,\mu,\nu}$, one notes that if there is an intertwiner C connecting the representation with its complex conjugate, then from $\overline{\kappa(g)}\,C = C\kappa(g)$, $g \in \mathrm{SL}_2(\mathbb{C})$, one obtains

$$C = \begin{pmatrix} 0 & \alpha I \\ \beta I & 0 \end{pmatrix}, \quad \text{where} \quad \alpha, \beta \in \mathbb{C}\,. \tag{16}$$

From this and by inspection of (12)–(14), one proves

Proposition 1. *The spinorial representations of the groups*

$\mathrm{Pin}_{\pm+-}$ *and* $\mathrm{Pin}_{\pm-+}$ *are of complex type,*

$\mathrm{Pin}_{---} = \mathrm{Pin}_{3,1}$ *and* Pin_{+--} *are of real type,*

$\mathrm{Pin}_{-++} = \mathrm{Pin}_{1,3}$ *and* Pin_{+++} *are of quaternionic type.*

The spinorial representations of the groups Pin_{+++} *and* Pin_{+--} *have a kernel generated by PT; the spinorial representations of the six other spinorial double covers are faithful.*

Whenever the spinorial representation is of real type, one can choose the intertwiner (16) to be such that $\alpha = \beta = 1$. The reality condition (7) applied to the Dirac spinor (11) gives then $v = u$. Following Staruszkiewicz, using (13) and (14), one can represent space and time reflections 'anti-linearly':

in Pin_{---} by $P : u \mapsto \mathrm{i}\varepsilon\bar{u}$, $T : u \mapsto \varepsilon\bar{u}$,

in Pin_{+--} by $P : u \mapsto \mathrm{i}\varepsilon\bar{u}$, $T : u \mapsto -\mathrm{i}\varepsilon\bar{u}$.

The last line differs from the corresponding entry in Table III of [1], but the 4 by 4 matrix

$$\begin{pmatrix} (1+\mathrm{i})I & 0 \\ 0 & (1-\mathrm{i})I \end{pmatrix}$$

can be easily seen to intertwine the representation κ_{+--} and the one used by Staruszkiewicz.

4. Vectorial double covers and the structure of the group Ext $\mathbb{L}$

The eight vectorial double covers of $\mathbb{L}$ can be obtained by a construction similar to the one for the spinorial covers. One defines the homomorphism

$$l : H_{\lambda,\mu,\nu} \to \mathrm{Aut}\,\mathbb{L}^0 \ \text{ so that } \ l(-1)A = A,\ l(P)A = l(T)A = \pi A \pi$$

for every $A \in \mathbb{L}^0$. The vectorial double covers are $\mathbb{L}_{\lambda\mu\nu} = H_{\lambda,\mu,\nu} \times_l \mathbb{L}^0$. The map $\mathbb{L}_{\lambda\mu\nu} \to \mathbb{L}$ is $(h, A) \mapsto A\eta(h)$ with kernel generated by $(-1, \mathrm{id})$. Every group $\mathbb{L}_{\lambda\mu\nu}$ has eight connected components, each diffeomorphic to $\mathbb{L}^0$.

Every connected component of $\mathbb{L}$ is covered by two copies of $\mathbb{L}^0$: the vectorial extensions are topologically trivial, but among them only $\boldsymbol{I} = \mathbb{L}_{+++}$ is trivial as an extension of $\mathbb{L}$ by $\mathbb{Z}_2$.

To describe the composition law $\bullet$ of the group $\mathrm{Ext}\,\mathbb{L}$, consider the extension $H = \mathrm{Pin}_{\lambda,\mu,\nu} \bullet \mathrm{Pin}_{\lambda',\mu',\nu'}$. According to (4), if $g, g' \in \mathrm{SL}_2(\mathbb{C})$ and $[(g,g')] \in H$, then $\rho(g) = \rho(g')$ and the map $[(g,g')] \mapsto (g'g^{-1}, \rho(g)) \in \mathbb{Z}_2 \times \mathbb{L}^0$ shows that $H \to \mathbb{L}$ is a vectorial double cover. If $P \in \mathrm{Pin}_{\lambda,\mu,\nu}$ and $P' \in \mathrm{Pin}_{\lambda',\mu',\nu'}$, then $([(P,P')])^2 = ([(\mu 1, \mu' 1)]) = ([(\mu\mu' 1, 1)])$, where it is understood that signs multiply in the natural manner, $- \cdot - = +$, *etc.* This and similar computations prove

Proposition 2. *The composition law in the group* $\mathrm{Ext}\,\mathbb{L}$ *is given by*

$$\begin{aligned} \mathrm{Pin}_{\lambda,\mu,\nu} \bullet \mathrm{Pin}_{\lambda',\mu',\nu'} &= \mathbb{L}_{\lambda\lambda',\mu\mu',\nu\nu'}\,, \\ \mathrm{Pin}_{\lambda,\mu,\nu} \bullet \mathbb{L}_{\lambda',\mu',\nu'} &= \mathrm{Pin}_{\lambda\lambda',\mu\mu',\nu\nu'}\,, \\ \mathbb{L}_{\lambda,\mu,\nu} \bullet \mathbb{L}_{\lambda',\mu',\nu'} &= \mathbb{L}_{\lambda\lambda',\mu\mu',\nu\nu'}\,. \end{aligned}$$

The vectorial extensions form a subgroup of $\mathrm{Ext}\,\mathbb{L}$. *The spinorial extensions generate the group* $\mathrm{Ext}\,\mathbb{L}$.

To conclude, one can say that physicists have been wise to restrict their attention to the groups $\mathrm{Pin}_{1,3} = \mathrm{Pin}_{-++}$ and $\mathrm{Pin}_{3,1} = \mathrm{Pin}_{---}$ as these are the only double covers of $\mathbb{L}$ that admit faithful irreducible spinorial representations and allow the construction of real invariants and covariant currents.

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