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ANALYTIC CONTINUATION FORMULAE FOR THE BPZ CONFORMAL BLOCK

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Using the techniques developed by Ponsot and Teschner we derive the formulae for analytic continuation of the general 4-point conformal block.

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1. Introduction

The operator product expansion of primary fields in the standard CFT [1] can be written as

$$\phi_{\Delta_1\bar{\Delta}_1}(z,\bar{z})\phi_{\Delta_2\bar{\Delta}_2}(0,0) = \sum_p C_{12p}\, z^{\Delta_p-\Delta_1-\Delta_2}\bar{z}^{\,\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2}\,\Psi_{12p}(z,\bar{z})\,, \quad (1)$$

where for each p the descendent field $\Psi_{12p}(z,\bar{z})$ is uniquely determined by the conformal invariance. Acting on the vacuum $|\,0\,\rangle$ it generates a state $\Psi_{12p}(z,\bar{z})|\,0\,\rangle = |\psi_{12p}(z)\rangle \otimes |\bar{\psi}_{12p}(\bar{z})\rangle$ in the tensor product $\mathcal{V}_{\Delta_p}\otimes\mathcal{V}_{\bar{\Delta}_p}$ of the Verma modules with the highest weights Δ_p and $\bar{\Delta}_p$, respectively. The z dependence of each component is uniquely determined by the conformal invariance. In the "left" (holomorphic) sector one has

$$|\psi_{12p}(z)\rangle \;=\; \nu_p + \sum_{n=1}^{\infty} z^n \beta^{\,n}_{\Delta_p}\!\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]\,,$$

where ν_p is the highest weight vector in $\mathcal{V}_{\Delta_p}$ and $\beta^n_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right] \in \mathcal{V}^n_{\Delta_p} \subset \mathcal{V}_{\Delta_p}$. In the n-th level subspace $\mathcal{V}^n_{\Delta_p}$ we shall use a standard basis consisting of vectors of the form

$$\nu^n_{p,I} = L_{-I}\nu_p = L_{-i_k}\,\ldots\,L_{-i_2}L_{-i_1}\nu_p\,, \tag{2}$$

where $I = \{i_k, \ldots, i_1\}$ is an ordered ($i_k \geq \ldots \geq i_1 \geq 1$) sequence of positive integers of the length $|I| \equiv i_k + \ldots + i_1 = n$.

The conformal Ward identity for the 3-point function implies the equations

$$L_i\,\beta^n_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right] = (\Delta_p + i\Delta_1 - \Delta_2 + n - i)\beta^{n-i}_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]\,,$$

which in the basis (2) take the form

$$\sum_{|J|=n}\Big[G_\Delta\Big]_{IJ}\beta^n_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^J = \gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_I\,, \tag{3}$$

where $\Big[G_\Delta\Big]_{IJ} = \langle\nu_I, \nu_J\rangle$ is the Gram matrix of the standard symmetric bilinear form in $\mathcal{V}_{\Delta_p}$ and

$$\begin{aligned}\gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_I = &(\Delta + i_k\Delta_1 - \Delta_2 + i_{k-1}+\ldots+i_1)\ldots\\ &\times(\Delta + i_2\Delta_1 - \Delta_2 + i_1)(\Delta + i_1\Delta_1 - \Delta_2)\,.\end{aligned} \tag{4}$$

For all values of the variables c and Δ for which the Gram matrices are invertible the equations (3) admit unique solutions

$$\beta^n_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]^I = \sum_{|J|=n}\Big[G_\Delta\Big]^{IJ}\gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_J\,.$$

In this range the 4-point conformal block is defined as a formal power series [1]

$$\begin{aligned}\mathcal{F}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) &= z^{\Delta-\Delta_2-\Delta_1}\left(1 + \sum_{n=1}^{\infty} z^n\mathcal{F}^n_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\right),\\ \mathcal{F}^n_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \sum_{|I|=|J|=n}\gamma_\Delta\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_I\Big[G_\Delta\Big]^{IJ}\gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_J\,.\end{aligned} \tag{5}$$

One of the long standing open problems has been the calculation of the radius of convergence of this series. The basic difficulty is that no closed formula for the coefficients is known, what makes a direct analysis prohibitively difficult.

As the dimension of $\mathcal{V}^n_\Delta$ grows rapidly with n also numerical calculations by inverting the Gram matrices became very laborious for higher orders.

A more efficient method based on a recurrence relation for the coefficients was developed by Zamolodchikov [2]. It should be stressed however that the analytic properties of the conformal block with respect to its parameters which are crucial for this method are derived under the assumption that the radius of convergence is nonzero. Also some conjectured formulae of this approach are still to be rigorously proved.

The commonly accepted hypothesis justified by all special cases where the conformal block can be explicitly calculated is that the series (5) converges for all $|z| < 1$. It has been recently suggested [3] to apply the free field representation of the chiral vertex operator [3, 4] to solve this problem. We assume in the present paper that the radius of convergence is 1.

Another hypothesis supported by all known examples concerns an analytic continuation of the function defined by the series (5). It states that the only singularities of the conformal block with respect to the z variable are branching points (in general of a transcendental kind) at 0, 1, and ∞ [5]. This in particular means that the conformal block is a single-valued analytic function on the universal covering of a 3-punctured Riemann sphere and can be expressed by a power series convergent in the entire domain of its analyticity [6]. A recurrence relation for calculating coefficients of this so called q-expansion [6] provides an efficient method for numerical analysis of conformal block and can be applied for testing the conformal bootstrap equations [7, 8].

In the RCFT models where the number of conformal blocks is finite the problem of analytic continuation is essentially equivalent to the problem of calculating the monodromy matrices relating conformal blocks in different channels [9–11]. A thorough analysis of the consistency conditions such matrices have to satisfy was done by Moore and Seiberg in terms of the braiding and fusion relations of chiral vertex operators [11].

The Moore–Seiberg formalism suitably generalized to the case of continuous spectrum was recently applied by Ponsot and Teschner to derive a system of functional equations for the braiding and fusion matrices of the Liouville theory. They also constructed explicit solutions to these equations by means of the representation theory of $\mathcal{U}_q(\mathrm{sl}(2,\mathbb{R}))$ [12, 13]. The exact form of the braiding and fusion matrices can be also derived by direct calculations of the exchange relation of chiral vertex operators in the free field representation [3, 4] (see also [14] for an earlier construction).

In the present note we use the results of [3, 4, 12, 13, 15, 16] to derive the formulae for the analytic continuation of the BPZ conformal block. To this end a choice of cuts for the function (5) has to be made. We pick the cut starting at the origin to run along the negative real axis, and the one starting at 1 to run along the positive real axis. With this choice the formulae read

$$\begin{aligned}\mathcal{F}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) &= z^{-2\Delta_2}\frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_u\, e^{-\varepsilon\pi i(\Delta_1+\Delta_4-\Delta_s-\Delta_u)}\\ &\quad\times B_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_1 & \Delta_4\end{smallmatrix}\right]\left(\frac{1}{z}\right),\end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{F}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) &= (1-z)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1}\, e^{-\varepsilon\pi i(\Delta_s-\Delta_2-\Delta_1)}\\ &\quad\times\mathcal{F}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_1\\ \Delta_4 & \Delta_2\end{smallmatrix}\right]\left(\frac{z}{z-1}\right),\end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{F}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) &= z^{\Delta_4-\Delta_3-\Delta_2-\Delta_1}\frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, e^{\varepsilon\pi i(\Delta_t-\Delta_3-\Delta_2)}\\ &\quad\times B_{\alpha_s\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{smallmatrix}\right]\mathcal{F}_{\Delta_t}\left[\begin{smallmatrix}\Delta_3 & \Delta_1\\ \Delta_2 & \Delta_4\end{smallmatrix}\right]\left(1-\frac{1}{z}\right),\end{aligned} \tag{8}$$

$$\mathcal{F}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = \frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, B_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{smallmatrix}\right]\mathcal{F}_{\Delta_t}\left[\begin{smallmatrix}\Delta_1 & \Delta_2\\ \Delta_4 & \Delta_3\end{smallmatrix}\right](1-z)\,, \tag{9}$$

where $\varepsilon = +1$ if $\arg z > 0$, $\varepsilon = -1$ if $\arg z < 0$, and we have used the standard parameterizations of the central charge and the conformal weights

$$\begin{aligned} c &= 1 + 6Q^2\,,\\ Q &= b + \frac{1}{b}\,,\\ \Delta_j &= \Delta(\alpha_j) = \alpha_j(Q-\alpha_j), \qquad j = s,\, t,\, u,\, 1, \ldots, 4. \end{aligned}$$

The first two equations are straightforward consequences of the braiding relation derived in [3, 4]. The next two are less obvious and their derivation is our main objective in this paper.

The braiding matrix calculated in the free field representation from the exchange relation of chiral vertex operators is given by [3, 4]:

$$B^{\varepsilon}_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = e^{\varepsilon\pi i(\Delta_1+\Delta_4-\Delta_s-\Delta_u)}B_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{10}$$

$$
\begin{aligned}
&B_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] \qquad\qquad (11)\\
&= \frac{\Gamma_b(\bar\alpha_4+\bar\alpha_2-\alpha_u)\Gamma_b(\alpha_4+\bar\alpha_2-\alpha_u)\Gamma_b(\bar\alpha_4-\alpha_2+\alpha_u)\Gamma_b(\alpha_4-\alpha_2+\alpha_u)}{\Gamma_b(\bar\alpha_4+\bar\alpha_3-\alpha_s)\Gamma_b(\alpha_4+\bar\alpha_3-\alpha_s)\Gamma_b(\bar\alpha_4-\alpha_3+\alpha_s)\Gamma_b(\alpha_4-\alpha_3+\alpha_s)}\\
&\times \frac{\Gamma_b(\bar\alpha_1+\bar\alpha_3-\alpha_u)\Gamma_b(\alpha_1+\bar\alpha_3-\alpha_u)\Gamma_b(\bar\alpha_1-\alpha_3+\alpha_u)\Gamma_b(\alpha_1-\alpha_3+\alpha_u)}{\Gamma_b(\bar\alpha_1+\bar\alpha_2-\alpha_s)\Gamma_b(\alpha_1+\bar\alpha_2-\alpha_s)\Gamma_b(\bar\alpha_1-\alpha_2+\alpha_s)\Gamma_b(\alpha_1-\alpha_2+\alpha_s)}\\
&\times \frac{\Gamma_b(2\alpha_s)\Gamma_b(2\bar\alpha_s)}{\Gamma_b(\bar\alpha_u-\alpha_u)\Gamma_b(\alpha_u-\bar\alpha_u)}\\
&\times \frac{1}{i}\int\limits_{i\mathbb{R}} dt\, \frac{S_b(\bar\alpha_1+t)S_b(\alpha_1+t)S_b(\bar\alpha_4-\alpha_3+\alpha_2+t)S_b(\alpha_4-\alpha_3+\alpha_2+t)}{S_b(\bar\alpha_s+\alpha_2+t)S_b(\alpha_s+\alpha_2+t)S_b(\bar\alpha_u+\bar\alpha_3+t)S_b(\alpha_u+\bar\alpha_3+t)}\,,
\end{aligned}
$$

where $\Gamma_b(z)$ is the Barnes double gamma function, $S_b(z) \equiv \Gamma_b(z)/\Gamma_b(Q-z)$[1], and we have use the abbreviated notation $\bar\alpha \equiv Q-\alpha$. For $\alpha_j \in \frac{Q}{2}+i\mathbb{R}$, $j = s, t, u, 1, \ldots, 4$ corresponding to the spectrum of the Liouville theory the integrand in (8) has simple poles on the imaginary axis. The contour of integration is located to the left of all these poles.

The matrix $B_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ is related to the fusion matrix for the normalized chiral vertex operators [15] by the simple exchange $\alpha_1 \leftrightarrow \alpha_2$ of its parameters

$$
B_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = F_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\bar\alpha_4 & \alpha_2\\ \bar\alpha_3 & \alpha_1\end{smallmatrix}\right] = F_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{smallmatrix}\right].
$$

It is symmetric with respect to the exchange of column and rows, as well as with respect to the change $\alpha_j \to Q-\alpha_j$ in each α_j separately [12, 13, 15, 16]. The latter property means that the matrices (10), (11) depend only on conformal weights. An explicit derivation of these properties along the line mentioned in [16] is presented in Appendix B.

Both the formulae (6)–(9) and the expressions (10), (11) admit an analytic continuation to generic values of α_j. We present a special example of such continuation to a degenerate weight in Sec. 3.

Finally let us note that possible applications of the analytic continuation formulae goes beyond the Liouville theory. They are powerful tools not only for analyzing general properties of the conformal block like the conjectured analytic structure but also for explicit calculations.

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[1] Some of the properties of these special functions are collected in Appendix A.

2. Derivation

We shall work with the matrix elements of the chiral vertex operators [11] rather than with the operators themselves. For a given triple $\Delta_1, \Delta_2, \Delta_3$ of conformal weights we define the matrix element of a single chiral vertex operator as a trilinear map

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0} : \mathcal{V}_{\Delta_3} \times \mathcal{V}_{\Delta_2} \times \mathcal{V}_{\Delta_1} \to \mathbb{C}$$

satisfying the following conditions[2] [3]

$$\begin{aligned} \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(L_{-n}\xi_3, \nu_2, \xi_1) &= \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \nu_2, L_n\xi_1) \\ &+ z^n(z\partial_z + (n+1)\Delta_2)\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \nu_2, \xi_1)\,, \end{aligned} \tag{12}$$

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, L_{-1}\xi_2, \xi_1) = \partial_z\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \xi_2, \xi_1)\,, \tag{13}$$

$$\begin{aligned} \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, L_n\xi_2, \xi_1) &= \sum_{k=0}^{n+1} \binom{n+1}{k}(-z)^k \Big(\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(L_{k-n}\xi_3, \xi_2, \xi_1) \\ &- \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \xi_2, L_{n-k}\xi_1)\Big)\,, \\ &\text{for } n > 1\,, \end{aligned} \tag{14}$$

$$\begin{aligned} \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, L_{-n}\xi_2, \xi_1) &= \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^k \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(L_{n+k}\xi_3, \xi_2, \xi_1) + (-1)^n \\ &\times \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^{-n+1-k} \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \xi_2, L_{k-1}\xi_1)\,, \\ &\text{for } n > 1\,, \end{aligned} \tag{15}$$

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\nu_3, \nu_2, \nu_1) = z^{\Delta_3-\Delta_2-\Delta_1}\,, \tag{16}$$

where ν_i is the highest weight state in $\mathcal{V}_{\Delta_i}$ $(i = 1, 2, 3)$.

The form $\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}$ is uniquely determined by the properties above. In particular, for L_0-eingenstates $L_0|\xi_i\rangle = \Delta_i(\xi_i)|\xi_i\rangle,\ i = 1, 2, 3$, one has

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,z\;\,0}(\xi_3, \xi_2, \xi_1) = z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\,1\;\,0}(\xi_3, \xi_2, \xi_1)\,, \tag{17}$$

[2] These are just the well known conditions for the chiral vertex operator written in terms of its matrix elements.

and

$$
\begin{aligned}
\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\;1\;\;\;0}(\nu_{3,I},\nu_2,\nu_1) &= \gamma_{\Delta_3}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_I ,\\
\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\;1\;\;\;0}(\nu_{3},\nu_2,\nu_{1,I}) &= \gamma_{\Delta_1}\begin{bmatrix}\Delta_2\\ \Delta_3\end{bmatrix}_I ,\\
\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\;1\;\;\;0}(\nu_{3},\nu_{2,I},\nu_1) &= (-1)^{|I|}\gamma_{\Delta_2}\begin{bmatrix}\Delta_1\\ \Delta_3\end{bmatrix}_I ,
\end{aligned}
\tag{18}
$$

for all vectors $\nu_{i,I}$ of the form (2).

In order to simplify notation we introduce a graphic representation of the form ρ and the matrix elements of two possible compositions of the chiral vertex operators

$$
\begin{array}{ccccc} & & 2 & & \\ & & | & & \\ 3 & \text{---} & z & \text{---} & 1 \end{array}
\;\equiv\; \rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\;z\;\;\;0}(\nu_3,\nu_2,\nu_1)\,,
$$

$$
\begin{array}{ccccccccc} & & 3 & & & & 2 & & \\ & & | & & & & | & & \\ 4 & \text{---} & z_3 & \text{---} & s & \text{---} & z_2 & \text{---} & 1 \end{array}
\;\equiv
$$

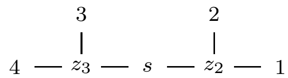
$$
\sum_{I,J}\rho^{\Delta_4\,\Delta_3\,\Delta_s}_{\infty\;\;z_3\;\;0}(\nu_4\,,\,\nu_3\,,\,\nu_{s,I})\,[G_{\Delta_s}]^{IJ}\rho^{\Delta_s\,\Delta_2\,\Delta_1}_{\infty\;\;z_2\;\;0}(\nu_{s,J}\,,\,\nu_2\,,\,\nu_1)\,,
$$

$$
\begin{array}{ccccc} & & 3 & & \\ & & | & & \\ t & \text{---} & z_{32} & \text{---} & 2 \\ | & & & & \\ z_2 & \text{---} & 1 & & \end{array}
\quad\text{with } 4 \text{---} z_2 \;\equiv
$$

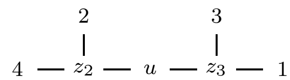
$$
\sum_{I,J}\rho^{\Delta_4\,\Delta_t\,\Delta_1}_{\infty\;\;z_2\;\;0}(\nu_4\,,\,\nu_{t,I}\,,\,\nu_1)\,[G_{\Delta_t}]^{IJ}\rho^{\Delta_t\,\Delta_3\,\Delta_2}_{\infty\;\;z_{32}\;\;0}(\nu_{t,J}\,,\,\nu_3\,,\,\nu_2)\,,
$$

where the abbreviation $z_{32} = z_3 - z_2$ has been used. Let us note that the compositions are well defined if the intermediate conformal weights are non-degenerate.

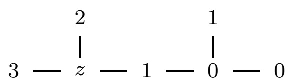
Our basic tool in further considerations is the braiding relation obtained in [3, 4]



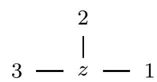
$$= \frac{1}{2i} \int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_u \; B^{\varepsilon_{32}}_{\alpha_s \alpha_u}\!\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right] \tag{19}$$

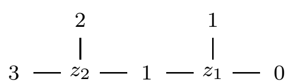


where $\varepsilon_{32} = +1$ if $\arg z_{32} > 0$, $\varepsilon_{32} = -1$ if $\arg z_{32} < 0$. We shall also need the formulae for the coupling to the vacuum ($\Delta_0 = 0$)

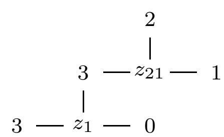


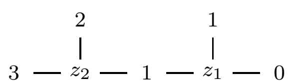
$$= \tag{20}$$



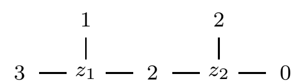


$$= \tag{21}$$





$$= \Omega^{\varepsilon_{21}}_{321} \tag{22}$$



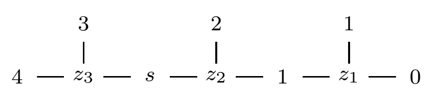
where

$$\Omega^{\varepsilon_{21}}_{321} = e^{\varepsilon_{21}\pi i(\Delta_3 - \Delta_2 - \Delta_1)}.$$

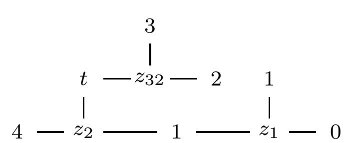
The first of these relations is a direct consequence of (17). The second follows from the fact that L_{-1} acts as the generator of translations (13) [4]. The third one can be derived as a special limiting case of the braiding relation by analyzing the analytic continuation of the formula (11) from $\alpha_1 \in \frac{Q}{2} + i\mathbb{R}$ to $\alpha_1 = 0$ [3, 4, 15].

Following [3, 4, 11] we define the generalized conformal blocks in each channel

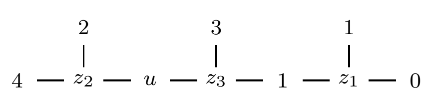
$$\mathcal{F}^s_{\Delta_s}\!\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\!(z_3, z_2, z_1) \;=$$



$$\mathcal{F}^t_{\Delta_t}\!\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\!(z_3, z_2, z_1) \;=$$



$$\mathcal{F}^u_{\Delta_u}\!\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\!(z_3, z_2, z_1) \;=$$



One can easily calculate their relations to the BPZ conformal block (5) using (17), (18) and (20)

$$\mathcal{F}^{s}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3,z_2,z_1) = (z_3-z_1)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1} \times \mathcal{F}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\left(\frac{z_2-z_1}{z_3-z_1}\right), \tag{23}$$

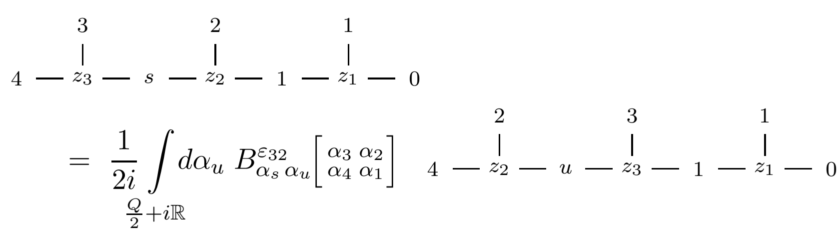
$$\mathcal{F}^{t}_{\Delta_t}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3,z_2,z_1) = e^{-\varepsilon_{32}\pi i(\Delta_t-\Delta_3-\Delta_2)}(z_2-z_1)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1} \times \mathcal{F}_{\Delta_t}\left[\begin{smallmatrix}\Delta_1 & \Delta_3\\ \Delta_4 & \Delta_2\end{smallmatrix}\right]\left(1-\frac{z_3-z_1}{z_2-z_1}\right), \tag{24}$$

$$\begin{aligned}\mathcal{F}^{u}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3,z_2,z_1) &= (z_2-z_1)^{\Delta_4-\Delta_3-\Delta_2-\Delta_1} \times \mathcal{F}_{\Delta_u}\left[\begin{smallmatrix}\Delta_2 & \Delta_3\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\left(\frac{z_3-z_1}{z_2-z_1}\right)\\ &= (z_2-z_1)^{-2\Delta_2}\mathcal{F}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_1 & \Delta_4\end{smallmatrix}\right]\left(\frac{z_3-z_1}{z_2-z_1}\right).\end{aligned} \tag{25}$$

The braiding relation (19) implies that the braiding matrix (10) can be seen as the $s-u$ monodromy matrix for generalized conformal blocks

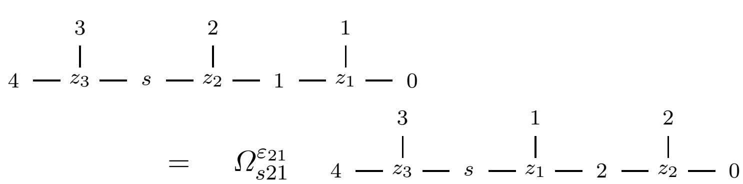
$$\mathcal{F}^{s}_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3,z_2,z_1) = \frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_u\, B^{\varepsilon_{32}}_{\alpha_s\,\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}^{u}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3,z_2,z_1),$$

or in the graphic representation



Using (10), (23), (25) and setting $(z_3,z_2,z_1)=(1,z,0)$ one gets the relation (6).

In a similar way the formula (7) can be derived from the special limiting case of the braiding relation (22)

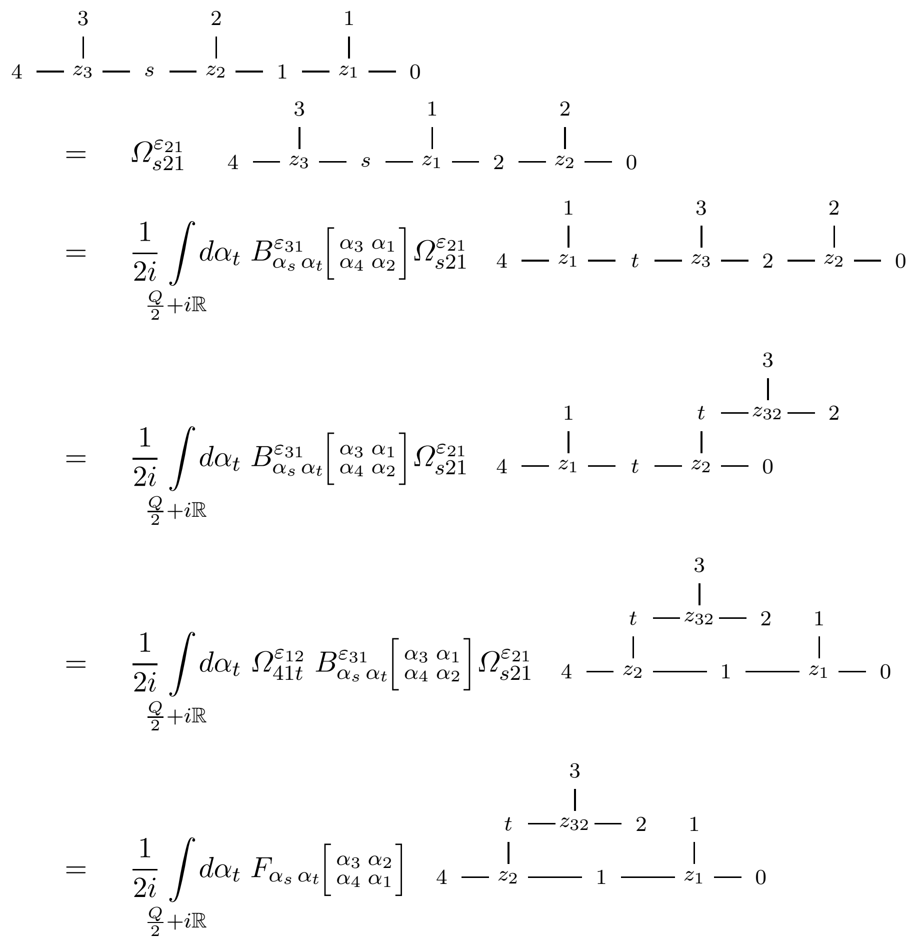


by using (23) and setting $(z_3, z_2, z_1) = (1, z, 0)$.

The fusion matrix can be defined as the $s-t$ monodromy matrix [3, 4]

$$\mathcal{F}^s_{\Delta_s}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3, z_2, z_1) = \frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, F_{\alpha_s\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}^t_{\Delta_t}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3, z_2, z_1)\,. \tag{26}$$

Using formulae (19), (21) and (22) one can find its relation to the braiding matrix [4, 11]



If we exclude the case when $\arg z_1$ lies between $\arg z_2$ and $\arg z_3$ this yields the relation mentioned in introduction

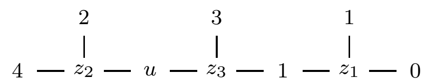
$$F_{\alpha_s\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = B_{\alpha_s\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{smallmatrix}\right]. \tag{27}$$

Setting $(z_3, z_2, z_1) = (1, z, 0)$ in (26) and using (23), (24), (27) one gets the formula (8).

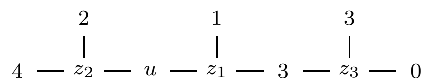
In order to derive the relation (9) we consider the $u - t$ monodromy matrix defined by

$$\mathcal{F}^{u}_{\Delta_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3, z_2, z_1) = \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_u\, A^{\varepsilon_{32}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}^{t}_{\Delta_t}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z_3, z_2, z_1)\,. \tag{28}$$

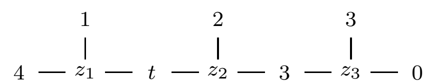
Using formulae (19), (21) and (22) one gets



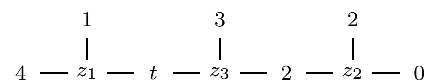
$$= \quad \Omega^{\varepsilon_{31}}_{u321}$$



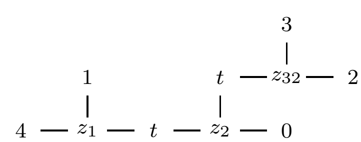
$$= \quad \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, B^{\varepsilon_{21}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\Omega^{\varepsilon_{31}}_{u31}$$



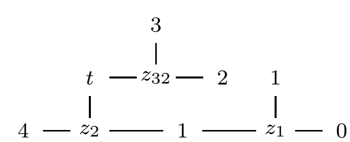
$$= \quad \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, \Omega^{\varepsilon_{23}}_{t23}\, B^{\varepsilon_{21}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\Omega^{\varepsilon_{31}}_{u31}$$



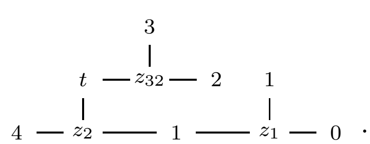
$$= \quad \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, \Omega^{\varepsilon_{23}}_{t23}\, B^{\varepsilon_{21}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\Omega^{\varepsilon_{31}}_{u31}$$



$$= \quad \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, \Omega^{\varepsilon_{12}}_{41t}\, \Omega^{\varepsilon_{23}}_{t23}\, B^{\varepsilon_{21}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]\Omega^{\varepsilon_{31}}_{u31}$$



$$= \quad \frac{1}{2i}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_t\, A^{\varepsilon_{32}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$$



With the same restriction for arguments of z_i as in the case of the fusion matrix this implies

$$A^{\varepsilon_{32}}_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = e^{\varepsilon_{32}\pi i(\Delta_3+\Delta_2-\Delta_t)} B_{\alpha_u\,\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_4 & \alpha_3\end{smallmatrix}\right]. \tag{29}$$

Setting $(z_3, z_2, z_1) = (z, 1, 0)$ in (28) and using (24), (25), (29) one gets the formula (9).

3. External weight $\Delta(1,2)$

If one of the external conformal weights $\Delta_i,\ i = 1, \ldots, 4$ corresponds to a degenerate Virasoro representation and the intermediate weight Δ_s satisfies an appropriate fusion rule then the conformal block is a solution of a certain ordinary differential equation and the analytic continuation formulae contain only finite number of conformal blocks [1].

In the simplest case of the conformal weight

$$\delta = \Delta\left(-\tfrac{b}{2}\right) = -\tfrac{3}{4}\,b^2 - \tfrac{1}{2} = \Delta(1,2)\,,$$

the conformal blocks are solutions of a second order differential equation and can be expressed in terms of hypergeometric functions [1]. For the weight δ located at $z_1 = 0$ one has

$$\begin{aligned}\mathcal{F}_{\Delta(\alpha_2-\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \delta\end{smallmatrix}\right](z) \;&=\; z^{\Delta(\alpha_2-\frac{b}{2})-\Delta_2-\delta}(1-z)^{\Delta(\alpha_4+\frac{b}{2})-\Delta_2-\Delta_3} \\ &\times {}_2F_1\Big(b(-\alpha_4+\alpha_3+\alpha_2-\tfrac{b}{2}),\, b(-\alpha_4+\bar\alpha_3+\alpha_2-\tfrac{b}{2});\, b(2\alpha_2-b);\, z\Big),\end{aligned} \tag{30}$$

$$\begin{aligned}\mathcal{F}_{\Delta(\alpha_2+\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \delta\end{smallmatrix}\right](z) \;&=\; z^{\Delta(\alpha_2+\frac{b}{2})-\Delta_2-\delta}(1-z)^{\Delta(\alpha_4+\frac{b}{2})-\Delta_2-\Delta_3} \\ &\times {}_2F_1\Big(b(-\alpha_4+\alpha_3+\bar\alpha_2-\tfrac{b}{2}),\, b(-\alpha_4+\bar\alpha_3+\bar\alpha_2-\tfrac{b}{2});\, b(2\bar\alpha_2-b);\, z\Big).\end{aligned} \tag{31}$$

Using the relation

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{32}$$

valid in the range $|\arg(1-z)| < \pi$, and taking into account our choice of cuts one gets

$$\begin{aligned}\mathcal{F}_{\Delta(\alpha_2\pm\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \delta\end{smallmatrix}\right](z) \;&=\; (1-z)^{\Delta_4-\Delta_3-\Delta_2-\delta} e^{-\varepsilon\pi i(\Delta(\alpha_2\pm\frac{b}{2})-\Delta_2-\delta)} \\ &\times \mathcal{F}_{\Delta(\alpha_2\pm\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \delta\\ \Delta_4 & \Delta_2\end{smallmatrix}\right]\left(\frac{z}{z-1}\right),\end{aligned} \tag{33}$$

where the conformal blocks with the weight δ at the location $z_2 = z$ are given by

$$\mathcal{F}_{\Delta(\alpha_1-\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \delta \\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; z^{\Delta(\alpha_1-\frac{b}{2})-\Delta_1-\delta}(1-z)^{\Delta(\alpha_3-\frac{b}{2})-\Delta_3-\delta}$$

$$\times\, {}_2F_1\Big(b(-\alpha_4+\alpha_3+\alpha_1-\tfrac{b}{2}), b(-\bar{\alpha}_4+\alpha_3+\alpha_1-\tfrac{b}{2}); b(2\alpha_1-b); z\Big),$$

$$\mathcal{F}_{\Delta(\alpha_1+\frac{b}{2})}\left[\begin{smallmatrix}\Delta_3 & \delta \\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; z^{\Delta(\alpha_1+\frac{b}{2})-\Delta_1-\delta}(1-z)^{\Delta(\alpha_3-\frac{b}{2})-\Delta_3-\delta}$$

$$\times\, {}_2F_1\Big(b(-\alpha_4+\alpha_3+\bar{\alpha}_1-\tfrac{b}{2}), b(\alpha_4+\alpha_3-\alpha_1-\tfrac{b}{2}); b(2\bar{\alpha}_1-b); z\Big).$$

Thus in the case under consideration the formula (7) can be seen as a generalization of the formula (33) for the hypergeometric functions.

This is also true for the other formulae. As an example we consider the $s-u$ monodromy (6) for the blocks (30), (31). It can be easily derived from the relation

$$F(\alpha,\beta;\gamma;z) \;=\; \frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; \frac{1}{z}\right)$$

$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \frac{1}{z}\right)$$

valid in the range $|\arg(-z)| < \pi$. One obtains

$$\mathcal{F}_{\Delta_s^\sigma}\left[\begin{smallmatrix}\Delta_3 & \Delta_2 \\ \Delta_4 & \delta\end{smallmatrix}\right](z) \;=\; z^{\Delta_4-\Delta_3-\Delta_2-\delta} \times \sum_{\tau=\pm} e^{-i\pi\varepsilon(\delta+\Delta_4-\Delta_s^\sigma-\Delta_u^\tau)}\, \mathcal{B}_{\sigma\,\tau}\;\; \mathcal{F}_{\Delta_u^\tau}\left[\begin{smallmatrix}\Delta_2 & \Delta_3 \\ \Delta_4 & \delta\end{smallmatrix}\right]\left(\frac{1}{z}\right) \tag{34}$$

$$=\; z^{-2\Delta_2}\sum_{\tau=\pm} e^{-i\pi\varepsilon(\delta+\Delta_4-\Delta_s^\sigma-\Delta_u^\tau)}\, \mathcal{B}_{\sigma\,\tau}\;\; \mathcal{F}_{\Delta_u^\tau}\left[\begin{smallmatrix}\Delta_3 & \Delta_2 \\ \delta & \Delta_4\end{smallmatrix}\right]\left(\frac{1}{z}\right),$$

$$\sigma \;=\; \pm\,,$$

where $\Delta_s^\pm = \Delta(\alpha_2 \pm \frac{b}{2})$, $\Delta_u^\pm = \Delta(\alpha_3 \pm \frac{b}{2})$, and

$$\mathcal{B}_{--} \;=\; \frac{\Gamma(b(2\alpha_2-b))\Gamma(b(\bar{\alpha}_3-\alpha_3))}{\Gamma(b(\bar{\alpha}_4-\alpha_3+\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\alpha_3+\alpha_2-\frac{b}{2}))}\,,$$

$$\mathcal{B}_{-+} \;=\; \frac{\Gamma(b(2\alpha_2-b))\Gamma(b(\alpha_3-\bar{\alpha}_3))}{\Gamma(b(\bar{\alpha}_4-\bar{\alpha}_3+\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\bar{\alpha}_3+\alpha_2-\frac{b}{2}))}\,,$$

$$\mathcal{B}_{+-} \;=\; \frac{\Gamma(b(2\bar\alpha_2-b))\Gamma(b(\bar\alpha_3-\alpha_3))}{\Gamma(b(\bar\alpha_4-\alpha_3+\bar\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\alpha_3+\bar\alpha_2-\frac{b}{2}))}\,,$$

$$\mathcal{B}_{++} \;=\; \frac{\Gamma(b(2\bar\alpha_2-b))\Gamma(b(\alpha_3-\bar\alpha_3))}{\Gamma(b(\bar\alpha_4-\bar\alpha_3+\bar\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\bar\alpha_3+\bar\alpha_2-\frac{b}{2}))}\,.$$

In order to derive the $s-u$ monodromy from the general expression (6) one needs to analytically continue the integral

$$\frac{1}{2i}\int\limits_{\frac{Q}{2}+i\mathbb{R}} d\alpha_u\; e^{-\varepsilon\pi i(\Delta_1+\Delta_4-\Delta_s-\Delta_u)}\; B_{\alpha_s\,\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\; \mathcal{F}_{\Delta_u}\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_1 & \Delta_4\end{smallmatrix}\right]\!\left(\frac{1}{z}\right) \tag{35}$$

from the "physical" values $\alpha_s,\alpha_1 \in Q/2+i\mathbb{R}$ to $\alpha_s=\alpha_2\pm\frac{b}{2}$, $\alpha_1=-\frac{b}{2}$. Let us note that for $\alpha_2,\alpha_3,\alpha_4\in Q/2+i\mathbb{R}$ the conformal block in the integrand of (35) is regular in this limit.

For $\alpha_1\in Q/2+i\mathbb{R}$ the continuation $\alpha_s\to\alpha_2\pm\frac{b}{2}$ of the braiding matrix (11) takes the form

$$B_{\alpha_2\pm\frac{b}{2},\,\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = W_\pm(\alpha_1)U_\pm(\alpha_1)I_\pm(\alpha_1)\,, \tag{36}$$

where

$$\begin{aligned} W_\pm(\alpha_1) \;=\; & \frac{\Gamma_b(2\alpha_2\pm b)\Gamma_b(2\bar\alpha_2\mp b)}{\Gamma_b(2\bar\alpha_2-\bar\alpha_1\mp\frac{b}{2})\Gamma_b(2\bar\alpha_2-\alpha_1\mp\frac{b}{2})\Gamma_b(\bar\alpha_1\pm\frac{b}{2})\Gamma_b(\alpha_1\pm\frac{b}{2})}\\ &\times\frac{1}{\Gamma_b(\bar\alpha_4+\bar\alpha_3-\alpha_2\mp\frac{b}{2})\Gamma_b(\alpha_4+\bar\alpha_3-\alpha_2\mp\frac{b}{2})}\\ &\times\frac{1}{\Gamma_b(\bar\alpha_4-\alpha_3+\alpha_2\pm\frac{b}{2})\Gamma_b(\alpha_4-\alpha_3+\alpha_2\pm\frac{b}{2})}\,, \end{aligned} \tag{37}$$

$$\begin{aligned} U_\pm(\alpha_1) \;=\; & \Gamma_b(\bar\alpha_4+\bar\alpha_2-\alpha_u)\Gamma_b(\alpha_4+\bar\alpha_2-\alpha_u)\Gamma_b(\bar\alpha_4-\alpha_2+\alpha_u)\Gamma_b(\alpha_4-\alpha_2+\alpha_u)\\ \times\; & \frac{\Gamma_b(\bar\alpha_1+\bar\alpha_3-\alpha_u)\Gamma_b(\alpha_1+\bar\alpha_3-\alpha_u)\Gamma_b(\bar\alpha_1-\alpha_3+\alpha_u)\Gamma_b(\alpha_1-\alpha_3+\alpha_u)}{\Gamma_b(\bar\alpha_u-\alpha_u)\Gamma_b(\alpha_u-\bar\alpha_u)}\,, \end{aligned} \tag{38}$$

$$I_\pm(\alpha_1) \;=\; \frac{1}{i}\int\limits_{\mathcal{C}} dt\; \frac{S_b(\bar\alpha_1+t)S_b(\alpha_1+t)S_b(\bar\alpha_4-\alpha_3+\alpha_2+t)S_b(\alpha_4-\alpha_3+\alpha_2+t)}{S_b(Q\mp\frac{b}{2}+t)S_b(2\alpha_2\pm\frac{b}{2}+t)S_b(\bar\alpha_u+\bar\alpha_3+t)S_b(\alpha_u+\bar\alpha_3+t)}\,. \tag{39}$$

The contour $\mathcal{C}$ in the integral $I_\pm(\alpha_1)$ is deformed such that the zeroes of the denominator are located to its right and all the poles of the numerator to its left.

Let us assume for a moment that the integral $I_\pm(\alpha_1)$ is regular in the limit $\alpha_1 \to -\frac{b}{2}$, *i.e.* there exists a finite limit $I_\pm(-\frac{b}{2})$ with no poles as a function of α_u. In this case the only poles of the integrand in (36) which locations depend on α_1 come from the factor $U_\pm(\alpha_1)$. One can easily verify that in the limit $\alpha_1 \to -\frac{b}{2}$ the contour of integration can be deformed such that the integral of $U_\pm I_\pm$ is finite. In this case all the integral vanish due to the factor $\Gamma_b(\alpha_1 \pm \frac{b}{2})^{-1}$ in $W_\pm$ (37).

It follows that the only contribution to the integral (35) comes from the non-regular part of the integral $I_\pm$. Such part can arise only if the contour $\mathcal{C}$ gets "pinched" between moving poles. For $\alpha_s = \alpha_2 + \frac{b}{2}$ this happens only for one pair of poles: the pole of the factor $S_b(\alpha_1 + t)$ at $t = -\alpha_1$, approaching the contour $\mathcal{C}$ from the left, and the pole of the factor $S_b(Q - \frac{b}{2} + t)^{-1}$ at $t = \frac{b}{2}$, located to the right of $\mathcal{C}$. Moving the contour to the right (or to the left) of this pair (*cf.* [13], Lemma (3)) one gets

$$\begin{aligned} I_+(\alpha_1) &= I_+^0(\alpha_1) + I_+^R(\alpha_1)\,, \\ I_+^0(\alpha_1) &= \frac{S_b(\bar\alpha_1 - \alpha_1) S_b(\bar\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1) S_b(\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1)}{S_b(Q - \frac{b}{2} - \alpha_1) S_b(2\alpha_2 + \frac{b}{2} - \alpha_1) S_b(\bar\alpha_3 + \bar\alpha_u - \alpha_1) S_b(\bar\alpha_3 + \alpha_u - \alpha_1)}\,, \end{aligned}$$

where $I_+^R(\alpha_1)$ denotes the regular part.

In the case $\alpha_s = \alpha_2 - \frac{b}{2}$ there are two pairs of colliding poles with the contour $\mathcal{C}$ in between and the integral $I_-(\alpha_1)$ can be written as

$$I_-(\alpha_1) = I_-^0(\alpha_1) + I_-^1(\alpha_1) + I_-^R(\alpha_1)\,,$$

$$I_-^0(\alpha_1) = \frac{S_b(\bar\alpha_1 - \alpha_1) S_b(\bar\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1) S_b(\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1)}{S_b(Q + \frac{b}{2} - \alpha_1) S_b(2\alpha_2 - \frac{b}{2} - \alpha_1) S_b(\bar\alpha_3 + \bar\alpha_u - \alpha_1) S_b(\bar\alpha_3 + \alpha_u - \alpha_1)}\,,$$

$$\begin{aligned} I_-^1(\alpha_1) &= -\frac{1}{2\sin(\pi b^2)} \frac{S_b(\bar\alpha_1 - \alpha_1 - b)}{S_b(Q - \frac{b}{2} - \alpha_1) S_b(2\alpha_2 - \frac{b}{2} - \alpha_1 - b)} \\ &\quad \times \frac{S_b(\bar\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1 - b) S_b(\alpha_4 + \alpha_2 - \alpha_3 - \alpha_1 - b)}{S_b(\bar\alpha_3 + \bar\alpha_u - \alpha_1 - b) S_b(\bar\alpha_3 + \alpha_u - \alpha_1 - b)}\,, \end{aligned}$$

where $I_-^0(\alpha_1)$ is the contribution from the pole of $S_b(\alpha_1 + t)$ at $t = -\alpha_1$ and the pole of $S_b(Q + \frac{b}{2} + t)^{-1}$ at $t = \frac{b}{2}$, $I_-^1(\alpha_1)$ is the contribution from the pole of $S_b(\alpha_1 + t)$ at $t = -\alpha_1 - b$ and the pole of $S_b(Q + \frac{b}{2} + t)^{-1}$ at $t = -\frac{b}{2}$, and $I_-^R(\alpha_1)$ is the regular part.

$I^0_+(\alpha_1)$, $I^0_-(\alpha_1)$ and $I^1_-(\alpha_1)$ are finite in the limit $\alpha_1 \to -\frac{b}{2}$ and by themselves do not provide a compensating factor for vanishing $W_\pm$. They however change the structure of poles in the integrand of (35) which is now determined by the factors $U_+I^0_+$, $U_+I^0_-$, and $U_+I^1_-$, respectively.

In the case of $U_+I^0_+$ there are four pairs of poles "pinching" the contour of integration. The corresponding contributions can be calculated as in the case of t-integration in terms of residues of the poles at the locations $\alpha_u = \alpha_3 - \alpha_1$, $\bar\alpha_u = \alpha_3 - \alpha_1$, $\alpha_u = \alpha_3 + \alpha_1$, $\bar\alpha_u = \alpha_3 + \alpha_1$. Due to the symmetry $\alpha_u \leftrightarrow \bar\alpha_u$ the first two and the last two residues are equal. Since

$$
\begin{aligned}
&\lim_{\alpha_1\to-\frac{b}{2}} W_+(\alpha_1)\,\frac{1}{i}\oint\limits_{\alpha_u=\alpha_3-\alpha_1} d\alpha_u\, U_+(\alpha_1) I^0_+(\alpha_1)\\
&\quad= \frac{\Gamma(b(2\bar\alpha_2-b))\Gamma(b(\alpha_3-\bar\alpha_3))}{\Gamma(b(\bar\alpha_4-\bar\alpha_3+\bar\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\bar\alpha_3+\bar\alpha_2-\frac{b}{2}))} = \mathcal{B}_{++}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\alpha_1\to-\frac{b}{2}} W_+(\alpha_1)\,\frac{1}{i}\oint\limits_{\alpha_u=\alpha_3+\alpha_1} d\alpha_u\, U_+(\alpha_1) I^0_+(\alpha_1)\\
&\quad= \frac{\Gamma(b(2\bar\alpha_2-b))\Gamma(b(\bar\alpha_3-\alpha_3))}{\Gamma(b(\bar\alpha_4-\alpha_3+\bar\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\alpha_3+\bar\alpha_2-\frac{b}{2}))} = \mathcal{B}_{+-}\,,
\end{aligned}
$$

one recovers the formula (34) for $\sigma = +$.

The same four pairs of colliding poles appear in the case of $U_-I^0_-$. The corresponding residua are given by

$$
\begin{aligned}
&\lim_{\alpha_1\to-\frac{b}{2}} W_-(\alpha_1)\,\frac{1}{i}\oint\limits_{\alpha_u=\alpha_3-\alpha_1} d\alpha_u\, U_-(\alpha_1) I^0_-(\alpha_1)\\
&\quad= \frac{\Gamma(b(2\alpha_2-b))\Gamma(b(\alpha_3-\bar\alpha_3))}{\Gamma(b(\bar\alpha_4-\alpha_3+\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\alpha_3+\alpha_2-\frac{b}{2}))}\\
&\qquad\times\frac{\Gamma(b(\bar\alpha_4+\bar\alpha_3-\alpha_2-\frac{b}{2})\Gamma(b(\alpha_4+\bar\alpha_3-\alpha_2-\frac{b}{2}))}{\Gamma(b(\bar\alpha_4+\alpha_3-\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4+\alpha_3-\alpha_2-\frac{b}{2}))} \;\equiv\; \mathcal{B}^{(0)}_{-+}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\alpha_1\to-\frac{b}{2}} W_-(\alpha_1)\,\frac{1}{i}\oint\limits_{\alpha_u=\alpha_3+\alpha_1} d\alpha_u\, U_-(\alpha_1) I^0_-(\alpha_1)\\
&\quad= \frac{\Gamma(b(2\alpha_2-b))\Gamma(b(\bar\alpha_3-\alpha_3))}{\Gamma(b(\bar\alpha_4-\alpha_3+\alpha_2-\frac{b}{2}))\Gamma(b(\alpha_4-\alpha_3+\alpha_2-\frac{b}{2}))} = \mathcal{B}_{--}\,.
\end{aligned}
$$

In the case of $U_- I^1_-$ there are only two pairs of colliding poles contributing the residua at $\alpha_u = \alpha_3 - \alpha_1$ and $\bar{\alpha}_u = \alpha_3 - \alpha_1$. By the $\alpha_3 \leftrightarrow \bar{\alpha}_3$ symmetry they are equal and take the form

$$\begin{aligned} &\lim_{\alpha_1 \to -\frac{b}{2}} W_-(\alpha_1) \frac{1}{i} \oint\limits_{\alpha_u = \alpha_3 - \alpha_1} d\alpha_u \, U_-(\alpha_1) I^1_-(\alpha_1) \\ &\quad = \frac{\Gamma(b(\bar{\alpha}_4 + \bar{\alpha}_3 - \alpha_2 - \frac{b}{2}))\Gamma(b(\alpha_4 + \bar{\alpha}_3 - \alpha_2 - \frac{b}{2}))}{\Gamma(b(\bar{\alpha}_2 - \alpha_2))\Gamma(b(2\bar{\alpha}_3 - b))} \equiv \mathcal{B}^{(1)}_{-+} \, . \end{aligned}$$

Using properties of the gamma functions and trigonometric identities one gets

$$\mathcal{B}^{(0)}_{-+} + \mathcal{B}^{(1)}_{-+} = \frac{\Gamma(b(2\alpha_2 - b))\Gamma(b(\alpha_3 - \bar{\alpha}_3))}{\Gamma(b(\bar{\alpha}_4 - \bar{\alpha}_3 + \alpha_2 - \frac{b}{2}))\Gamma(b(\alpha_4 - \bar{\alpha}_3 + \alpha_2 - \frac{b}{2}))} = \mathcal{B}_{-+} \, ,$$

what agrees with (34) for $\sigma = -$.

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Appendix A

For $\Re x > 0$ the Barnes double gamma function has an integral representation

$$\log \Gamma_b(x) = \int\limits_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Q}{2}t}}{(1 - e^{-tb})\left(1 - e^{-t/b}\right)} - \frac{\left(\frac{Q}{2} - x\right)^2}{2e^t} - \frac{\frac{Q}{2} - x}{t} \right] .$$

It satisfies functional relations of the form

$$\begin{aligned} \Gamma_b(x + b) &= \frac{\sqrt{2\pi}\, b^{bx - \frac{1}{2}}}{\Gamma(bx)} \Gamma_b(x) \, , \\ \Gamma_b\left(x + \frac{1}{b}\right) &= \frac{\sqrt{2\pi}\, b^{-\frac{x}{b} + \frac{1}{2}}}{\Gamma(\frac{x}{b})} \Gamma_b(x) \, , \end{aligned}$$

and can be analytically continued to the whole complex x plane as a meromorphic function with poles located at $x = -mb - n\frac{1}{b}$, $m, n \in \mathbb{N}$.

For $x \to 0$:

$$\Gamma_b(x) \;=\; \frac{\Gamma_b(Q)}{2\pi x} + \mathcal{O}(1)\,.$$

For $0 < \Re x < Q$ the function $S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$ can be represented as

$$\log\, S_b(x) \;=\; \int\limits_0^\infty \frac{dt}{t}\left[\frac{\sinh\left(\frac{Q}{2}-x\right)t}{2\sinh\frac{bt}{2}\sinh\frac{t}{2b}} - \frac{Q-2x}{t}\right].$$

S_b is a meromorphic function of x with poles located at $x = -\,mb - n\frac{1}{b},\ m,$ $n \in \mathbb{N}$, and zeroes at $x = Q + mb + n\frac{1}{b},\ m,n \in \mathbb{N}$. It satisfies functional relations of the form

$$S_b(x+b) \;=\; 2\sin\left(\pi b x\right) S_b(x)\,,$$

$$S_b\left(x+\frac{1}{b}\right) \;=\; 2\sin\left(\frac{\pi x}{b}\right) S_b(x)\,,$$

and for $x \to 0$:

$$S_b(x) = \frac{1}{2\pi x} + \mathcal{O}(1)\,, \qquad\qquad S_b(Q+x) = -2\pi x + \mathcal{O}\left(x^2\right)\,.$$

Both $\Gamma_b(x)$ and $S_b(x)$ are invariant under $b \to \frac{1}{b}$.

Appendix B

Our aim is to demonstrate the symmetry properties of the matrix $B_{\alpha_s\,\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ (11). The conjugations $\alpha_1 \to \bar{\alpha}_1,\ \alpha_4 \to \bar{\alpha}_4,\ \alpha_s \to \bar{\alpha}_s$ and $\alpha_u \to \bar{\alpha}_u$ are explicit. To see the symmetry of the braiding matrix under the and exchange of the columns, *i.e.* simultaneous transformation $\alpha_1 \leftrightarrow \alpha_4$, $\alpha_2 \leftrightarrow \alpha_3$ one only needs to shift the integration variable $t \to t + \alpha_3 - \alpha_2$.

To find the remaining symmetries one can start from the following identity satisfied by the deformed hypergeometric function

$$F_b(\alpha,\beta;\gamma;-iy) \;=\; e^{\pi y(\alpha+\beta-\gamma)}\frac{S_b\left(-iy+\frac{Q}{2}-\frac{\alpha+\beta-\gamma}{2}\right)}{S_b\left(-iy+\frac{Q}{2}+\frac{\alpha+\beta-\gamma}{2}\right)} \times F_b(\gamma-\alpha,\gamma-\beta;\gamma;-iy)\,. \tag{40}$$

From its integral representation

$$F_b(\alpha,\beta;\gamma;y) = \frac{1}{i}\frac{S_b(\gamma)}{S_b(\alpha)S_b(\beta)}\int\limits_{i\mathbb{R}} ds\; e^{2\pi i s y}\,\frac{S_b(\alpha+s)S_b(\beta+s)}{S_b(\gamma+s)S_b(Q+s)}$$

it follows that the integral that appears in the expression for the braiding matrix,

$$I = \frac{1}{i}\int_{i\mathbb{R}} dt \frac{S_b(\bar{\alpha}_1+t)S_b(\alpha_1+t)S_b(\bar{\alpha}_4-\alpha_3+\alpha_2+t)S_b(\alpha_4-\alpha_3+\alpha_2+t)}{S_b(\bar{\alpha}_u+\bar{\alpha}_3+t)S_b(\alpha_u+\bar{\alpha}_3+t)S_b(\bar{\alpha}_s+\alpha_2+t)S_b(\alpha_s+\alpha_2+t)},$$

can be expressed as

$$\begin{aligned} I \;=\; & \frac{S_b(\alpha_4-\alpha_3+\alpha_s)S_b(\bar{\alpha}_1-\alpha_2+\alpha_s)S_b(\bar{\alpha}_4-\bar{\alpha}_2+\bar{\alpha}_u)S_b(\alpha_1-\bar{\alpha}_3+\bar{\alpha}_u)}{S_b(2\bar{\alpha}_2)S_b(2\alpha_3)} \\ & \times \int_{\mathbb{R}} dy\; e^{2\pi y(\alpha_s+\alpha_3-\alpha_u-\alpha_2)} F_b(\alpha_4-\alpha_3+\alpha_s,\bar{\alpha}_1-\alpha_2+\alpha_s;2\bar{\alpha}_2;-iy) \\ & \times F_b(\bar{\alpha}_4-\bar{\alpha}_2+\bar{\alpha}_u,\alpha_1-\bar{\alpha}_3+\bar{\alpha}_u;2\alpha_3;iy)\,. \end{aligned}$$

Using in this expression the identity (40) we get

$$\begin{aligned} I \;=\; & \frac{S_b\,(\alpha_s+\alpha_4-\alpha_3)\,S_b\,(\alpha_s+\bar{\alpha}_1-\alpha_2)\,S_b\,(\bar{\alpha}_4+\alpha_2-\alpha_u)\,S_b\,(\alpha_1+\alpha_3-\alpha_u)}{S_b\,(\alpha_s+\alpha_3-\alpha_4)\,S_b\,(\alpha_s+\alpha_2-\bar{\alpha}_1)\,S_b\,(\alpha_4+\bar{\alpha}_2-\alpha_u)\,S_b\,(\bar{\alpha}_1+\bar{\alpha}_3-\alpha_u)} \\ & \times \frac{1}{i}\int_{i\mathbb{R}} dt\; \frac{S_b\,(\alpha_3+t)\,S_b\,(\bar{\alpha}_3+t)\,S_b\,(\alpha_4+\alpha_1-\alpha_2+t)\,S_b\,(\alpha_4+\alpha_1-\bar{\alpha}_2+t)}{S_b\,(\alpha_s+\alpha_4+t)\,S_b\,(\bar{\alpha}_s+\alpha_4+t)\,S_b\,(\alpha_u+\alpha_1+t)\,S_b\,(\bar{\alpha}_u+\alpha_1+t)}, \end{aligned}$$

what gives

$$\begin{aligned} & B_{\alpha_s\,\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \\ & = \frac{\Gamma_b(\bar{\alpha}_2+\bar{\alpha}_4-\alpha_u)\Gamma_b(\alpha_2+\bar{\alpha}_4-\alpha_u)\Gamma_b(\bar{\alpha}_2-\alpha_4+\alpha_u)\Gamma_b(\alpha_2-\alpha_4+\alpha_u)}{\Gamma_b(\bar{\alpha}_2+\alpha_1-\alpha_s)\Gamma_b(\alpha_2+\alpha_1-\alpha_s)\Gamma_b(\bar{\alpha}_2-\bar{\alpha}_1+\alpha_s)\Gamma_b(\alpha_2-\bar{\alpha}_1+\alpha_s)} \\ & \times \frac{\Gamma_b(\alpha_3+\alpha_1-\alpha_u)\Gamma_b(\bar{\alpha}_3+\alpha_1-\alpha_u)\Gamma_b(\alpha_3-\bar{\alpha}_1+\alpha_u)\Gamma_b(\bar{\alpha}_3-\bar{\alpha}_1+\alpha_u)}{\Gamma_b(\alpha_3+\bar{\alpha}_4-\alpha_s)\Gamma_b(\bar{\alpha}_3+\bar{\alpha}_4-\alpha_s)\Gamma_b(\alpha_3-\alpha_4+\alpha_s)\Gamma_b(\bar{\alpha}_3-\alpha_4+\alpha_s)} \\ & \times \frac{\Gamma_b(2\alpha_s)\Gamma_b(2Q-2\alpha_s)}{\Gamma_b(Q-2\alpha_u)\Gamma_b(2\alpha_u-Q)} \\ & \times \frac{1}{i}\int_{i\mathbb{R}} dt\; \frac{S_b\,(\alpha_3+t)\,S_b\,(\bar{\alpha}_3+t)\,S_b\,(\bar{\alpha}_2-\bar{\alpha}_1+\alpha_4+t)\,S_b\,(\alpha_2-\bar{\alpha}_1+\alpha_4+t)}{S_b\,(\bar{\alpha}_u+\alpha_1+t)\,S_b\,(\alpha_u+\alpha_1+t)\,S_b\,(\bar{\alpha}_s+\alpha_4+t)\,S_b\,(\alpha_s+\alpha_4+t)} \\ & = B_{\alpha_s\,\alpha_u}\begin{bmatrix} \bar{\alpha}_1 & \alpha_4 \\ \alpha_2 & \bar{\alpha}_3 \end{bmatrix}. \end{aligned} \tag{41}$$

This form of the braiding matrix is explicitly invariant under the conjugations $\alpha_2 \to \bar{\alpha}_2$ and $\alpha_3 \to \bar{\alpha}_3$. Shifting the integration variable $t \to t+\bar{\alpha}_1-\alpha_4$ we get

$$B_{\alpha_s\,\alpha_u}\begin{bmatrix} \bar{\alpha}_1 & \alpha_4 \\ \alpha_2 & \bar{\alpha}_3 \end{bmatrix} = B_{\alpha_s\,\alpha_u}\begin{bmatrix} \alpha_4 & \bar{\alpha}_1 \\ \bar{\alpha}_3 & \alpha_2 \end{bmatrix}$$

and this, together with (41), finally proves the symmetry of the braiding matrix with respect to the exchange of its rows

$$B_{\alpha_s\,\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = B_{\alpha_s\,\alpha_u}\!\left[\begin{smallmatrix}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{smallmatrix}\right].$$

REFERENCES

[1] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl. Phys.* **B241**, 333 (1984).

[2] A.B. Zamolodchikov, *Commun. Math. Phys.* **96**, 419 (1984).

[3] J. Teschner, *Class. Quant. Grav.* **18**, R153 (2001).

[4] J. Teschner, `hep-th/0303150`.

[5] A.B. Zamolodchikov, A.B. Zamolodchikov *Sov. Sci. Rev. A. Phys.* **10**, 269 (1989).

[6] A.B. Zamolodchikov, *Theor. Math. Phys.* **73**, 1088 (1987).

[7] A.B. Zamolodchikov, A.B. Zamolodchikov, *Nucl. Phys.* **B477**, 577 (1996).

[8] I. Runkel, G.M.T. Watts, *J. High Energy Phys.* **0109**, 006 (2001).

[9] D.C. Lewellen, *Nucl. Phys.* **B320**, 345 (1989).

[10] B. Blok, S. Yankielowicz, *Nucl. Phys.* **B321**, 717 (1989).

[11] G.W. Moore, N. Seiberg, *Phys. Lett.* **B212**, 451 (1988); *Commun. Math. Phys.* **123**, 177 (1989).

[12] B. Ponsot, J. Teschner, `hep-th/9911110`.

[13] B. Ponsot, J. Teschner, *Commun. Math. Phys.* **224**, 613 (2001).

[14] J.L. Gervais, J. Schnittger, *Phys. Lett.* **B315**, 258 (1993).

[15] B. Ponsot, J. Teschner, *Nucl. Phys.* **B622**, 309 (2002).

[16] B. Ponsot, `hep-th/0301193`.