

TRANSPORT OF FORCED QUANTUM MOTORS  
IN THE STRONG FRICTION LIMIT\*

ŁUKASZ MACHURA, JERZY ŁUCZKA

Institute of Physics, University of Silesia, 40-007 Katowice, Poland

PETER TALKNER, PETER HÄNGGI

Institut für Physik, Universität Augsburg, 86135 Augsburg, Germany

*(Received November 9, 2006)*

The directed transport of an overdamped Brownian motor moving in a spatially periodic potential that lacks reflection symmetry (*i.e.* a ratchet potential) is studied when driven by thermal and dichotomic nonequilibrium noise in the presence of an external, constant load force. We consider both, the classical and the quantum tunneling assisted regimes. The current-load characteristics are investigated as a function of the system parameters like the load force, the temperature and the amplitude strength of the applied two-state noise.

PACS numbers: 05.40.-a, 05.60.Gg, 02.50.Ey, 05.60.-k

**1. Introduction**

Classical regimes of transport of microscopic objects like Brownian particles are well elaborated in the previous literature (for a historical overview see in Ref. [1]). In the last decade, special interest has been devoted to transport in ratchet systems (also termed Brownian motor systems), *i.e.* to the phenomenon of noise assisted, directed motion of particles in spatially periodic structures which possess a broken reflection symmetry [2–4]. In contrast, the quantum properties of directed transport are only partially elaborated in such Brownian motor systems [5–8]. Challenges arise in the quantum regime because the transport can strongly depend on the mutual interplay of pure quantum effects like tunneling and particle wave interference with dissipation processes, nonequilibrium fluctuations and external driving [9]. Moreover, there exist typically no analogous closed evolution

---

\* Presented at the XIX Marian Smoluchowski Symposium on Statistical Physics, Kraków, Poland, May 14–17, 2006.

equations of such system as in the classical regimes, which are based for example on Langevin or Fokker–Planck equations. However, special quantum regimes can be described nevertheless by use of effectively classical methods. For example, if the system strongly interacts with a thermostat, quantum diffusive dynamics can be described by an effective classical Smoluchowski equation for the diagonal part of the statistical operator in the position representation [10], in which the potential and diffusion coefficient are modified due to quantum effects (Section 2). This so called quantum Smoluchowski equation has been applied to describe activation processes, quantum diffusion, and Brownian motors [5–8,10]. In this work, we employ it to study the transport properties of an overdamped Brownian motor moving in a spatially periodic potential  $U(x) = U(x+L)$  of the period  $L$  under the influence of an external, constant bias force when driven by both, thermal equilibrium and nonequilibrium fluctuations. We analyze the classical as well as the quantum regimes. In particular, the resulting *current-load characteristics* are investigated as functions of the system parameters like load, temperature and amplitude of the nonequilibrium noise (Section 3).

## 2. Quantum Smoluchowski equation

For systems strongly interacting with a thermostat, which in turn implies a strong friction limit, the quantum dynamics above the crossover temperature to pure quantum tunneling [11] can be described in terms of a generalized Smoluchowski equation which accounts for the leading quantum corrections. For a particle of mass  $M$  moving in the potential  $V(x)$ , this quantum Smoluchowski equation (QSE) for the coordinate-diagonal elements of the density operator  $\rho(t)$ , *i.e.* for the probability density function  $P(x, t) = \langle x | \rho(t) | x \rangle$  in position space  $x$ , takes the form [10]:

$$\Gamma \frac{\partial}{\partial t} P(x, t) = \frac{\partial}{\partial x} V'_{\text{eff}}(x) P(x, t) + \frac{\partial^2}{\partial x^2} D_{\text{eff}}(x) P(x, t), \quad (1)$$

where  $\Gamma$  denotes the friction coefficient. The effective potential reads

$$V_{\text{eff}}(x) = V(x) + \frac{1}{2} \lambda V''(x), \quad (2)$$

where the prime denotes the derivative with respect to the coordinate  $x$ . The quantum correction parameter  $\lambda$  describes quantum fluctuations in position space and reads

$$\lambda = \frac{\hbar}{\pi\Gamma} \left[ \gamma + \Psi \left( 1 + \frac{\hbar\beta\Gamma}{2\pi M} \right) \right], \quad \beta = \frac{1}{k_{\text{B}}T}. \quad (3)$$

Here,  $\Psi(z)$  is the digamma function,  $\gamma \simeq 0.5772$  the Euler–Mascheroni constant,  $T$  is the temperature and  $k_B$  denotes the Boltzmann constant. The parameter  $\lambda$  depends nonlinearly on the Planck constant  $\hbar$  and on the mass  $M$  of the Brownian particle (let us remind that in the classical case, the overdamped dynamics does not depend on the mass  $M$ ).

The effective diffusion coefficient reads [7, 12]

$$D_{\text{eff}}(x) = \frac{1}{\beta[1 - \lambda\beta V''(x)]}. \quad (4)$$

Note that for  $k_B T \ll \hbar\Gamma/M$ ,  $\lambda$  becomes

$$\lambda = \frac{\hbar}{\pi T} \left[ \gamma + \ln \left( \frac{\hbar\beta\Gamma}{2\pi M} \right) \right]. \quad (5)$$

From the mathematical point of view, the Smoluchowski equation (1) corresponds to the classical Langevin equation in the Ito interpretation [13],

$$\Gamma \dot{x} = -V'_{\text{eff}}(x) + \sqrt{2\Gamma D_{\text{eff}}(x)} \xi(t). \quad (6)$$

The zero-mean and the  $\delta$ -correlated Gaussian white noise  $\xi(t)$ , meaning that  $\langle \xi(t)\xi(s) \rangle = \delta(t-s)$ , models the influence of a thermostat of temperature  $T$  on the system.

### 3. Biased quantum motor transport

We focus on the dynamics of overdamped quantum Brownian motors [5–8] moving in a spatially periodic potential  $U(x) = U(x+L)$  and driven by nonequilibrium fluctuations  $\eta(t)$ . The quantum thermal fluctuations are determined by the parameter  $\lambda$  (see Eq. 3). Additionally, a constant bias force  $F_0$  is applied to the system. The dynamics can then be described by the Langevin equation

$$\Gamma \dot{x} = -V'_{\text{eff}}(x) + \sqrt{2\Gamma D_{\text{eff}}(x)} \xi(t) + \eta(t), \quad (7)$$

where  $V_{\text{eff}}(x)$  is given by Eq. (2) with

$$V(x) = U(x) - F_0 x. \quad (8)$$

We rewrite Eq. (7) in the dimensionless form, namely,

$$\dot{y} = -W'_{\text{eff}}(y) + F + \sqrt{2\mathcal{D}_{\text{eff}}(y)} \hat{\xi}(s) + \hat{\eta}(s), \quad (9)$$

where the position of the Brownian motor is scaled as  $y = x/L$ , time is rescaled as  $s = t/\tau_0$ , with the characteristic time scale reading  $\tau_0 = \Gamma L^2/\Delta V$

(the barrier height  $\Delta V$  is the difference between the maximal and minimal values of the unbiased potential  $V(x)$ ). During this time span, a classical, overdamped particle moves a distance of length  $L$  under the influence of the constant force  $\Delta V/L$ . The effective potential is  $W_{\text{eff}}(y) = W(y) + (1/2)\lambda_0 W''(y)$ , where the rescaled periodic potential  $W(y) = U(yL)/\Delta V = W(y+1)$  possesses unit period and a unit barrier height. The dimensionless parameter  $\lambda_0 = \lambda/L^2$  describes quantum fluctuations over the characteristic length  $L$ , see in Ref. [7] for further details.

The rescaled diffusion function  $\mathcal{D}_{\text{eff}}(y)$  reads,

$$\mathcal{D}_{\text{eff}}(y) = \frac{1}{\beta_0[1 - \lambda_0\beta_0 W''(y)]}. \quad (10)$$

The dimensionless, inverse temperature  $\beta_0 = \Delta V/k_B T$  is the ratio of the activation energy in the non-scaled potential and the thermal energy. The rescaled Gaussian white noise reads  $\hat{\xi}(s) = (L/\Delta V)\xi(t)$ , the rescaled, non-thermal stochastic force is  $\hat{\eta}(s) = (L/\Delta V)\eta(t)$  and the rescaled constant force stands for  $F = (L/\Delta V)F_0$ .

The nonequilibrium fluctuations  $\hat{\eta}(s)$  in Eq. (9) are described by symmetric Markovian dichotomic noise

$$\hat{\eta}(s) = \{-a, a\}, \quad (11)$$

which jumps between two states  $a$  and  $-a$  with a rate  $\nu$ . The induced stationary probability current  $J$ , or equivalently the asymptotic average velocity of the Brownian motor can then be determined in the adiabatic limit in a closed form. Put differently, the above stated problem can be solved analytically in the limit  $\nu \rightarrow 0$ . In the above introduced dimensionless variables the probability current takes the form

$$\langle \dot{y} \rangle = J = \frac{1}{2} [J(a) + J(-a)], \quad (12)$$

$$J(a) = \frac{1 - \exp[-\beta_0(F + a)]}{\int_0^1 dy \mathcal{D}_{\text{eff}}^{-1}(y) \exp[-\beta_0\Phi(y, a)] \int_y^{y+1} dz \exp[\beta_0\Phi(z, a)]}, \quad (13)$$

with the biased, generalized thermodynamic potential reading

$$\begin{aligned} \Phi(y, a) &= \int \frac{W'(y) - (F + a)y}{\mathcal{D}_{\text{eff}}(y)} dy \\ &= W(y) + \frac{1}{2} \lambda_0 W''(y) - \frac{1}{2} \lambda_0 \beta_0 [W'(y)]^2 - \frac{1}{4} \lambda_0^2 \beta_0 [W''(y)]^2 \\ &\quad + (F + a)\lambda_0 \beta_0 W'(y) - (F + a)y. \end{aligned} \quad (14)$$

Unbiased transport properties driven by such dichotomic noise particle have been elaborated for classical particles in Ref. [14, 15] and in the quantum regime in Ref. [7].

This analytic expression for the current allows one to study directed transport in *arbitrarily shaped* ratchet potentials. As an example, we consider a family of asymmetric periodic potentials of the form

$$W(y) = W_0 \{ \sin(2\pi y) + 0.4 \sin[4\pi(y - 0.45)] + B \sin[6\pi(y - 0.45)] \}, \quad (15)$$

where  $B$  denotes a shape parameter and  $W_0$  is chosen in such a way that the maximal variation of the potential is normalized to unity.

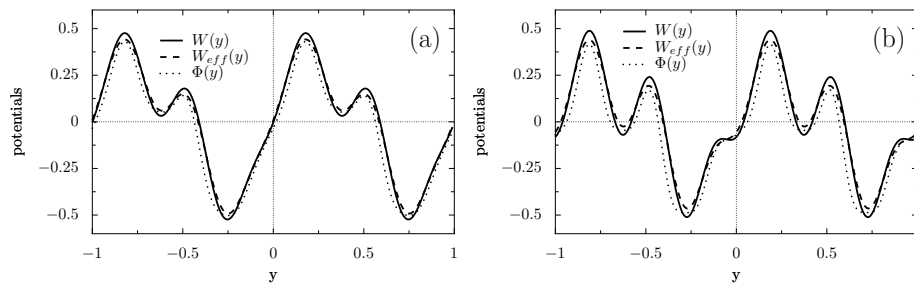


Fig. 1. The unbiased classical ratchet potential  $W(y)$  (solid line) together with the corresponding unbiased quantum potential  $W_{\text{eff}}(y)$  (dashed line) and the generalized thermodynamic potential  $\Phi(y) \equiv \Phi(y, 0)$  (dotted line) are depicted as functions of the scaled position  $y$  for  $F = 0$ ,  $\beta_0 = 10$  and two values of the shape parameter  $B$ . The left panel (a) refers to  $B = 0.3$  and the panel (b) to  $B = 0.62$ .

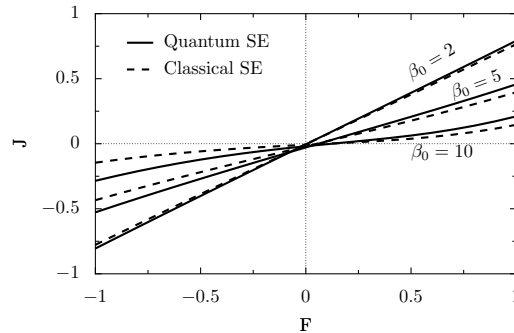


Fig. 2. (Color online) The directed quantum noise-induced transport  $J$  of the quantum Brownian motor (solid line) versus the constant bias force  $F$  is compared with its classical limit (dashed line). The current-load characteristics is studied here for several values of the dimensionless inverse temperature  $\beta_0 = 2, 5, 10$ . The dichotomic noise level is set to  $a = 1$ . The ratchet potential is defined by  $B = 0.3$  (see panel (a) in Fig. 1).

The influence of quantum corrections on the potential shape is displayed in Fig. 1 and on transport of the biased Brownian motor is presented in Figs. 2–4. For the rescaled quantum fluctuations, we set the temperature-dependent quantum parameter (3) equal to  $\lambda_0 = 10^{-4}[\gamma + \Psi(1 + 10^4\beta_0)]$ . We study the induced current-load characteristics as a function of all system

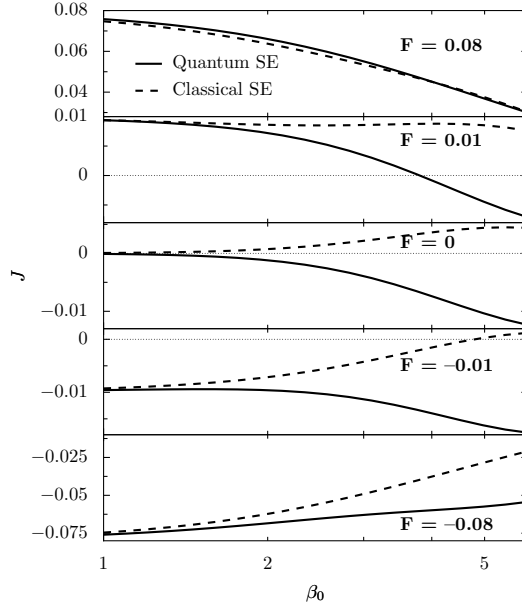


Fig. 3. The directed quantum noise-induced transport  $J$  of the quantum Brownian motor (solid line) versus the dimensionless inverse temperature  $\beta_0$  is compared with its classical limit (dashed line). The classical and quantum currents are depicted in the various panels for five different values of the applied external bias  $F$ ; *i.e.* for bottom to top  $F = -0.08 - 0.01, 0, 0.01, 0.08$ . The dichotomic noise level is set to  $a = 1$ . The ratchet potential is defined by  $B = 0.62$  (see panel (b) in Fig. 1).

parameters and elucidate how one can control the directed transport by adjusting the (inverse) temperature  $\beta_0$ , the dichotomic noise strength  $a$  and the strength of the constant force  $F$ .

#### 4. Quantum transport characteristics

We next address the question of how the constant bias load  $F$  affects the directed transport properties of the quantum and the corresponding classical Brownian motors that are driven out of equilibrium by a nonthermal dichotomic random force  $\eta(t)$ . In Fig. 2 we present the current-load characteristics for the dichotomic noise level  $a = 1.0$  and a ratchet potential with  $B = 0.3$ , see panel (a) in Fig. 1. The three sets of curves correspond to three different values of the dimensionless inverse temperature  $\beta_0 = 2, 5, 10$ .

In the absence of thermal (Gaussian) noise, the dynamics of the driven particle is confined to a single period as long as the bias forces remain limited to the interval  $[F_1, F_2]$  with  $F_1 \simeq -3.6$  and  $F_2 \simeq 4.42$  denoting the two threshold values. Then, the dichotomic noise alone with  $a = 1$  is not able to induce transitions to the neighboring periods; this becomes possible only in the presence of additional, thermal Gaussian noise of unbounded amplitude which in turn induces a finite probability current. For larger thermal noise strength (*i.e.* for smaller  $\beta_0$  or higher temperature), the quantum corrections seemingly play only a minor role for the probability current, see in Fig. 3. It is only for lower temperatures  $T$  that the influence of quantum effects become more pronounced and distinct deviations from the classical response behavior become detectable.

The value of the constant bias, for which the current vanishes, is termed the *stall force*  $F_{\text{stall}}$ . Generally, this stall force depends on the temperature  $\beta_0^{-1}$  and on the other system parameters as well. Fig. 2 depicts that by cooling down the system (*i.e.* increasing the inverse temperature  $\beta_0$ ) the stall force becomes shifted toward larger positive loads. This means that for lower temperatures the ratchet effect becomes more pronounced. Moreover, only for small enough temperatures one can resolve the different values of the stall force for the classical motor and the quantum motor dynamics. If the temperature is high, then both the quantum and the classical characteristics are very similar and, additionally, both current-load characteristics cross the zero-current axis at values that are close to zero. This corresponds to a rather weak ratchet effect.

With a value for the two-state noise fluctuations set at  $a = 1.0$  and for the ratchet potential defined by  $B = 0.62$  depicted in panel (b) in Fig. 1, we observe a pronounced influence of the quantum corrections on the transport [7]. Also in this case with  $B = 0.62$  the dichotomic force amplitude  $\eta(t)$  alone cannot induce transport, and the transitions over the potential barriers are triggered by thermal activation. The limiting force thresholds in this case read:  $F_1 \simeq -4.86$  and  $F_2 \simeq 5.5$ .

At zero bias ( $F = 0$ ), the quantum and classical motors now proceed in the *opposite directions* within a large range of temperatures, see in the central panel in Fig. 3. By applying a large enough constant load either into positive or negative direction, this feature is seemingly destroyed. The motors are forced to transport accordingly to the applied bias. For very small values of the force, however, this very intriguing behavior induced by quantum fluctuations is still preserved at low temperatures.

In the Fig. 4 we plot the current as a function of the dichotomic noise amplitude  $a$ . The ratchet potential is the same as the one used in Fig. 2. The rescaled inverse temperature is set rather large at  $\beta_0 = 2$ . We compare the resulting classical and quantum currents for five various values of the constant force  $F = -1, -0.2, 0, 0.2, 1$ .

The middle panel of Fig. 4 depicts the Brownian motor currents for the unbiased case  $F = 0$ . We recover a distinct feature of the ratchet dynamics, namely, the occurrence of current reversals [15], both in the classical and in the quantum case, located however at different  $a$ -values. In presence of a sufficiently large bias force these current reversals disappear and the classical and quantum currents approach each other.

For small values of the dichotomic noise strength  $a$  and in the absence of the external load, classical and quantum currents assume almost identical values, see the central panel in Fig. 4. In this regime of small dichotomic noise strengths and non-zero bias, the absolute value of the quantum current is always larger than its classical counterpart.

### 5. Summary

Quantum noise induced, directed transport features of an overdamped Brownian motor moving in a spatially periodic ratchet potential, that is exposed to the constant load in the presence of nonequilibrium, adiabatically

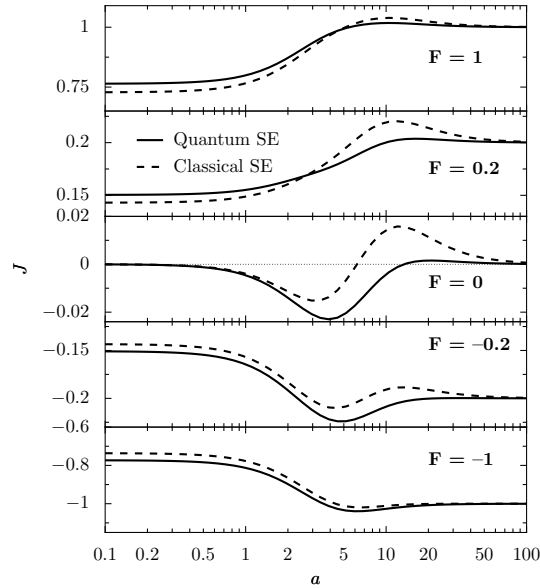


Fig. 4. The average current  $J$  of the quantum (solid line) and classical (dashed line) Brownian motor versus the two-state noise amplitude  $a$  is shown in order to elucidate the influence of quantum fluctuations on the directed transport. We present the current for several values of constant load force  $F = -1, -0.2, 0, 0.2, 1$ , from bottom toward top. The dimensionless inverse temperature reads  $\beta_0 = 2$  and the potential shape parameter  $B$  is set to 0.3, *cf.* panel (a) in Fig. 1.



varying dichotomic fluctuations, are investigated with this work. The classical and the quantum regimes, being determined by the ratio of two energy scales (the energy of thermal equilibrium fluctuations and the activation energy over barriers), are analyzed in detail. There is no a general rule on the influence of quantum corrections on current: as a function of the system parameters, the quantum effects may either increase or as well decrease the average current of the so forced Brownian motors. The impact of quantum corrections is also clearly encoded in a shift of position of the respective values for the stall force.

This work has been supported by the ESF programme: Stochastic Dynamics, the Polish Ministry of Science and Higher Education via the grant No PBZ-MIN-008/P03/2003.

#### REFERENCES

- [1] P. Hänggi, F. Marchesoni, *Chaos* **15**, 026101 (2005).
- [2] P. Reimann, P. Hänggi, *Appl. Phys.* **A75**, 169 (2002); P. Reimann, *Phys. Rep.* **361**, 57 (2002).
- [3] R.D. Astumian, P. Hänggi, *Physics Today* **55**, 33 (2002).
- [4] P. Hänggi, F. Marchesoni, F. Nori, *Ann. Phys.* **14**, 51 (2005).
- [5] P. Reimann, M. Grifoni, P. Hänggi, *Phys. Rev. Lett.* **79**, 10 (1997); P. Reimann, P. Hänggi, *Chaos* **8**, 629 (1998).
- [6] I. Goychuk, M. Grifoni, P. Hänggi, *Phys. Rev. Lett.* **81**, 649 (1998); I. Goychuk, M. Grifoni, P. Hänggi, *Phys. Rev. Lett.* **81**, 2837 (1998) (Erratum); I. Goychuk, P. Hänggi, *Europhys. Lett.* **43**, 503 (1998); M. Grifoni, M.S. Ferreira, J. Peguiron, J.B. Majer, *Phys. Rev. Lett.* **89**, 146801 (2002); S. Scheidl, V.M. Vinokur, *Phys. Rev.* **B65**, 195305 (2002).
- [7] L. Machura, M. Kostur, P. Hänggi, P. Talkner, J. Łuczka, *Phys. Rev.* **E70**, 031107 (2004).
- [8] L. Machura, M. Kostur, P. Talkner, J. Łuczka, P. Hänggi, *Phys. Rev.* **E73**, 031105 (2006).
- [9] P. Hänggi, G.L. Ingold, *Chaos* **15**, 026105 (2005).
- [10] J. Ankerhold, P. Pechukas, H. Grabert, *Phys. Rev. Lett.* **87**, 086802 (2001); J. Ankerhold, *Phys. Rev.* **E64**, 060102 (2001).
- [11] P. Hänggi, H. Grabert, G.L. Ingold, U. Weiss, *Phys. Rev. Lett.* **55**, 761 (1985).
- [12] J. Łuczka, R. Rudnicki, P. Hänggi, *Physica A* **351**, 60 (2005).
- [13] P. Hänggi, H. Thomas, *Phys. Rep.* **88**, 207 (1982).
- [14] J. Kula, T. Czernik, J. Łuczka, *Phys. Rev. Lett.* **80**, 1377 (1998).
- [15] M. Kostur, J. Łuczka, *Phys. Rev.* **E63**, 021101 (2001).