TOWARDS NON-HERMITIAN RANDOM LÉVY MATRICES*

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We review a new technique for calculating spectral properties of infinite non-Hermitian random matrix models, and we present an algorithm for calculating bulk spectral properties of ensembles of the type $H_1 + iH_2$, where H_1 and H_2 are arbitrary free (in the sense of Voiculescu) ensembles, including cases of the Lévy (heavy-tailed) spectra. As a particular example, we solve analytically the ensemble $C_1 + iC_2$, where C_1 and C_2 are free centered random matrix ensembles of the Cauchy class.

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1. Introduction

Most of the ensembles considered in Random Matrix Theory (RMT) and its applications concern the case where real spectra have finite supports. There exists, however, a non-trivial class where the support of the real spectrum is infinite. A notable subclass is comprised of the so-called Lévy matrices, as they are stable under convolution. They are a subject of vigorous research — first, they are interesting from the point of view of numerous applications, since heavy-tailed processes appear in a majority of complex systems; second, they are considerably more difficult on the technical level, since the absence of finite moments invalidates several standard

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tools of RMT. The studied problems include the Wigner-type Lévy matrices proposed by Bouchaud and Cizeau [1], free Lévy measures constructed by Voiculescu and Bercovici [2], and some Free Random Variables matrix realizations [3], interrelations between the above two approaches [4, 5], and some applications in financial engineering [6].

As far as we know, non-Hermitian Lévy ensembles have not so far been studied in the literature. Non-Hermiticity itself is a problem of considerable difficulty, because in this case the spectrum forms two-dimensional islands on the complex plane, and the tools of RMT based on holomorphic functions (*e.g.* the Green's function, the R-transform, *etc.*) are no longer applicable. Nevertheless, several sophisticated tools have been proposed to study this problem, triggered by important applications in quantum chaotic scattering [7], dissipative systems [8], Euclidean QCD in the presence of chemical potential [9], growth problems [10] and mesoscopic systems [11–14]. All of the developed methods rely, however, heavily on the existence of finite moments, prohibiting therefore an extension to heavy-tailed complex spectra.

In this note we circumvent this problem by using the novel formulation for non-Hermitian random ensembles based on the *quaternion extension* developed recently by two of the present authors [15]. First, in Sec. 2, we recall very briefly the connection between infinite-size random ensembles and Free Random Variables (FRV). Next, in Sec. 3, we show how the concepts of FRV can be generalized to the non-Hermitian case. To avoid the unnecessary repetitions, we show only some major formulae, referring for details and proofs to the original work [15]. Afterwards, using a simple operational algorithm (Subsec. 4.1), we solve two important cases. The first one (Subsec. 4.2) is intended to advocate the power of the quaternion method — by adding the Hermitian and anti-Hermitian GUE ensembles (*i.e.* H_1+iH_2, where H_1 and H_2 are free GUE ensembles), we recover the Ginibre–Girko circle [16, 17], or rather its slight generalization to the elliptic case [18]. As the second example (Subsec. 4.3), we present a new result considering the model $C_1 + iC_2$, where C_1 and C_2 are free Cauchy ensembles. In this case, we obtain the support of the complex spectrum bounded by two pairs of hyperbolas, and also the analytic form of the spectral density. We also check how well this non-Hermitian "Cauchy cross" is reproduced by a numerical simulation. Finally, in Sec. 5, we provide a list of several known and unknown results obtained by us using this quaternion technique.

2. Free random variables

2.1. Freeness

Two Hermitian random matrices H_1 and H_2 are called *free* [19] if

$$\langle p_1(H_1)r_1(H_2)p_2(H_1)r_2(H_2)\dots\rangle = 0\,, \quad \text{if} \quad \langle p_i(H_1)\rangle = \langle r_j(H_2)\rangle = 0\,, \tag{1}$$

where p_i and r_j are polynomials. The basic feature of this definition is that consecutive polynomials should depend on different variables. Note that the expectation value of Hermitian random matrices used here is defined as

$$\langle H\rangle \equiv \left\langle \frac{1}{N}\operatorname{Tr} H \right\rangle_{\text{cl}}\,, \tag{2}$$

with $\langle\dots\rangle_{\text{cl}}$ just some classical (commutative) expectation value, which we take to have a generic form

$$\langle f(H)\rangle_{\text{cl}} \equiv \int dH e^{-N\operatorname{Tr} V(H)} f(H)\,, \tag{3}$$

where $V(H)$ is some (usually polynomial) potential.

This definition (1) gives the rule of how to calculate the mixed moments out of the separate moments (if the matrices are not centered, *i.e.* if $\langle H_i\rangle \neq 0$, we use the trick of renaming them as $\tilde{H}_i \equiv H_i - \langle H_i\rangle$); since by definition $\langle \tilde{H}_1\tilde{H}_2\rangle = 0$, the first mixed moments are

$$\begin{aligned}
\langle H_1H_2\rangle &= \langle H_1\rangle\langle H_2\rangle\,,\\
\langle H_1H_2H_1H_2\rangle &= \langle H_1^2\rangle\langle H_2\rangle^2 + \langle H_1\rangle^2\langle H_2^2\rangle - \langle H_1\rangle^2\langle H_2\rangle^2\,,\\
\langle H_1H_1H_2H_2\rangle &= \langle H_1^2\rangle\langle H_2^2\rangle\,.
\end{aligned}$$

Note that freeness is a much more restrictive property than well-known independence in classical probability theory; mixed moments are combinations of products of the individual moments, and not just products. It turns out that it is precisely freeness that extends all the important features of independence to the non-commutative case; non-commutative probability together with the notion of freeness is known under the name of the *Free Random Variables calculus* (FRV).

2.2. The addition law

We will now introduce the concept of additivity and the R-transformation. The fundamental result in the FRV calculus [19] is that one can define (via the so-called *non-crossing partitions* [20]) an analogue of the cumulants, the so-called *free cumulants* $\kappa_{H,n}$, which obey the *additivity property*,

$$\kappa_{H_1+H_2,n} = \kappa_{H_1,n} + \kappa_{H_2,n}\,, \tag{4}$$

where H_1 and H_2 are two free Hermitian random matrices, just as the usual cumulants obey the additivity property in classical probability. The point is that we cannot simply relate the moments of $H_1 + H_2$ to the moments of the separate H_1 and H_2, as is done in the commutative case via the Fourier transformation, since now mixed moments of centered variables do not factorize; nevertheless, it is possible to construct other objects, the free cumulants, that possess this property and thus lead to the addition law, but their relation to the moments is far more involved than classically.

The essence of the construction is as follows: First, we introduce the generating function of the moments,

$$G_H(z) \equiv \sum_{n\geq 0} \frac{\langle H^n \rangle}{z^{n+1}} = \frac{1}{N} \left\langle \mathrm{Tr}\, \frac{1}{z1_N - H} \right\rangle_{\mathrm{cl}} , \tag{5}$$

which is called the *Green's function*. (Here "1_N" is the $N \times N$ unit matrix.) Second, we define an *a priori* unknown function, called the Voiculescu's R-transform, which is the generating function of the free cumulants,

$$R_H(z) \equiv \sum_{n\geq 0} \kappa_{H,n+1} z^n \,. \tag{6}$$

Now the additivity property (4) is simply translated in the language of the R-transform to the so-called *FRV addition law*,

$$R_{H_1+H_2}(z) = R_{H_1}(z) + R_{H_2}(z) \,. \tag{7}$$

Now because the moments and the free cumulants are related in a complicated way to each other, so do the complex functions $G_H(z)$ and $R_H(z)$ seem to be related in an involved manner. However, a very non-trivial result of FRV is that their relation is actually very simple — indeed, it turns out that a slight redefinition of the R-transform,

$$B_H(z) \equiv R_H(z) + \frac{1}{z} \,, \tag{8}$$

allows to write the relation between both functions in a compact form,

$$B_H(G_H(z)) = G_H(B_H(z)) = z \,, \tag{9}$$

i.e. the function $B_H(z) = R_H(z) + 1/z$ is the *functional inverse* of the Green's function. We like to call $B_H(z)$ the *Blue's function*, after Tony Zee, who has proposed this name and popularized FRV among physicists [21].

The construction looks thus as follows: Once we know the Green's functions $G_{H_1}(z)$, $G_{H_2}(z)$ of two given free Hermitian random matrix models H_1 and H_2, *i.e.* once we know all their moments, it is straightforward (namely,

by functional inversion) to get their Blue's functions $B_{H_1}(z)$, $B_{H_2}(z)$, *i.e.* also all their free cumulants, which are additive. Now the addition law (7) can be rewritten in the language of the Blue's function

$$B_{H_1+H_2}(z) = B_{H_1}(z) + B_{H_2}(z) - \frac{1}{z} \tag{10}$$

and it gives the Blue's function of the sum H_1+H_2. Now one more functional inversion leads to the Green's function $G_{H_1+H_2}(z)$.

So the *algorithm of adding two free Hermitian random matrices* may be summarized as follows:

1. Assuming we can construct the Green's functions $G_{H_1}(z)$ and $G_{H_2}(z)$, we obtain the Blue's functions $B_{H_1}(z)$ and $B_{H_2}(z)$ by functional inversion (9).

2. It amounts to apply the addition law (10) to get $B_{H_1+H_2}(z)$.

3. Finally, we again invert it functionally to find $G_{H_1+H_2}(z)$.

Let us recall how to use the Green's function to obtain the spectral properties of H. The fundamental problem in RMT is to find the averaged distribution of the eigenvalues λ_i of H,

$$\rho_H(\lambda) \equiv \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle_{\text{cl}} . \tag{11}$$

The Green's function can be easily used to reconstruct this density according to

$$\rho_H(\lambda) = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im} G_H(\lambda + i\varepsilon) , \tag{12}$$

which stems from the well-known formula $1/(\lambda+i\varepsilon) = \mathrm{pv}(1/\lambda) - i\pi\delta(\lambda)$. This means that the density of the eigenvalues can be read from the discontinuities of the Green's function.

2.3. Examples

For the sake of the further part of this note, let us now discuss two examples of Hermitian ensembles. As the first one, consider the *Gaussian Unitary Ensemble* (GUE). Since centered Gaussianity is characterized just by a single non-vanishing second cumulant (*i.e.* dispersion, $\kappa_{\text{GUE},2} \equiv d$), the R-transform of GUE is simply

$$R_{\text{GUE}}(z) = dz , \tag{13}$$

or equivalently

$$B_{\mathrm{GUE}}(z) = dz + \frac{1}{z}\,. \tag{14}$$

We functionally invert this Blue's function by the substitution $z \to G_{\mathrm{GUE}}(z)$, obtaining the quadratic equation

$$dG^2_{\mathrm{GUE}}(z) - zG_{\mathrm{GUE}}(z) + 1 = 0\,,$$

which is readily solved by

$$G_{\mathrm{GUE}}(z) = \frac{1}{2d}\left(z \pm \sqrt{z^2 - 4d}\right)\,. \tag{15}$$

Only the lower sign reproduces the correct asymptotic behavior, $G_{\mathrm{GUE}}(z) \sim 1/z$ for large z. The imaginary part of this Green's function, according to (12), yields the famous Wigner's semicircle law,

$$\rho_{\mathrm{GUE}}(\lambda) = \frac{1}{2\pi d}\sqrt{4d - \lambda^2}\,. \tag{16}$$

As the second example, let us take the *Cauchy ensemble*. Remarkably, the FRV calculus can be extended to *unbounded spectra*. In particular, one can introduce the notion of *free stability* in analogy to classical Lévy stability in probability theory [22], and even more surprisingly, classify all the stability classes (*i.e.* the functional forms of the R-transform) in full analogy to the Lévy classification of characteristics functions of power-like probability distributions. This is the so-called *Bercovici–Pata bijection* [23]. From the broad class of the free Lévy processes, we quote here the simplest one, *i.e.* the *centered free Cauchy model*. Its R-transform reads

$$R_{\mathrm{Cauchy}}(z) = -i\gamma\,, \tag{17}$$

and thus the Blue's function is

$$B_{\mathrm{Cauchy}}(z) = -i\gamma + \frac{1}{z}\,, \tag{18}$$

where γ is a positive real number. The Green's function is then

$$G_{\mathrm{Cauchy}}(z) = \frac{1}{z + i\gamma}\,, \tag{19}$$

and the spectral density stems quickly from it

$$\rho_{\mathrm{Cauchy}}(\lambda) = \frac{1}{\pi}\frac{\gamma}{\lambda^2 + \gamma^2}\,. \tag{20}$$

The fact that the spectral density of the free Cauchy model is functionally identical to the Cauchy density in classical probability is actually accidental; in a few other solvable cases there is only a one-to-one correspondence between the asymptotic forms, *i.e.* heavy tails. Note also that the potential $V(H)$ reproducing the spectral density of the Cauchy ensemble (20) is highly non-trivial and non-polynomial [3].

3. Addition of non-Hermitian ensembles

The crucial difference which arises in the non-Hermitian case (we denote an arbitrary non-Hermitian random matrix by X) is that the eigenvalues of X are complex in general; in the large-N limit they form two-dimensional domains ("islands") on the complex plane, in contrary to one-dimensional cuts in the Hermitian case. The Green's function loses its analyticity, *i.e.* it is analytic (holomorphic) only outside the eigenvalues' domains, whereas in the Hermitian case it is holomorphic everywhere except some one-dimensional cuts; hence the power series expansion no longer captures the full information about the Green's function, and it is exactly its non-holomorphic behavior that determines the eigenvalues' distribution on the two-dimensional supports. In the Hermitian case, working with the *complex* Green's function allowed us to infer *real* spectral distributions from the discontinuities on the complex plane. It is tempting to find a similar method in the non-Hermitian case of *complex* spectra. A natural generalization is the algebra of *quaternions*. Even though such speculations have appeared in the literature [28], an explicit realization appeared only very recently [15]. We are exploiting the following correspondence as a guiding tool,

$$\begin{array}{ccc} \text{Hermitian} & \longrightarrow & \text{non-Hermitian} \\ \downarrow & & \downarrow \\ \text{real spectrum} & \longrightarrow & \text{complex spectrum} \\ \downarrow & & \downarrow \\ \text{complex Green's function} & \longrightarrow & \text{quaternion Green's function} \end{array} \tag{21}$$

Here we present only the final results, referring for details to the original work [15] or to more pedagogical lectures [24].

First, we define the *quaternion Green's function* as

$$\mathcal{G}_X(Q) \equiv \frac{1}{N}\left\langle \mathrm{bTr}\frac{1}{Q\otimes 1_N - X^{\mathrm{D}}}\right\rangle_{\mathrm{cl}}, \tag{22}$$

where Q is an *arbitrary quaternion*,

$$Q = \left(\begin{array}{c|c} a & i\bar{b} \\ \hline ib & \bar{a} \end{array}\right)_{2\times 2} = x_0 1_2 + i\vec{x}\cdot\vec{\sigma} \tag{23}$$

(here $\sigma_{1,2,3}$ are the usual Pauli matrices, and $a \equiv x_0 + ix_3$, $b \equiv x_1 + ix_2$). The block-trace bTr and the operation X^{D} are defined as

$$\mathrm{bTr}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)_{2N\times 2N} \equiv \left(\begin{array}{c|c} \mathrm{Tr}\, A & \mathrm{Tr}\, B \\ \hline \mathrm{Tr}\, C & \mathrm{Tr}\, D \end{array}\right)_{2\times 2} \tag{24}$$

and

$$X^{\mathrm{D}} \equiv \left(\begin{array}{c|c} X & \\ \hline & X^{\dagger} \end{array}\right)_{2N\times 2N}, \tag{25}$$

respectively. This defines $\mathcal{G}_X$ as a quaternion function of a quaternion variable. The gain is the analogy to the Hermitian case — in particular, the crucial linearity in X^{D} of the denominator of this quaternion Green's function. We mention here that this approach can be viewed as a generalization of the matrix Green's function introduced by two of the present authors some time ago [25, 26], provided one identifies a with the complex variable z and b with the infinitesimal regulator $|\varepsilon|$. The advantage of working with quaternions is that $\mathcal{G}_X$ has several non-trivial properties as a function of Q which cannot be seen when we are infinitesimally close to the complex plane, as considered in [25, 26]. For the exact relations and correspondence to the so-called "Hermitization method" [27–29], we refer to the original works.

Similarly to the Hermitian case (see (9)), one introduces the functional inverse of the quaternion Green's function, for which we naturally coin the name "*quaternion Blue's function*",

$$\mathcal{G}_X(\mathcal{B}_X(Q)) = \mathcal{B}_X(\mathcal{G}_X(Q)) = Q\,, \tag{26}$$

for *any* quaternion Q. Again (compare (10)), the analogous *addition law* holds [15],

$$\mathcal{B}_{X_1+X_2}(Q) = \mathcal{B}_{X_1}(Q) + \mathcal{B}_{X_2}(Q) - Q^{-1}\,, \tag{27}$$

for any Q, and any two free non-Hermitian random matrices X_1 and X_2.

In particular, the advocated method provides a way to solve any non-Hermitian problem of the type $X = H_1 + iH_2$, for H_1 and H_2 free [15]. The resulting quaternion Blue's function reads [15]

$$\begin{aligned} \mathcal{B}_X(Q) \;&=\; \beta_{H_1}(q,\bar{q})1_2 + \beta_{H_2}q^{\mathrm{I}}, \overline{q^{\mathrm{I}}})\, i\sigma_3 \\ &\quad - \left(\beta'_{H_1}(q,\bar{q}) + \beta'_{H_2}(q^{\mathrm{I}}, \overline{q^{\mathrm{I}}}) + \frac{1}{\det Q}\right) Q^{\dagger}\,, \end{aligned} \tag{28}$$

where $q = x_0 + i|\vec{x}|$ and $\bar{q} = x_0 - vi|\vec{x}|$ are the two eigenvalues of the quaternion Q (23), while q^{I} and $\overline{q^{\mathrm{I}}}$ are the eigenvalues of the quaternion $Q^{\mathrm{I}} \equiv Qi\sigma_3$,

and the scalar functions β_H and β'_H are defined as

$$\beta_H(q,\bar{q}) \;\equiv\; \frac{qB_H(q)-\bar{q}B_H(\bar{q})}{q-\bar{q}}\,, \tag{29}$$

$$\beta'_H(q,\bar{q}) \;\equiv\; \frac{B_H(q)-B_H(\bar{q})}{q-\bar{q}}\,. \tag{30}$$

To recover the spectral density and the support of the eigenvalues, one has first to invert functionally the above equation for any quaternion Q, obtaining thus the quaternion Green's function. Second, one substitutes an arbitrary quaternion Q by the diagonal 2×2 matrix $\mathrm{diag}(z,\bar{z})$, *i.e.* one makes the projection from the quaternion space onto the complex plane. This completes the solution of the problem. The resulting Green's function is the 2×2 matrix

$$\mathcal{G}_X(z,\bar{z}) = \left(\begin{array}{c|c} \mathcal{G}_X^{11}(z,\bar{z}) & \mathcal{G}_X^{1\bar{1}}(z,\bar{z}) \\ \hline \mathcal{G}_X^{\bar{1}1}(z,\bar{z}) & \mathcal{G}_X^{\bar{1}\bar{1}}(z,\bar{z}) \end{array}\right)_{2\times 2}\,. \tag{31}$$

The upper left entry,

$$\mathcal{G}_X^{11}(z,\bar{z}) \equiv G_X(z,\bar{z})\,, \tag{32}$$

is the desired non-holomorphic resolvent, and the two-dimensional spectral density follows from it as

$$\rho_X(z,\bar{z}) = \frac{1}{\pi}\partial_{\bar{z}}\, G_X(z,\bar{z})\,. \tag{33}$$

The product of the off-diagonal elements carries also an interesting spectral information [30], namely it is the *correlator between the left and right eigenvectors of* X, introduced in [31]. Explicitly,

$$C_X(z,\bar{z}) \equiv \mathcal{G}_X^{1\bar{1}}(z,\bar{z})\mathcal{G}_X^{\bar{1}1}(z,\bar{z}) = -\frac{\pi}{N}\left\langle \sum_{i=1}^{N} (L_i|L_i)(R_i|R_i)\delta^{(2)}(z-\lambda_i)\right\rangle_{\mathrm{cl}}\,. \tag{34}$$

In particular, this correlator allows to find the shape of the bordering "coastline" of the eigenvalues' "islands" — indeed, on the borderline the correlator must vanish, which gives us the equation of the borderline of the eigenvalues' domains to be

$$C_X(z,\bar{z}) = 0\,. \tag{35}$$

4. Explicit calculations

4.1. The algorithm

The construction presented in the previous chapter solves *any* problem of the type $X = H_1 + iH_2$, where H_1 and H_2 are Hermitian and free. In practice, explicit calculations might, however, be quite tedious. To avoid the procedure of functional inverting of quaternion functions, we present an operational algorithm [15] which allows to solve the problem without any explicit reference to quaternions (of course, it is obtained from the quaternion construction by some tedious algebra). The advantage is that the only necessary input comes from Hermitian random matrix theory, *i.e.* only the knowledge of Hermitian Green's functions $G_{H_1}(z)$ and $G_{H_2}(z)$, or rather their functional inverses, the Blue's functions $B_{H_1}(z)$ and $B_{H_2}(z)$, is required.

The algorithm goes as follows:

1. Write down the two equations
$$\begin{aligned} B_{H_1}(g) &= x + \frac{m}{g}\,, \\ B_{H_2}(g^{\mathrm{I}}) &= y + \frac{1-m}{g^{\mathrm{I}}}\,, \end{aligned} \tag{36}$$
where $z \equiv x + iy$, with three unknown quantities, complex g, g^{I}, and real m. Find $g + \bar{g}$, $g^{\mathrm{I}} + \overline{g^{\mathrm{I}}}$, $g\bar{g}$, $g^{\mathrm{I}}\overline{g^{\mathrm{I}}}$ as functions of m.

2. Compute m from the third equation,
$$g\bar{g} = g^{\mathrm{I}}\overline{g^{\mathrm{I}}}. \tag{37}$$

3. Derive $g + \bar{g}$, $g^{\mathrm{I}} + \overline{g^{\mathrm{I}}}$ and $|g|^2$ from the above two steps.

4. The non-holomorphic Green's function and the correlator between the left and right eigenvectors of $X = H_1 + iH_2$ are given by
$$G_X(x,y) = \frac{g+\bar{g}}{2} - i\frac{g^{\mathrm{I}} + \overline{g^{\mathrm{I}}}}{2} = \mathrm{Re}\, g - i\mathrm{Re}\, g^{\mathrm{I}}\,, \tag{38}$$
$$C_X(x,y) = \left(\frac{g+\bar{g}}{2}\right)^2 + \left(\frac{g^{\mathrm{I}}+\overline{g^{\mathrm{I}}}}{2}\right)^2 - |g|^2 = (\mathrm{Re}\, g)^2 - (\mathrm{Im}\, g^{\mathrm{I}})^2\,. \tag{39}$$

5. The spectral density and the borderline of the support of the eigenvalues are now easily computed via, respectively
$$\rho_X(x,y) = \frac{1}{\pi}\,\partial_{\bar{z}} G_X(z,\bar{z})\,, \tag{40}$$
$$C_X(z,\bar{z}) = 0\,. \tag{41}$$

4.2. The elliptic Gaussian case

As the first example, let us consider the non-Hermitian model $X = H_1 + iH_2$, where both H_1, H_2 are the free centered GUE random matrices with dispersions d_1 and d_2, respectively. Both Blue's functions read therefore (see (14)), $B_{H_i}(z) = d_i z + 1/z$, for $i = 1, 2$, hence Step 1 of the algorithm (36) gives two quadratic equations,

$$\begin{aligned} d_1 g^2 - xg + 1 - m &= 0\,, \\ d_2 (g^{\mathrm{I}})^2 - y g^{\mathrm{I}} + m &= 0\,. \end{aligned} \tag{42}$$

Even without solving them, but just from the Vieta's rules, we get

$$\begin{aligned} g + \bar{g} &= \frac{x}{d_1}, \qquad |g|^2 = \frac{1-m}{d_1}\,, \\ g^{\mathrm{I}} + \overline{g^{\mathrm{I}}} &= \frac{y}{d_2}\,, \qquad |g^{\mathrm{I}}|^2 = \frac{m}{d_2}\,. \end{aligned} \tag{43}$$

Step 2 (37) leads, therefore, to a linear equation for m,

$$\frac{1-m}{d_1} = \frac{m}{d_2} \quad \Rightarrow \quad m = \frac{d_2}{d_1 + d_2}\,, \tag{44}$$

In Step 3 we write thus easily,

$$g + \bar{g} = \frac{x}{d_1}\,, \qquad g^{\mathrm{I}} + \overline{g^{\mathrm{I}}} = \frac{y}{d_2}\,, \qquad |g|^2 = |g^{\mathrm{I}}|^2 = \frac{1}{d_1 + d_2}\,. \tag{45}$$

Step 4 completes the problem,

$$\begin{aligned} G_{H_1+iH_2}(x,y) &= \frac{1}{2}\left(\frac{x}{d_1} - i\frac{y}{d_2}\right), \\ C_{H_1+iH_2}(x,y) &= \frac{1}{4}\left(\frac{x^2}{d_1^2} + \frac{y^2}{d_2^2}\right) - \frac{1}{d_1 + d_2}\,. \end{aligned} \tag{46}$$

Finally, Step 5 gives the uniform spectral density

$$\rho_{H_1+iH_2}(x,y) = \frac{1}{4\pi}\left(\frac{1}{d_1} + \frac{1}{d_2}\right) \tag{47}$$

and the borderline of the support,

$$\frac{x^2}{d_1^2} + \frac{y^2}{d_2^2} = \frac{4}{d_1 + d_2}\,, \tag{48}$$

which is the ellipse with semi-axes $2d_1/\sqrt{d_1 + d_2}$ and $2d_2/\sqrt{d_1 + d_2}$. In such a simple way we have reproduced the classical results for the circular ($d_1 = d_2$) and elliptic Gaussian models [16–18, 31].

4.3. The hyperbolic Cauchy case

After this presentation of how our quaternion algorithm provides a fast way of reproducing the Ginibre–Girko ellipse, let us move to another example, this time of a highly non-trivial non-Hermitian random matrix model with heavy tails, namely $X = C_1 + iC_2$, where both C_1, C_2 are free centered Cauchy ensembles with ranges γ_1 and γ_2, respectively. Both Blue's functions are thus (see (18)), $B_{C_i}(z) = -i\gamma_i + 1/z$, for $i = 1, 2$. In Step 1 (36) we get two linear equations which give

$$g = \frac{1-m}{x+i\gamma_1}\,, \qquad g^{\mathrm{I}} = \frac{m}{y+i\gamma_2}\,, \tag{49}$$

and, therefore,

$$\begin{aligned} g+\bar{g} &= \frac{2x(1-m)}{X^2}\,, & |g|^2 &= \frac{(1-m)^2}{X^2}\,, \\ g^{\mathrm{I}}+\overline{g^{\mathrm{I}}} &= \frac{2ym}{Y^2}\,, & |g^{\mathrm{I}}|^2 &= \frac{m^2}{Y^2}\,. \end{aligned} \tag{50}$$

where for short, $X \equiv \sqrt{x^2+\gamma_1^2}$ and $Y \equiv \sqrt{y^2+\gamma_2^2}$. Step 2 yields a quadratic equation for m, with solutions

$$m = \frac{Y}{Y \pm X} \tag{51}$$

(only the upper sign turns out to reproduce the positive spectral density), and Step 3 follows trivially. Step 4 thus completes the solution of the problem

$$\begin{aligned} G_{C_1+iC_2}(x,y) &= \frac{1}{X+Y}\left(\frac{x}{X} - i\frac{y}{Y}\right)\,, \\ C_{C_1+iC_2}(x,y) &= \frac{1}{(X+Y)^2}\left(-\frac{\gamma_1^2}{X^2} + \frac{y^2}{Y^2}\right)\,. \end{aligned} \tag{52}$$

Finally, Step 5 gives the non-uniform spectral density

$$2\pi\rho_{C_1+iC_2}(x,y) = -\frac{1}{(X+Y)^2}\left(\frac{x^2}{X^2}+\frac{y^2}{Y^2}\right) + \frac{1}{X+Y}\left(\frac{\gamma_1^2}{X^3}+\frac{\gamma_2^2}{Y^3}\right) \tag{53}$$

and the borderline of the support,

$$xy = \pm\gamma_1\gamma_2\,, \tag{54}$$

which consists of two pairs of hyperbolas. The spectrum is therefore localized on the infinite "hyperbolic cross". Fig. 1 shows three dimensional plot of the density (53) in case of equal ranges $\gamma_1 = \gamma_2$.

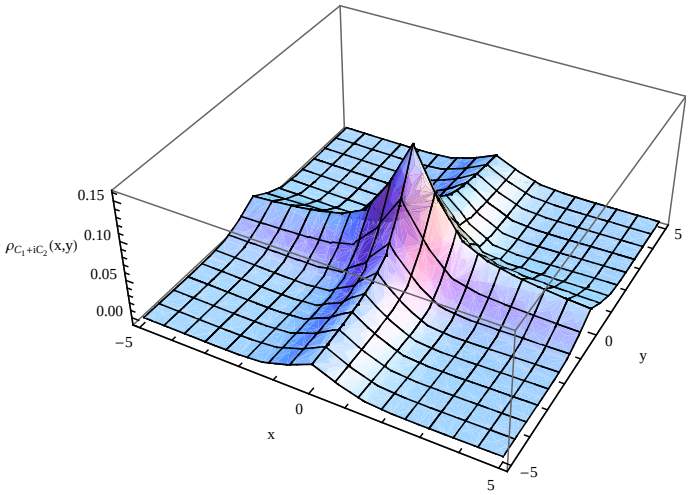

Fig. 1. Three-dimensional plot of the spectral density for non-Hermitian Cauchy ensemble $C_1 + iC_2$. Both ranges γ_i are equal, hence resulting "Cauchy cross" is symmetric.

We are not aware of the appearance of these formulae in the literature, which is also one of the reasons why we have performed numerical checks to verify these analytical results. Certainly, the above expressions are valid for infinite matrices, and since the ways of assessing non-Hermitian $1/N$-corrections in this case are so far unknown, numerical simulation can be viewed as a verification of the bulk properties of the ensemble only. Another complication comes from the fact that very simple spectrum of the free Cauchy model corresponds to a highly non-trivial non-polynomial measure [3] of the type $V(\lambda) = \ln(\lambda^2 + \gamma^2)$ (compare *e.g.* to the well-known Gaussian case of $V(\lambda) = \lambda^2$). We avoid the problem of such a measure by using the recently established link [5] between the so-called *Wigner Lévy matrices* and *Free Lévy matrices*. The Wigner Lévy matrices, introduced by Bouchaud and Cizeau [1], are obtained by filling the elements of the matrix with random numbers generated from a classical pdf of the Lévy type. By construction, such matrices are stable, but they are not rotationally invariant. One may, however, enforce the rotational invariance in the following way: We take a set of the Wigner Lévy matrices $\{A_k : k = 1, \ldots, M\}$, then rotate each one of them with an independent random orthogonal transformation O_k, and finally add them to each other,

$$B \equiv \frac{1}{M^{1/\alpha}} \sum_{k=1}^{M} O_k \, A_k \, O_k^{\mathrm{T}} \,, \tag{55}$$

where α is the stability index, which for the Cauchy case equals to one. The resulting matrix B is rotationally invariant and represents a Free Random Variable. We have performed [5] extensive numerical checks verifying this observation. Since numerical algorithms for Lévy pdfs and random orthogonalization are well-developed, the above method of enforcing the rotational invariance is the fastest and most reliable way of generating symmetric free Lévy ensembles. We use it here to verify the "Cauchy cross". First, we generate free Hermitian ensembles C_1 and C_2, checking that each of them reproduces the correct spectral density (20). Next, we generate the non-Hermitian ensemble $C_1 + iC_2$, and plot the resulting complex spectrum. Fig. 2 shows the comparison of the numerical simulation *versus* analytical result for the symmetric ($\gamma_1 = \gamma_2$) cross. Although we have used only 500 $N \times N$ matrices, with $N = 50$, the agreement is fair. We are also planning a higher-statistics simulation to confirm the analytical shape of the spectral distribution.

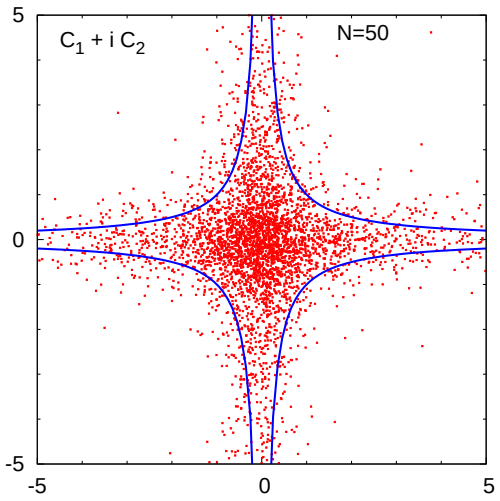

Fig. 2. A numerical simulation *versus* the analytical prediction for the hyperbolic Cauchy cross.

5. Conclusions and prospects

We have presented and solved a sample of a non-Hermitian random matrix ensemble with a heavy tail; we have chosen to consider the Cauchy model because of two reasons — first, the model has surprising calculational simplicity within the quaternion scheme; second, it represents the "extreme" heavy-tailed case, since even the first moment is divergent. Because the quaternion technique does not rely on the existence of the moments, it is

easy to extend several classical Gaussian non-Hermitian ensembles to the Lévy domain, including the generic cases of "quantum chaotic scattering" [7] (H_1 Gaussian, H_2 Wishart), the complex Pastur ensemble (H_1 deterministic, H_2 Gaussian), "bridging via dissipation" [8] (H_1 deterministic, H_2 Wishart), to mention only a few. Some of these extensions have already been presented at this conference [32]. The method is also applicable to circular unitary ensembles (CUE) [33], enlarging the class of solvable non-Hermitian models. Finally, the presented method offers, probably for the first time, a way to study genuine non-Hermitian Lévy ensembles, *i.e.* of the type $L_1 + iL_2$, where L_1 and L_2 are arbitrary free Lévy random matrices — in hope to shed more light on the difficult subject of non-Hermiticity on infinite supports.

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