ON ENERGY OF THE FRIEDMAN UNIVERSES IN CONFORMALLY FLAT COORDINATES

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Recently many authors have calculated energy of the Friedman universes by using coordinate-dependent double index energy-momentum complexes in Cartesian comoving coordinates (t, x, y, z) and concluded that the flat and closed Friedman universes are energy-free. In this paper by using Einstein canonical energy-momentum complex and by doing calculations in conformally flat coordinates we show that such conclusion is incorrect. The results obtained in this paper are compatible with the results of our previous paper, see J. Garecki, *Found. Phys.* **37**, 341 (2007), where we have used coordinate-independent averaged energy-momentum tensors to analyze the energy of Friedman universes.

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1. Introduction

A spacetime is *conformally flat* if there exist coordinates (τ, x, y, z) in which the line element ds^2 reads

$$\begin{aligned} ds^2 &= \Omega^2(\tau, x, y, z)\left(d\tau^2 - dx^2 - dy^2 - dz^2\right) \\ &= \Omega^2(\tau, x, y, z)\eta_{ik}dx^i dx^k\,, \end{aligned} \tag{1}$$

where η_{ik} means the Minkowski metric, *i.e.*, $\eta_{ik} = diag(1, -1, -1, -1)$[1]. $\Omega(\tau, x, y, z)$ is a sufficiently smooth and positive-definite function called *conformal factor*.

We will call the coordinates (τ, x, y, z) *the conformally flat or conformally inertial* coordinates.

[1] We prefer signature $(+---)$ and we will use geometrized units in which $G = c = 1$.

The conformally flat coordinates are determined up to 15-parameters Lie group of the *conformal transformations*. This group contains, as a subgroup, the 10-parameters Poincare' group [1, 2].

It is obvious that the conformally flat coordinates are *geometrically and physically distinguished*, like inertial coordinates (t, x, y, z) in a Minkowski spacetime[2].

The necessary and sufficient condition for a four (or more) dimensional spacetime to be conformally flat is that *Weyl conformal curvature tensor* [3] vanishes. Physically, the Weyl tensor describes source-free, *i.e.*, independent of matter, gravitational field.

If a spacetime is neither *flat* nor *asymptotically flat* (at spatial or at null infinity), but it is only *conformally flat*, then one should choose conformally flat coordinates to analyze energy and momentum of such a spacetime by using *coordinate-dependent*[3], double index energy-momentum complexes, matter and gravitation.

In this context we would like to remark that already in the case of a *Minkowski* spacetime the energy-momentum complexes can be reasonably used only in an "affine" coordinates in which the metric components are constant, *e.g.*, in an inertial (= Lorentzian) coordinates (t, x, y, z) in which the line element ds^2 reads

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \,. \tag{2}$$

On the other hand, in an *asymptotically flat* spacetime one can reasonably use these complexes only in an asymptotically flat (= asymptotically inertial or asymptotically Lorentzian) coordinates. So, in the case of a *conformally flat spacetime* one should use the energy-momentum complexes in the conformally flat coordinates, *i.e.*, in the *conformally inertial coordinates*.

If a spacetime is neither asymptotically flat nor conformally flat, then one can reasonably use the energy-momentum complexes to an averaged and covariant analyzis of the gravitational and matter fields in a normal Riemann coordinates [8, 9].

It is commonly known that the Friedman universes are conformally flat [4–6] so that it is natural to analyze their energetic content in conformally flat coordinates (τ, x, y, z).

Recently many authors have calculated the energy of the Friedman and also more general, only spatially homogeneous universes [7] mainly by using coordinate-dependent double index energy-momentum complexes. These

[2] For example, they determine the same causal structure of the spacetime as inertial coordinates (t, x, y, z) in a Minkowski spacetime.

[3] By "coordinate dependent quantity" we mean a quantity which is not a tensor (in general–which is not a tensor valued p-form). By "coordinate independent quantity" we mean a tensor quantity (in general — a tensor valued p-form).

authors did not perform their calculations in the conformally flat coordinates (τ, x, y, z), but in the so-called *Cartesian comoving coordinates* (t, x, y, z) in which the line element ds^2 of the Friedman universes has the form

$$ds^2 = dt^2 - \frac{a^2(t)\left(dx^2 + dy^2 + dz^2\right)}{\left[1 + \frac{k(x^2+y^2+z^2)}{4}\right]^2}, \tag{3}$$

where $a = a(t)$ is the *scale factor*, and $k = 0, \pm 1$ is the *normalized curvature* of the slices $t = \text{const}$, t denotes the universal time parameter called *cosmic time*.

In the Cartesian comoving coordinates (t, x, y, z) only spatial part of the full metric is conformally flat.

The above mentioned authors have concluded that the closed Friedman universes have zero net global energy and that the flat Friedman universes are energy free, locally and globally[4]. Of course, the results which one obtains in comoving coordinates other than (3) are radically different (see, *e.g.*, [8]).

For an open Friedman universe one gets divergent global results already in the Cartesian comoving coordinates (t, x, y, z).

It seems that the problem of the global quantities of the Friedman and more general spatially homogeneous, universes *is not a well-defined* physical problem, because *one cannot measure* the global energy and momentum of the whole universe. The global energy and momentum, the global angular momentum have their physical meaning only in the case of an asymptotically flat spacetime (at spatial or at null infinities) where these global quantities can be measured. So, the calculations of the global energy and momentum, the global angular momentum of a universe have only *pure mathematical sense*.

In the case of a universe the physically sensible are only the *local quantities* (*e.g.*, energy density and its flux) and *global quantities of an isolated part of the universe* (*e.g.*, global energy of the Solar System). If we use a coordinate-dependent double index energy-momentum complex, then all these quantities should be calculated in *a privileged coordinates*, *e.g.*, in the case of a Friedman universe one should use for this purpose the geometrically and physically favored *conformally flat coordinates* (τ, x, y, z).

We would like to emphasize that the global result $E = 0$ obtained in the Cartesian comoving coordinates (t, x, y, z) for a closed Friedman universe is obtained if we take the limit $r \longrightarrow \infty$ for integration over the slice $t = \text{const}$, where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial coordinate. But if $r \longrightarrow \infty$, then

[4] I must admit that in my old papers I also followed this conclusion. Now I think that it was incorrect.

the spatial conformal factor $a^2(t)/[(1+\frac{r^2}{4})^2]$ *tends to zero* which leads to singularity.

To summarize, one should doubt in physical validity of the conclusion that the closed and flat Friedman universes and spatially homogeneous Kasner and Bianchi universes are energy–free.

In this context, we would like to remark that by using our coordinate independent *averaged relative energy-momentum tensors* [8] or *superenergy tensors* [9] one can make *mathematically correct and coordinate independent* local analysis of the Friedman and more general universes. One can also formally calculate, correctly from the mathematical point of view, the global, integral quantities for such universes.

It is interesting that going this way one gets a *positive-definite* energy for the all Friedman universes and also for Kasner and Bianchi type I universes[5]. So, in our opinion, all these universes *do not have to be* energetic emptiness.

In this paper we present the results of the analysis of the energetic content of the Friedman universes in the *distinguished* conformally flat coordinates (τ, x, y, z). These coordinates are the most appropriate for this goal, if one uses an energy-momentum complex. Our analysis will be done with the help of the most important in general relativity Einstein's canonical double index energy-momentum complex

$$
{}_{\mathrm{E}}K_i^{\ k} := \sqrt{|g|}\left(T_i^{\ k} + {}_{\mathrm{E}}t_i^{\ k}\right) = {}_{\mathrm{F}}U_i^{\ [kl]}{}_{,l}\,, \tag{4}
$$

where ${}_{\mathrm{F}}U_i^{\ [kl]} = (-){}_{\mathrm{F}}U_i^{\ [lk]}$ are Freud's superpotentials, and ${}_{\mathrm{E}}t_i^{\ k}$ are the components of the canonical Einstein's energy-momentum pseudotensor of the gravitational field [10–12]. $T_i^{\ k}$ are the components of the symmetric energy-momentum tensor of matter.

As we will see, by using this energy-momentum complex in the conformally flat coordinates (τ, x, y, z), *one cannot conclude* that the Friedman universes have zero net energy, locally or globally.

The analogous result one can obtain by using any other reasonable double index energy-momentum complex.

We hope that this paper and the our previous paper [8] convincingly show that the Friedman universes *are not energetic emptiness*, neither locally nor globally.

Finishing this section we would like to emphasize an important advantage of the conformally flat coordinates (τ, x, y, z) over the Cartesian comoving coordinates (t, x, y, z). Namely, solving the energy-momentum problem of the Friedman universes in Cartesian comoving coordinates (t, x, y, z) one uses *only the line element* (3) *independently* of the Einstein equations and

[5] More general spatially homogeneous universes have not been considered yet.

their solutions. On the other hand, the results obtained in conformally flat coordinates (τ, x, y, z) *explicitly depend* not only on the Friedman–Lemaitre–Robertson–Walker line element ds^2 but also on *the solutions* of the Einstein equations.

Without losing generality we will consider in this paper only dust the Friedman universes.

The paper is organized as follows. In Section 2 we discuss the dust Friedman universes in conformally flat coordinates (τ, x, y, z), and in Section 3 we analyze the energy and its flux for dust Friedman universes in these coordinates. Our analysis will be performed with the help of the Einstein canonical energy-momentum complex. Finally, in Section 4 we give our conclusion.

2. Dust Friedman universes in the conformally flat coordinates $(\boldsymbol{\tau}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$

2.1. Closed dust Friedman universes ($k = 1$)

Let us consider the Friedman–Lemaitre–Robertson–Walker (FLRW) line element

$$ds^2 = a^2(\eta)\left\{d\eta^2 - d\chi^2 - sin^2\chi\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right\} \tag{5}$$

with the following ranges of the coordinates $(\eta, \chi, \theta.\varphi)$:

$$0 < \chi < \pi\,, \qquad 0 < \theta < \pi\,, \qquad 0 < \varphi < 2\pi\,, \qquad \chi - \pi < \eta < \pi - \chi\,. \tag{6}$$

Physically, the coordinate η is the *conformal time*, χ is a *radial coordinate*, and θ, φ are ordinary *spherical angular coordinates* (see, *e.g.*, [13]).

The bijective transformation

$$\begin{aligned}
&\tau + r = \tan\left(\frac{\eta+\chi}{2}\right), \qquad \tau - r = \tan\left(\frac{\eta-\chi}{2}\right),\\
&\theta' = \theta\,, \qquad \varphi' = \varphi\,,\\
&0 < \chi < \pi\,, \qquad \chi - \pi < \eta < \pi - \chi\,, \qquad 0 < \theta < \pi\,, \qquad 0 < \varphi < 2\pi\,,
\end{aligned} \tag{7}$$

with inverse

$$\begin{aligned}
\eta &= \arctan(\tau + r) + \arctan(\tau - r)\,, \quad -\infty < \tau + r < \infty\,,\\
\chi &= \arctan(\tau + r) - \arctan(\tau - r)\,, \quad -\infty < \tau - r < \infty\,,\ 0 < r < \infty,\\
\theta &= \theta'\,, \qquad \varphi = \varphi'\,, \qquad 0 < \theta' < \pi\,, \qquad 0 < \varphi' < 2\pi\,,
\end{aligned} \tag{8}$$

map this spacetime onto a conformally flat spacetime with the following line element

$$\begin{aligned}
ds^2 &= \frac{4a^2(\tau, x, y, z)}{[1 + (\tau + r)^2][1 + (\tau - r)^2]}\,\eta_{ik}\, dx^i dx^k\\
&=: \Omega^2(\tau, x, y, z)\,\eta_{ik}\, dx^i dx^k\,,
\end{aligned} \tag{9}$$

where

$$\begin{aligned} x &= r\sin\theta\cos\varphi\,, \qquad y = r\sin\theta\sin\varphi\,, \qquad z = r\cos\theta\,, \\ r &= \sqrt{x^2+y^2+z^2}\,. \end{aligned} \tag{10}$$

This means that the transformation (7) covers the region of the spacetime (5)–(6) with the conformally flat coordinates (τ, x, y, z). One can call these coordinates *the conformally inertial coordinates*.

If we omit the angular coordinates (θ, φ) then this region is the triangle

$$0 < \chi < \pi\,, \qquad \chi - \pi < \eta < \pi - \chi \tag{11}$$

on the plane (η, χ).

Now let us consider the closed dust Friedman universe with the following line element in the same coordinates $(\eta, \chi, \theta, \varphi)$

$$ds^2 = a^2(\eta)\left\{d\eta^2 - d\chi^2 - \sin^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\}\,, \tag{12}$$

with

$$a(\eta) = a_0\left(1 + \cos\eta\right)\,, \qquad t(\eta) = a_0\left(\eta + \pi + \sin\eta\right)\,, \tag{13}$$

and with the following ranges of the coordinates $(\eta, \chi, \theta, \varphi)$

$$-\pi < \eta < \pi\,, \qquad 0 < \chi < \pi\,, \qquad 0 < \theta < \pi\,, \qquad 0 < \varphi < 2\pi\,. \tag{14}$$

The coordinates $(\eta, \chi, \theta, \varphi)$ are comoving, *i.e.*, the dust particles and the *fundamental observers* are at rest in these coordinates.

Here $a(\eta)$ is the *scale factor* and t is the *cosmic time*; $a_0 = \frac{4}{3}\pi\rho a^3 = \mathrm{const}$ is the first integral of the Friedman equations. If we omit the angular coordinates (θ, φ), then this universe is a rectangle $(-)\pi < \eta < \pi,\ 0 < \chi < \pi$ on the plane of the variables $\eta,\ \chi$. Comparing this rectangle with the previous triangle one can easily see that the conformally flat coordinates (τ, x, y, z) cover only a half of the closed dust Friedman universe which is determined by the following ranges of the coordinates $(\chi, \eta, \theta, \varphi)$

$$0 < \chi < \pi\,, \qquad \chi - \pi < \eta < \pi - \chi\,, \qquad 0 < \theta < \pi\,, \qquad 0 < \varphi < 2\pi\,. \tag{15}$$

It is worth emphasizing that only one slice, $\eta = 0$, of the closed dust Friedman universe is *entirely covered* by the conformally flat coordinates $(\tau = 0, x, y, z)$. Any other slice $\eta = \eta_0 \neq 0$ is only *partially covered* by these coordinates.

Applying an active point of view one can say that this distinguished slice $\eta = 0$ is mapped onto the subspace

$$\tau = 0\,, \qquad -\infty < x < \infty\,, \qquad -\infty < y < \infty\,, \qquad -\infty < z < \infty \tag{16}$$

of the conformally flat spacetime (τ, x, y, z) which has the line element (9) with

$$a(\tau, x, y, z) = a_0 \{1 + \cos[\arctan(\tau + r) + \arctan(\tau - r)]\} \, . \tag{17}$$

The limiting values

$$x =^{+}_{-} \infty \, , \qquad y =^{+}_{-} \infty \, , \qquad z =^{+}_{-} \infty \tag{18}$$

are not admissible by the condition $\Omega(0, x, y, z) > 0$.

It follows that in conformally flat coordinates it is possible to calculate integrals only over the distinguished spatial slice $\eta = 0$[6]. This fact is very important, *e.g.*, for formal calculation of the global energy and momentum of a closed dust Friedman universe.

It is very interesting that in the conformally flat coordinates (τ, x, y, z) the initial singularity at $\eta = (-)\pi$ and the final singularity at $\eta = \pi$ are moved to $\tau = -\infty$ and to $\tau = \infty$ respectively, *i.e.*, we *have no cosmological singularity* in this case at a finite moment of the conformal time coordinate τ.

Matter and comoving observers are not at rest in the conformally flat coordinates (τ, x, y, z). They both move with the same 4-velocity

$$\begin{aligned} u^0 &= \frac{1 + \tau^2 + r^2}{2a(\tau, x, y, z)} \, , \qquad u^1 = \frac{\sin\theta \cos\varphi \cdot \tau \cdot r}{a(\tau, x, y, z)}, \\ u^2 &= \frac{\sin\theta \cos\varphi \cdot \tau \cdot r}{a(\tau, x, y, z)} \, , \quad u^3 = \frac{\cos\theta \cdot \tau \cdot r}{a(\tau, x, y, z)} \, , \end{aligned} \tag{19}$$

where

$$\begin{aligned} a &= a_0 \{1 + \cos[\arctan(\tau + r) + \arctan(\tau - r)]\} \, , \\ \sin\theta &= \frac{\sqrt{x^2 + y^2}}{r} \, , \quad \cos\theta = \frac{z}{r} \, , \quad \cos\varphi = \frac{x}{\sqrt{x^2 + y^2}} \, , \\ \sin\varphi &= \frac{y}{\sqrt{x^2 + y^2}} \, , \quad r = \sqrt{x^2 + y^2 + z^2} \, . \end{aligned} \tag{20}$$

Only fundamental observers which lie on the distinguished slice $\eta = 0$ also remain at rest in the conformally flat coordinates (τ, x, y, z) in the slice $\tau = 0$.

[6] $\eta = 0$ corresponds to space $\tau = 0$ in the conformally flat coordinates (τ, x, y, z) as it was already mentioned.

2.2. Open dust Friedman universes ($k=-1$)

Now, let us consider an open dust Friedman universe endowed with the same comoving coordinates $(\eta, \chi, \theta, \varphi)$ as in the closed case.

We have (see, *e.g.*, [13])

$$\begin{aligned} ds^2 &= a^2(\eta)\left\{d\eta^2 - d\chi^2 - \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\}, \\ a &= a_0(\cosh\eta - 1), \quad t = a_0(\sinh\eta - \eta), \end{aligned} \tag{21}$$

where $a_0 = \frac{4}{3}\pi\rho a^3 =$const and

$$0 < \eta < \infty, \qquad 0 < \chi < \infty, \qquad 0 < \theta < \pi, \qquad 0 < \varphi < 2\pi. \tag{22}$$

Then, the transformation

$$\begin{aligned} r &= \frac{a_0}{2}e^{\eta}\sinh\chi, \qquad \tau = \frac{a_0}{2}e^{\eta}\cosh\chi, \qquad \tau > \frac{a_0}{2}, \qquad r > 0, \\ \theta' &= \theta, \qquad \varphi' = \varphi, \end{aligned} \tag{23}$$

with inverse

$$\begin{aligned} &\eta = \ln\left(\frac{2\sqrt{\tau^2 - r^2}}{a_0}\right), \qquad \tau^2 - r^2 > \frac{a_0^2}{4}, \qquad \theta = \theta', \qquad \varphi = \varphi', \\ &\tanh\chi = \frac{r}{\tau} \longrightarrow \sinh\chi = \frac{\tau^2}{\tau^2 - r^2}, \end{aligned} \tag{24}$$

brings the line element (21)–(22) to the conformally flat form

$$\begin{aligned} ds^2 &= \left(1 - \frac{a_0}{2\sqrt{\tau^2 - r^2}}\right)^4 \eta_{ik}dx^i dx^k \\ &=: \Omega^2(\tau, x, y, z)\eta_{ik}\, dx^i dx^k. \end{aligned} \tag{25}$$

Here the *conformal factor* $\Omega = \left(1 - \frac{a_0}{2\sqrt{\tau^2 - r^2}}\right)^2$, and $\tau^2 - r^2 > \frac{a_0}{2}$.
$r = \sqrt{x^2 + y^2 + z^2}$, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.

From an active point of view the transformation (23) maps the open dust Friedman universe (21)–(22) onto interior of the future light cone $\tau^2 - x^2 - y^2 - z^2 = 0$ of a Minkowskian spacetime which line element in an inertial coordinates reads

$$ds^2 = \eta_{ik}dx^i dx^k. \tag{26}$$

Under this mapping, a slice $0 < \eta = \eta_0$ of the open dust Friedman universe is mapped onto a hyperboloid $\tau^2 - r^2 = B^2$, $B^2 := \frac{a_0^2 e^{2\eta_0}}{4}$ in the spacetime with the line element (26).

In the conformally flat coordinates (τ, x, y, z) the dust matter filling the open Friedman universe and comoving fundamental observers also *are not at rest*. Namely, they have the following 4-velocity in these coordinates

$$u^0 = \frac{\tau}{a}\,, \qquad u^1 = \frac{x}{a}\,, \qquad u^2 = \frac{y}{a}\,, \qquad u^3 = \frac{z}{a}\,, \tag{27}$$

where

$$a = a_0 \left(\frac{\tau^2 - r^2 + a_0^2/4}{a_0\sqrt{\tau^2 - r^2}} - 1 \right)\,, \qquad r^2 = x^2 + y^2 + z^2\,. \tag{28}$$

2.3. Flat dust Friedman universes $(k = 0)$

Finally, let us consider a flat Friedman universe filled with dust matter in the Cartesian comoving coordinates (t, x, y, z).

We have (see, *e.g.*, [13])

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)\,, \tag{29}$$

where

$$a(t) = At^{2/3}\,, \qquad A = (6\pi\rho a^3)^{1/3} = \text{const} > 0\,, \quad 0 < t < \infty\,. \tag{30}$$

The parameter t is the *cosmic time* and $a(t)$ denotes as usual the *scale factor*.

In order to pass to the conformally flat coordinates (τ, x, y, z) it is sufficient in the case only to change the time coordinate t onto *conformal time* τ following the scheme

$$d\tau = \frac{dt}{a(t)}\,. \tag{31}$$

From (30)–(31) it follows that

$$\tau = \frac{3}{A}t^{1/3} \equiv t = \frac{A^3}{27}\tau^3\,, \tag{32}$$

and

$$a(\tau) := a[t(\tau)] = \frac{A^3}{9}\tau^2\,, \qquad 0 < \tau < \infty\,. \tag{33}$$

Now the line element (29) is

$$ds^2 = a^2(\tau)\left(d\tau^2 - dx^2 - dy^2 - dz^2\right)\,, \tag{34}$$

i.e., we get the line element (29)–(30) in the *conformally flat form* with the *conformal factor*

$$\Omega = \Omega(\tau) = a(\tau) = \frac{A^3}{9}\tau^2\,, \qquad 0 < \tau < \infty. \tag{35}$$

From geometrical point of view the flat dust Friedman universe in conformally flat coordinates (τ, x, y, z) is identical with the upper half $(\tau > 0)$ of the conformally flat spacetime which has the following line element

$$ds^2 = a^2(\tau)\left(d\tau^2 - dx^2 - dy^2 - dz^2\right) . \tag{36}$$

It is interesting that in this case the conformally flat coordinates (τ, x, y, z) are also *comoving coordinates*, like the initial Cartesian coordinates (t, x, y, z).

A 4-velocity of a particle in the flat Friedman universe (identical with the 4-velocity of a fundamental observer) in the conformally flat coordinates (τ, x, y, z) is

$$u^i = \frac{\delta^i_0}{a(\tau)} \equiv u_i = a(\tau)\eta_{io} . \tag{37}$$

It results that the dust and the fundamental observers both *are at rest* in these coordinates, like as in the Cartesian comoving coordinates (t, x, y, z).

3. Energy of the Friedman universes in the conformally flat coordinates (τ, x, y, z)

In this section we will consider the energetic content of the Friedman universes in the physically and geometrically distinguished *conformally flat coordinates* (τ, x, y, z). In our analysis we will use the double index Einstein's canonical energy-momentum complex of matter and gravitation,

$$_{\mathrm{E}}K_i^{\ k} := \sqrt{|g|}\left(T_i^{\ k} + {}_{\mathrm{E}}t_i^{\ k}\right) , \tag{38}$$

where $T_i^{\ k}$ are the components of the symmetric energy-momentum tensor of matter and $_{\mathrm{E}}t_i^{\ k}$ mean the components of the *Einstein gravitational energy-momentum pseudotensor* (see, *e.g.*, [10–13]).

It is known that

$$\sqrt{|g|}\left(T_i^{\ k} + {}_{\mathrm{E}}t_i^{\ k}\right) = {}_{\mathrm{F}}U_i^{\ [kl]}{}_{,l} , \tag{39}$$

where $_{\mathrm{F}}U_i^{\ [kl]} = (-)_{\mathrm{F}}U_i^{\ [lk]}$ are *Freud's superpotentials* which in a coordinate basis read

$$_{\mathrm{F}}U_i^{\ [kl]} = \alpha\left\{\frac{g_{ia}}{\sqrt{|g|}}\left[(-g)\left(g^{ka}g^{lb} - g^{la}g^{kb}\right)\right]_{,b}\right\} , \qquad \alpha = \frac{1}{16\pi} , \tag{40}$$

and that the equations (39) represent special form of the Einstein equations (in a mixed form and multiplied by $\sqrt{|g|}$).

Owing to antisymmetry of the Freud's superpotentials from (39) one can easily obtain the *local energy-momentum conservation laws*, for matter and gravitation

$$ {}_{\mathrm{E}}K_i{}^k{}_{,k} = 0 \,. \tag{41}$$

By using Stokes theorem one can obtain from (41) meaningful *integral conservation laws* for a closed system in an asymptotically flat coordinates.

Of course, in GR one can consider many other energy-momentum complexes. However, the Einstein expressions seems to be the best of all variety of the energy-momentum complexes (see, *e.g.*, [11]). In consequence, in this paper we restrict ourselves, like in our previous papers, only to this double index energy-momentum complex[7].

For a conformally flat spacetime with

$$\begin{aligned} g_{ik} &= \Omega^2\eta_{ik} \equiv g^{ik} = \Omega^{-2}\eta^{ik}\,, \quad \Omega = \Omega(\tau,x,y,z)\,, \\ \sqrt{|g|} &= \Omega^4\,, \end{aligned} \tag{42}$$

from (39)–(40) one obtains

$$ {}_{\mathrm{E}}K_i{}^k = 4\alpha\left(\delta_i^k\eta^{lb} - \delta_i^l\eta^{kb}\right)\left(\Omega_{,l}\Omega_{,b} + \Omega\Omega_{,bl}\right). \tag{43}$$

As a trivial conclusion from (43) we get

$$ {}_{\mathrm{E}}K_0{}^0 = 0 \tag{44}$$

if $\Omega = \Omega(x^0) \equiv \Omega(\tau)$.

It is exactly what happens in the case of a flat Friedman universe.

Note that in this case the component ${}_{\mathrm{E}}K_0{}^0$ has physical meaning of the total "energy density" of matter and gravitation for comoving observers which have 4-velocities $u^i = \delta_0^i/a(\tau)$.

In general, the simple calculations performed by using of (43) and earlier given forms of the conformal factor $\Omega(\tau,x,y,z)$ for the considered dust Friedman universes lead to the following results:

1. In the case of a flat, dust Friedman universe only the components

$$ {}_{\mathrm{E}}K_1{}^1 = {}_{\mathrm{E}}K_2{}^2 = {}_{\mathrm{E}}K_3{}^3 = 4\alpha\left(\dot{a}^2 + a\ddot{a}\right) \tag{45}$$

of the canonical energy-momentum complex ${}_{\mathrm{E}}K_i{}^k$ are different from zero in the conformally flat coordinates (τ,x,y,z). Here $\dot{a} := da/d\tau$,

[7] But using of another reasonable double index energy-momentum complex will lead to analogous results.

$\ddot{a} := d^2a/d\tau^2$. Thus, in this case, *not all the components* of the complex ${}_{\mathrm{E}}K_i{}^k$ *vanish*.

In consequence, there exist observers with 4-velocities

$$\begin{aligned} u^i &= \left(\frac{1}{a\sqrt{1-v^2}}\,, \quad \frac{v_x}{a\sqrt{1-v^2}}\,, \quad \frac{v_y}{a\sqrt{1-v^2}},\ \frac{v_z}{a\sqrt{1-v^2}}\right)\,, \\ v_x &= \frac{dx}{d\tau}\,, \quad v_y = \frac{dy}{d\tau}\,, \quad v_z = \frac{dz}{d\tau}\,, \quad v^2 = v_x^2 + v_y^2 + v_z^2\,, \end{aligned} \tag{46}$$

for which the "energy density" $\epsilon := {}_{\mathrm{E}}K_i{}^k u^i u_k$ and its flux (= Poynting's vector)

$$P^i = \left(\delta^i_k - u^i u_k\right){}_{\mathrm{E}}K_l{}^k u^l \tag{47}$$

are different from zero.

For such observers we have

$$\begin{aligned} \epsilon &= \frac{(-)8}{27}\,\alpha A^6 \tau^2 \frac{v^2}{(1-v^2)} < 0\,, \\ P^0 &= \frac{4\alpha(\dot{a}^2 + a\ddot{a})v^2}{a(\tau)(1-v^2)^{3/2}}\,, \quad P^\beta = \frac{4\alpha(\dot{a}^2 + a\ddot{a})v^\beta}{a(\tau)(1-v^2)^{3/2}}\,, \end{aligned} \tag{48}$$

where

$$a(\tau) = \frac{A^3}{9}\tau^2 > 0\,, \quad \dot{a} = \frac{2A^3}{9}\tau > 0\,, \quad \ddot{a} = \frac{2A^3}{9} > 0\,, \quad \beta = 1,2,3, \tag{49}$$

and the integral

$$E = \int\limits_{\tau=\mathrm{const}} \epsilon\, dxdydz \tag{50}$$

is divergent to minus infinity.

We would like to remark that the spatial velocity $v^2 = v_x^2 + v_y^2 + v_z^2$ of these observers can be *infinitesimally small*, *i.e.*, these observers can *infinitesimally* differ from comoving observers.

Only for *comoving observers* which have their 4-velocity of the form $u^i = \delta^i_0/a$ we have

$$\epsilon = {}_{\mathrm{E}}K_0{}^0 = 0 \ \longrightarrow E = 0\,. \tag{51}$$

So, the physical situation in this case is *qualitatively* and *quantitatively* different from the case of a Minkowski spacetime endowed with inertial coordinates (t, x, y, z). Namely, in Minkowski spacetime covered by inertial coordinates (t, x, y, z), the canonical energy-momentum complex

${}_{\mathrm{E}}K_i{}^k$ (and other energy-momentum complexes too) *vanishes identically* and for *any observers* we have: $\epsilon = 0,\ P^i = 0$.

Thus, by using double index energy-momentum complexes, one *cannot conclude* that the flat Friedman universes are *energetic emptiness*, like in a Minkowski spacetime. In the case all depends on the family of the used observers.

2. An open dust Friedman universe.

In this case all the components of the canonical energy-momentum complex ${}_{\mathrm{E}}K_i{}^k$ are different from zero in the conformally flat coordinates (τ, x, y, z). So, an open dust Friedman universe surely is not an *energetic emptiness.*

If one calculates the "total energy density" $\epsilon = {}_{\mathrm{E}}K_i{}^k u^i u_k$, matter and gravitation, for family of the observers which are at rest in the conformally flat coordinates (τ, x, y, z) (*i.e.*, for observers which have their 4-velocities of the form $u^i = \delta^i_0/\Omega$ in these coordinates), then one gets

$$\begin{aligned}\epsilon \ = \ {}_{\mathrm{E}}K_0{}^0 = (-)\frac{3}{2}\alpha a_0^2\,\frac{\left(2\sqrt{\tau^2 - r^2} - a_0\right)^2}{(\tau^2 - r^2)} \\ \times\left[\frac{r^2}{(\tau^2 - r^2)^3} - \frac{\tau^2(a_0 - 2\sqrt{\tau^2 - r^2})}{a_0(\tau^2 - r^2)^3}\right].\end{aligned} \tag{52}$$

This expression is *negative-definite* and the integral

$$E = \int\limits_{\tau^2 - r^2 = B^2} {}_{\mathrm{E}}K_0{}^0 d^3S \tag{53}$$

over hypersurface $\tau^2 - r^2 = B^2,\ B := \frac{a_0}{2}e^{\eta_0} > \frac{a_0}{2}$ is divergent to minus infinity[8].

The integral (53) has mathematical meaning of the *global energy*, matter and gravitation, contained on the hypersurface $\tau^2 - r^2 = B^2, B > \frac{a_0}{2}$ [for observers which are at rest in the conformally flat coordinates (τ, x, y, x) in which the line element ds^2 is given by (25)].

[8] The hypersurface $\tau^2 - r^2 = B^2$ is a map in the conformally flat coordinates (τ, x, y, z) of the spatial slice $\eta = \eta_0$ of the Friedman universe in the initial coordinates $(\eta, \chi, \theta, \varphi)$.

3. A closed dust Friedman universe.

   In this case also all the components of the canonical energy-momentum complex ${}_{\mathrm{E}}K_i{}^k$ are different from zero in the conformally flat coordinates (τ, x, y, z). Thus, this universe, like an open Friedman universe, has *non-zero* "energy density" for an arbitrary set of observers, *i.e.*, a closed dust Friedman universe *is not an energetic emptiness.*

   Concerning global energy of a closed dust Friedman universe we must remember that this notion has only some *mathematical meaning*, and that *the conformally flat coordinates* (τ, x, y, z) *cover entirely only one distinguished slice* $\eta = 0$ of a closed dust Friedman universe.

   In conformally flat coordinates (τ, x, y, z) this slice is given by

$$\tau = 0\,, \quad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad -\infty < z < \infty\,. \tag{54}$$

   At the moment $\tau = 0$, the fundamental observers which were at rest in the initial coordinates $(\eta, \chi, \theta, \varphi)$ are also at rest in the conformally flat coordinates (τ, x, y, z). It is easily seen from the formulas (19)–(20) of the Section 2A. So, for these observers the component ${}_{\mathrm{E}}K_0{}^0(\tau = 0, x, y, z)$ represents the *total "energy density"* of matter and gravitation at the moment $\tau = 0$.

   By a simple calculation one can easily get that this component is

$${}_{\mathrm{E}}K_0{}^0 = \frac{(-)384\alpha a_0^2(r^2-1)}{(r^2+1)^4}\,. \tag{55}$$

   Formal calculation of the energy contained inside of the distinguished slice $\tau = 0$ in the conformally flat coordinates (τ, x, y, z) gives

$$\begin{aligned} E &= \int\limits_{\tau=0} {}_{\mathrm{E}}K_0{}^0 dxdydz = (-)1536\pi\alpha a_0^2 \int\limits_0^A \frac{(r^4 - r^2)}{(r^2+1)^4}\,dr \\ &= \frac{512\pi\alpha a_0^2 A^3}{(1+A^2)^3} > 0\,. \end{aligned} \tag{56}$$

   Apparently, A can be *arbitrary big*, but it always should be finite, because $A \longrightarrow \infty$ would lead to $\Omega \longrightarrow 0$, *i.e.*, in the limit it would lead to a singularity. Despite that, if we take the formal limit $A \longrightarrow \infty$, then we will get $E = 0$.

   But one *cannot conclude from this result that the closed dust Friedman universe really has zero net global energy.*

The reasons are as follows. At first, *one cannot calculate* analogous global integral over any other spatial slice $\eta = \eta_0 = \text{const} \neq 0, \ -\pi < \eta_0 < \pi$ of the closed dust Friedman universe because other slices *are not entirely covered* by the conformally flat coordinates (τ, x, y, z). We have already mentioned about this important fact in Section 2A. Secondly, *we have no global conservation laws in the relativistic cosmology*, *i.e.*, vanishing global energy at $\tau = 0$ *does not result* in $E = 0$ at $\tau \neq 0$.

Thirdly, if we use another set of observers, *e.g.*, the set of observers which have their 4-velocities

$$u^0 = \frac{1}{\Omega\sqrt{1-v^2}}\,, \qquad u^1 = \frac{v}{\Omega\sqrt{1-v^2}}\,, \qquad u^2 = u^3 = 0\,, \tag{57}$$

at $\tau = 0$, where $v = \sqrt{(\frac{dx}{d\tau})^2}$, then for such observers we will obtain (for simplicity we put $v = \text{const} > 0$)

$$\epsilon = {}_{\mathrm{E}}K_i{}^k u^i u_k = (-)\frac{384\alpha a_0^2}{(1-v^2)}\left[\frac{r^2(1-v^2) + 2v^2x^2 - 1}{(r^2+1)^4}\right]. \tag{58}$$

It follows from the above expression that for these observers the "global energy" E contained in the subspace $\tau = 0$ reads

$$\begin{aligned} E &= (-)\frac{384\alpha a_0^2}{(1-v^2)}\int\limits_0^\infty\int\limits_0^\pi\int\limits_0^{2\pi}\frac{[(1-v^2)r^2 + 2v^2x^2 - 1]}{(1+r^2)^4}\,r^2\sin\theta dr d\theta d\varphi \\ &= \frac{16\pi^2\alpha a_0^2 v^2}{(1-v^2)} > 0\,, \end{aligned} \tag{59}$$

i.e., it is *positive-definite* even for infinitesimally small v.

Thus, the "global energetic content" in the subspace $\tau = 0$ *depends on the used set of the observers* which evolve in the closed dust Friedman universe.

Once more we met a situation which is *qualitatively and quantitatively different* from the situation in Minkowski spacetime endowed with inertial coordinates (t, x, y, z).

4. Conclusion

Our conclusion is that the Friedman universes *are not energetic emptiness* even if we analyze these universes only with the help of a double index energy-momentum complex. Because these universes *are not asymptotically flat*, such analysis should be performed in the *geometrically and physically distinguished conformally flat coordinates* (τ, x, y, z).

We hope that we have convincingly justified this conclusion in the paper.

Our conclusion is in full agreement with our previous analysis of the Friedman (and also more general) universes with the help of the *averaged relative energy-momentum tensors* [8].

Of course, our conclusion contradicts the recently very popular opinion that the *Friedman universes are energy-free.* Such opinion originated from *incomplete analysis* of these universes performed in the Cartesian comoving coordinates (t, x, y, z) in which only the spatial part of the FLRW line element is conformally flat.

By *incomplete analysis* we mean the fact *of using only the comoving observers* to analyze the energetic content of the Friedman (and also more general) universes. As we have seen, using different set of the observers gives different, non-zero local and global results for flat Friedman universes and non-zero global results for a closed Friedman universe.

In fact, only by using the non-comoving observers one is able to show that the flat Friedman universes *are not energetic nonentity emptiness neither locally nor globally* and that the closed Friedman universes *are not global energetic emptiness.*

Restricting to the comoving observers only *is not justified physically*, *e.g.*, an Earth's observer *is not a comoving observer* in the real Universe.

We think that the conformally flat coordinates (τ, x, y, z) *have much more profound geometrical and physical meaning* than the Cartesian comoving coordinates. Thus, in order to correctly analyze the energy and momentum of the Friedman universes with the help of a coordinate-dependent energy-momentum complex, one should work in these coordinates. We have done this in the present paper for the energy.

We hope that this paper and our previous paper [8] will conclude the discussion about energetic content of the Friedman universes.

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