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EXOTIC SMOOTH 4-MANIFOLDS AND GERBES AS GEOMETRY FOR QUANTUM GRAVITY*

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The relation of some small exotic smooth $\mathbb{R}^4$ with Abelian gerbes and H-deformed generalized Hitchin's structures on $S^3 \subset \mathbb{R}^4$ is discussed. Exotic smoothness of $\mathbb{R}^4$ appears as some fundamental phenomenon related to string theory and which has not been taken into account yet in construction of any QG theory.

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1. Introduction

Path integral over spacetime geometries, at least formally, has to deal with different smooth structure of spacetime. Thus, a final version of quantum gravity should include possible effects from the so-called exotic smoothness of background manifolds [13]. The development of mathematical ideas relating the subject of "exotica" shows that there exist indeed many of unexplored, and potentially valid also for physics, effects just in the dimension 4. Only in this dimension one can find exotic $\mathbb{R}^n$-manifolds non-diffeomorphic but homeomorphic to $\mathbb{R}^4$. Any other $\mathbb{R}^n$, $n \neq 4$ is uniquely smooth. Moreover, there are at least two families of infinite continuum many different non-diffeomorphic smooth $\mathbb{R}^4$s. First is related with the failure of the smooth h cobordism theorem in dimension 5 and is called the family of *small* exotic $\mathbb{R}^4$s, and the second appears as the result of some failure the smooth surgery and contains *large* exotic $\mathbb{R}^4$s. Most open 4-manifolds have infinite continuum of different smoothings [2,8]. Also compact 4-manifolds can carry at most infinite *countably* many non-diffeomorphic smoothings, whereas in any other dimension $n \neq 4$ every M^n is smoothable in at most *finitely*

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many different ways [2]. Again the physical dimension 4 is distinguished. The "old-standing" problem in mathematics (since 1982) is to give a local coordinate description of exotic $\mathbb{R}^4$ such that a global exotic function is explicitly given. Such functions could be further applied to produce many physically valid exactly solvable examples where effects of exotic 4-manifolds (not only gravitational) are calculated. Even though such an explicit presentation is missing, two remarkable hypothesis emerged instead. First is the Brans hypothesis (1994) [5, 6] relating classical general relativity: *Certain exotic smooth 4-structures can be considered as sources for the external gravitational field in spacetime.* This hypothesis was successfully proved for compact smooth 4-manifolds by Asselmeyer-Maluga in 1996 [1], and for open 4-manifolds by Sładkowski in 1999 [2, 16, 17]. The second, are the hypothesis relating quantum mechanics: *Exotic $\mathbb{R}^4$s generate algebras which can naturally represent non-commutative algebras of quantum observables*, and quantum gravity: *Exotic smooth $\mathbb{R}^4$s are fundamental for the description of effects of QG in spacetime, similarly as the standard $\mathbb{R}^4$ is in the description of classical GR.* The QM case was formulated and analyzed both from the point of view of mathematical model theory [10] and differential geometry [4]. The QG formulation of the hypothesis appeared on the base of the results from model theory and topos theory [12]. However, the general difficulty was the missing of the explicit calculations supporting these quantum connections. Only recently there appeared the proposal serving as possible breakthrough in the subject. The connection of small exotic $\mathbb{R}^4$s with WZW models of conformal field theory (CFT) and Verlinde algebras of SU(2) at the suitable level and with gerbes and foliations was presented [3]. This shows indeed deep connection of 4-exotica with quantum gravity regime as in string theory and enables one for computations of the effects. Moreover, via the techniques developed it is possible to show that the quantization of electric charge in spacetime is forced by the exoticness of some 4-regions of it, without referring to magnetic monopoles. In this paper we focus on the relation of exotic smoothness with generalized Hitchin's structures on S^3 and the deformation of these by Abelian gerbes. This kind of structures appeared as fundamental in supersymmetric string theory and flux compactification herein. The geometric approach shows that in the dimension 4 we have peculiar extension of differential geometry over some quantum regions of string theory.

2. String geometry, B-field and gerbes

Einsteinian GR is essentially tied to geometry of some 4-dimensional pseudo-Riemannian manifolds. The critical point of the action of GR

$$S(M,g)=\int_M R dvol_g \tag{1}$$

in the absence of any other matter fields, is the pair (M,g) where M is the pseudo-Riemannian manifold and g a metric on it, R is the scalar curvature of g.

In the case of string theory, the concept of spacetime as a smooth manifold is not valid any longer in general. Rather we have string backgrounds which are described by 2-dimensional conformal field theory and σ models in a suitable targets. However, these string backgrounds still have well-defined geometric classical limits which appear to be the triples (M,g,B) where in addition to the pseudo-Riemannian smooth manifold M and metric g we have B-field, *i.e.* local 2-form on M. Conversely, every full string background, hence 2-dimensional CFT plus σ-model with the target M, can be derived from some limiting classical geometry (M,g,B) [14]. Hence the difference between geometries of classical and quantum gravity is based on the existence of B field in the case of superstring theory.

From the other side, B-field on a manifold M is the same as an *Abelian gerbe* with the connection [14]. Abelian gerbes are classified by the integral classes of $H^3(M,\mathbb{Z})$. These are geometric objects representing the third cohomologies similarly as complex line bundles represent the second cohomologies from $H^2(M,\mathbb{Z})$.

The important feature of 3-rd cohomology classes on manifolds is the deformation of generalized Hitchin's structures by these classes. In a special case of integral 3-rd cohomologies one has the twisted by gerbes generalized Dirac structures, or Courant brackets on manifolds. The relation of these with exotic $\mathbb{R}^4$s is the topic of the following section.

3. Exotic $\mathbb{R}^4$ and B-fields on S^3

The deep relation between the real 3-rd deRham cohomology classes of the 3-sphere embedded in $\mathbb{R}^4$ and small exotic $\mathbb{R}^4$s, was established in [3]: the classes from $H^3(S^3,\mathbb{R})$ correspond uniquely to different smoothings of $\mathbb{R}^4$ when S^3 is the boundary of the so-called Akbulut cork. The Akbulut cork is some compact contractible 4-submanifold with a boundary of $\mathbb{R}^4$. The boundary is the cohomology 3-sphere and the Akbulut cork determines uniquely (up to the isotopy) the exotic smooth structure on $\mathbb{R}^4$.

The generalized Hitchin's structure on S^3 arises when one replaces the tangent space TS^3 of S^3 by the sum $TS^3 \oplus T^\star S^3$ of the tangent and cotangent bundles, the spin structure for such generalized "tangent" bundle now becomes the bundle of all forms $\bigwedge^\bullet S^3$ on S^3. Then one defines the *Courant bracket* on the smooth sections of $TS^3 \oplus T^\star S^3$

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi)\,, \tag{2}$$

where $X+\xi$, $Y+\eta \in C^\infty(TS^3 \oplus T^\star S^3)$, $\mathcal{L}_X$ is the Lie derivative in the direction of the field X, $i_X\eta$ is the inner product of a 1-form η and a vector field X. On the r.h.s. of (2) $[\,,\,]$ is the Lie bracket on fields. The Courant bracket is skew symmetric and vanishes on 1-forms. However, the Courant bracket is not a Lie bracket, since the first does not fulfill the Jacobi identity. The expression measuring the failure of the identity is the Jacobiator:

$$\mathrm{Jac}(X,Y,Z) = [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]\,. \tag{3}$$

One can define the inner product on $TS^3 \oplus T^\star S^3$ as

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}(\xi(Y) + \eta(X))\,. \tag{4}$$

This product is symmetric and has the signature (n,n), where $n = \dim(M)$. In our case of S^3 the non-compact ortogonal group reads $\mathrm{O}(TS^3 \oplus T^\star S^3) = \mathrm{O}(3,3)$. A subbundle $L < TM \oplus T^\star M$ is *involutive* if it is closed under the Courant bracket defined on its smooth sections, and is *isotropic* when $\langle X,Y\rangle = 0$ for X, Y smooth sections of L. In the case that $\dim(L) = n$, hence is maximal, we call such an isotropic subbundle a *maximal isotropic subbundle*. The following property characterizes these subbundles [9]:

Given L a maximal isotropic subbundle of $TM \oplus T^\star M$, then L is involutive if $Jac_L = 0$.

A *Dirac structure* on $TM \oplus T^\star M$ is a *maximal isotropic and involutive subbundle* $L < TM \oplus T^\star M$.

The importance of using the Dirac structures is much generality of these in contrast to Poisson geometry, complex structures, foliated or symplectic geometries. Dirac structures unify all these cases and give rise to new ones. We are especially interested in the H-deformed Dirac structures where H is the 3-form on S^3. The H-deformed Dirac structures include also generalized complex structures which are well defined on some manifolds without any complex or symplectic structures at all. Moreover, this kind of geometry became extremely important in string theory (flux compactification, mirror symmetry, branes in YM manifolds) and related WZW models. These appear also as the important tool for the correct recognition of exotic smooth $\mathbb{R}^4$s [3].

The suitable modification of Lie product of fields on smooth S^3 is required. The modification is given firstly by the Courant bracket on $TS^3 \oplus T^\star S^3$ and then by the H-deformation of it.

In differential geometry, Lie bracket of smooth vector fields on a smooth manifold M is invariant under diffeomorphisms, and there are no other symmetries of the tangent bundle preserving the Lie bracket.

In the case of the extended "tangent space", which is $TM \oplus T^\star M$, the Courant bracket and the inner product are diffeomorphisms invariant. However, there exists another symmetry extending the diffeomorphisms: this is the *B-field transformation.* Given a two-form B on M one can think of it as the map $TM \to T^\star M$ by contracting B with X, $X \to i_X B$. The transformation of $TM \oplus T^\star M$ given by $e^B : X + \xi \to X + \xi + i_X B$ has the property that

the e^B *extension of diffeomorphisms are the only allowed symmetries of the Courant bracket.*

Or more precisely

the group of orthogonal Courant automorphisms of $TS^3 \oplus T^\star S^3$ *is the semidirect product of* $Diff(S^3)$ *and* Ω^2_{closed} [9].

We see that B-field extends the diffeomorphisms of TS^3 which is understood as the preparation for grasping the exotic diffeomorphisms of $\mathbb{R}^4$. For this we need rather deformed Courant bracket.

Given a Courant bracket on $TM \oplus T^\star M$ we are able to deform it by a real closed 3-form H from $H^3(M, \mathbb{R})$. For any real 3-form H one has the twisted Courant bracket on $TM \oplus T^\star M$ defined as

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + i_Y i_X H \,, \tag{5}$$

where $[,]$ on the RHS is the non-twisted Courant bracket. This can be also restated as the splitting condition in non-trivial twisted Courant algebroid [9].

This deformed bracket allows for defining various involutive and (maximal) isotropic structures with respect to $[\,,\,]_H$. These structures correspond to new H-twisted geometries which are different to the case of Dirac structures for the untwisted Courant bracket. Thus B-fields on S^3 deform Dirac structures on the 3-sphere. This generalized geometry on S^3 is better suited for the description of the differences between various smoothings of $\mathbb{R}^4$. We state here a general correspondence [3] leaving more explicit geometric analysis for a separate work:

Continuum of many distinct small exotic smooth structures on $\mathbb{R}^4$ *correspond* $1 \div 1$ *to the* H*-deformed classes of generalized Hitchin's geometries on* S^3*, where* $[H] \in H^3(S^3, \mathbb{R})$ *and* S^3 *is the boundary of the Akbulut cork.*

One would like to construct some exotic $\mathbb{R}^4$ from the specific generalized Hitchin's geometry taken from the class of H-deformed and integrable Dirac structures. This has not, however, been performed yet.

In the special case of integral 3-rd cohomologies of S^3 we get whole spectrum of possible geometrical interpretations. Namely, Abelian gerbes on S^3 as being classified by classes from $H^3(S^3, \mathbb{Z})$ should correspond somehow to smoothings of $\mathbb{R}^4$. In fact it holds:

Different (small) exotic smooth $\mathbb{R}^4$s correspond to different classes of Abelian gerbes on $S^3 \subset \mathbb{R}^4$.

Next, one could take a generalized Hitchin's structure on S^3 which would be deformed by the Abelian gerbes, and the result follows:

The deformed Hitchin's structures on S^3 by the S^1-gerbes on S^3 correspond to different exotic smooth (small) $\mathbb{R}^4$.

Or

the changes of certain smoothings of $\mathbb{R}^4$ correspond to the deformations by S^1-gerbes of the generalized Hitchin's structure on S^3, the boundary of the Akbulut cork.

Next, let us consider S^3 as the group SU(2). Defining WZW model on this SU(2) and observing that different gerbes on S^3 correspond to the integral levels $k \in \mathbb{Z}$, one has:

Different integral levels k of the WZW model on SU(2) correspond to different smoothings of $\mathbb{R}^4$.

One can also obtain the relation of 4-exotics with the levels of the Verlinde algebra of SU(2). This can be obtained by considering gerbes on orbifolds, in particular Abelian gerbes on SU(2) $\times$ SU(2) $\rightrightarrows$ SU(2) groupoid, where SU(2) acts on itself by conjugation [7].

The changes between some small exotic $\mathbb{R}^4$s can be correlated with the suitable changes of the level k of the Verlinde algebra of SU(2), V_k(SU(2)).

One can also show the peculiar relation of 4-exotica with non-commutative C^*-algebras and that the exotic smoothing of $\mathbb{R}^4$ twists the K-theory on S^3 toward the K-theory of some non-commutative Banach algebra (see also [15] and [14]).

$\mathbb{R}^4$ is the simplest 4-dimensional manifold which models spacetime locally. The broad spectrum of the relations of non-standard smooth structures on $\mathbb{R}^4$ with mathematical constructions of string theory and QFT indicates that fundamental physics, where QG becomes important, should be focused rather on exotic than standard smoothness of $\mathbb{R}^4$. This more than a satisfactory theory of QG in dimension 4 is still missing.

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