

## EXOTIC SMOOTHNESS AND ASTROPHYSICS\*

JAN SŁADKOWSKI

Institute of Physics, University of Silesia  
Uniwersytecka 4, 40-007 Katowice, Poland*(Received October 19, 2009)*

The problem of possible astrophysical consequences of the existence of exotic differential structures on manifolds is discussed. It is argued that corrections to the curvature of the form of a source like terms should be expected in the Einstein equations if they are written in the “wrong” differential structure. Examples of topologically trivial spaces on which exotic differential structures act as a source of gravitational force even in the absence of matter are given. Propagation of light in the presence of such phenomena is also discussed. A brief review of exotic smoothness is added for completeness.

PACS numbers: 98.80.Jk, 95.30.Sf

**1. Introduction**

In 1854 B. Riemann, the father of contemporary differential geometry, suggested that the geometry of space may be more than just a mathematical tool defining a stage for physical phenomena, and may in fact have profound physical meaning in its own right [1]. But is it reasonable to contemplate *to what extent the choice of mathematical model for spacetime has important physical significance?* With the advent of general relativity physicists began to think of the spacetime as a differential manifold. Since then various assumptions about the spacetime topology and geometry have been put forward. But why should the choice of differential structure of the spacetime manifold matter? Most of topological spaces used for modelling spacetime have natural differential structures and the question of their non-uniqueness seemed to be extravagant. Therefore, the counterintuitive discovery of exotic four dimensional Euclidean spaces following from the work of Freedman [2] and Donaldson [3] raised various discussions about the possible physical consequences of this discovery [4–21]. It has been shown that exotic (nonunique)

---

\* Presented at the XXXIII International Conference of Theoretical Physics, “Matter to the Deepest”, Ustroń, Poland, September 11–16, 2009.

smooth structures are especially abundant in dimension four [15] and there is at least a two parameter family of exotic  $\mathbb{R}^4$ s. Such manifolds play important rôle in theoretical physics and astrophysics and it became necessary to investigate the physical meaning of exotic smoothness. Unfortunately, this is not an easy task: we only know few complicated coordinate descriptions [16] and most mathematicians believe that in most cases there might not be any finite atlas on an exotic  $\mathbb{R}^4$  and other exotic four-manifolds (three coordinate patches description seems to be the best achievement). Since their discovery, exotic  $\mathbb{R}^4$  have revealed themselves in various physical contexts. For example, some non-perturbative limits of a QCD-like YM theory can be reached when the theory is formulated on 4-manifolds which are locally exotic  $\mathbb{R}^4$ ; exotic  $\mathbb{R}^4$ s in a region of the spacetime can act as the sources of the magnetic field in this spacetime and imply electric charge quantization; calculations on exotic  $\mathbb{R}^4$ s can be formulated in covariant cohomologies (sheaves and gerbes on groupoids) or in 2-dimensional quantum CFT and in that way imply quantum gravity corrections if compared to the calculations done in the standard  $\mathbb{R}^4$ . Such phenomena are counterintuitive but I am aware of no physical principle that would require rejection of such spacetimes or solutions.

## 2. Astrophysical consequences of the existence of exotic smoothness

### *2.1. Nonequivalent differential structures*

It is possible that two manifolds  $M_1$  and  $M_2$  are homeomorphic but not diffeomorphic, that is they are identical as topological spaces ( $M$ ) but no bijection (1–1 mapping) between them is differentiable<sup>1</sup>. In that case, we say that the underlying topological space  $M$  has nonequivalent differential structures often referred to as exotic differential structures. Therefore, exotic  $\mathbb{R}^4$ s are defined as four-manifolds that are homeomorphic to the four-dimensional Euclidean space  $\mathbb{R}^4$  but not diffeomorphic to it. The existence of nonequivalent differential structures does not change the definition of the derivative. The essential difference is that the sets (actually algebras) of real differentiable functions are different on non-diffeomorphic manifolds. In the case of exotic  $\mathbb{R}^4$ s this means that there are continuous functions  $\mathbb{R}^4 \mapsto \mathbb{R}$  that are smooth on one exotic  $\mathbb{R}^4$  and only continuous on another and *vice versa* [17].

---

<sup>1</sup> To be precise, these functions must not be differentiable at least at one point.

2.2. General relativity on exotic  $\mathbb{R}^4$ s with few symmetries

Suppose we are given an exotic  $\mathbb{R}^4_\theta$  with few symmetries<sup>2</sup>. We can try to solve the Einstein equations on this  $\mathbb{R}^4_\theta$ . Suppose we have found such a solution. Whatever the boundary conditions be we would face one of the two following situations [21].

The isometry group  $G$  of the solution acts properly<sup>3</sup> on  $\mathbb{R}^4_\theta$ . Then  $G$  is finite. There is no nontrivial Killing vector field and no solution to Einstein equations can be stationary. The gravity is quite “complicated” and even empty spaces do evolve.

The isometry group  $G$  of the solution acts nonproperly on  $\mathbb{R}^4_\theta$ . Then  $G$  is locally isomorphic to  $SO(n, 1)$  or  $SO(n, 2)$ . But the nonproper action of  $G$  on  $\mathbb{R}^4_\theta$  means that there are points infinitely close together in  $\mathbb{R}^4_\theta$  ( $x_n \rightarrow x$ ) such that arbitrary large different isometries ( $g_n \rightarrow \infty$ ) in  $G$  maps them into infinitely close points in  $\mathbb{R}^4_\theta$  ( $g_n x_n \rightarrow y \in \mathbb{R}^4_\theta$ ). There must exist quite strong gravity centers to force such convergence (even in empty spacetimes).

We see that in both cases Einstein gravity is quite nontrivial even in the absence of matter. Recall that if a spacetime has a Killing vector field  $\zeta^a$ , then every covering manifold admits appropriate Killing vector field  $\zeta'^a$  such that it is projected onto  $\zeta^a$  by the differential of the covering map. This means that discussed above properties are inherited by any space that has exotic  $\mathbb{R}^4$  with few symmetries as a covering manifold *e.g.* quotient manifolds obtained by a smooth action of some finite group.

2.3. String-like gravitational sources

Asselmeyer considered a topological manifold  $M$  that can be given two nonequivalent differential structures  $M'$  and  $M''$  and found the change in covariant derivative induced by exoticness [18]. Consider a 1–1 map  $\alpha : M' \rightarrow M''$  that is not a diffeomorphism at some point  $p_0 \in M'$ . The splitting of the map  $d\alpha : TM' \rightarrow TM''$  in some neighbourhood  $U(p_0)$  of the point  $p_0$ :

$$d\alpha |_{U(p_0)} = (b_1, b_2)$$

allows us to express the change in the covariant derivative in the following form:

$$\nabla'' = \nabla' + (b_1^{-1}db_1) \oplus (b_2^{-1}db_2) .$$

---

<sup>2</sup> We say that a smooth manifold has few symmetries provided that for every choice of differentiable metric tensor, the isometry group is finite.

<sup>3</sup> We say that  $G$  acts properly on  $X$  if and only if for all compact subsets  $Y \subset X$ , the set  $\{g \in G : gY \cap Y \neq \emptyset\}$  is also compact.

This made it possible to calculate the corresponding change in the curvature tensor and Einstein equations. Recall that the curvature tensor is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z,$$

where  $X, Y, Z$  are vector fields. This means that we should expect that

$$\text{Ric}(X, Y) - \frac{1}{2}g(X, Y)R \neq 0$$

in  $M''$  even if the right hand side vanishes on  $M'$ ! Asselmeyer argues for a string-like interpretation of this source term. Suppose we have discovered some strange astrophysical source of gravitation that do not fit to any acceptable solution of the Einstein equations<sup>4</sup>. This may mean that we are using wrong differential structure on the spacetime manifold and this strange source is sort of an artefact of this mistake. If we change the differential structure then everything would be OK.

#### 2.4. Why should exotic structures matter?

A manifold  $M$  can be equivalently described by its algebra of real differentiable functions  $C(M, \mathbb{R})$ . Connes managed to generalize this result for a much larger class of algebras, not necessarily commutative [22]. The idea behind this is that one have to define the appropriate Dirac operator and the differential calculus is recovered by commutators of functions with the Dirac operator. This means that the spacetime structure is actually given by those properties of matter fields that are governed by Dirac equations. It is possible that there are some subtleties in fundamental interaction that reveal themselves only on astrophysical scales and exotic differential structures might be necessary to take them into account.

### 3. Astrophysical observations of exotic smoothness

#### 3.1. Maxwell equations in gravitational background

Maxwell's equations in the background metric in empty space can be written as [23]<sup>5</sup>:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} - \frac{1}{c\sqrt{\gamma}} \frac{\partial}{\partial t} \sqrt{\gamma} \mathbf{D} - \frac{4\pi}{c} \mathbf{s} &= 0, \\ \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} + \frac{1}{c\sqrt{\gamma}} \frac{\partial}{\partial t} \sqrt{\gamma} \mathbf{B} &= 0, \end{aligned}$$

<sup>4</sup> For example, a cosmic string-like lensing is possible. *A priori*, it should be possible to distinguish between standard cosmic string and Asselmeyer's strings because cosmic strings forms conic spacetimes with rather trivial gravity.

<sup>5</sup> There still are controversies concerning definition of electromagnetic field in gravitational background but such details are unimportant here.

where  $\gamma = \det g_{ik}$ ,  $\mathbf{s}^i = \rho \frac{dx^i}{dt}$ . The fields  $\mathbf{B}$  and  $\mathbf{D}$  should be modified as:

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{h}} + \mathbf{H} \times \mathbf{G}, \quad \mathbf{B} = \frac{\mathbf{H}}{\sqrt{h}} + \mathbf{G} \times \mathbf{E},$$

with  $h = g_{00}$  and  $\mathbf{G}_i = -\frac{g_{0i}}{g_{00}}$ . Maxwell's equations in the background metric in empty space can be written as:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{D} &= 0, \\ \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} &= 0. \end{aligned}$$

Simple calculations [24] show that one should expect modification of the dispersion relation of the form:

$$\mathbf{k}^2 - \omega^2 - 2\mathbf{G} \cdot \mathbf{k}\omega = 0.$$

This corresponds to a subluminal propagation of electromagnetic radiation. Such effects are observable but without explicit solution for the metric tensor it would be difficult to ascribe them to the exoticness.

### 3.2. Non-gravitational effects and observations?

Every measurement performed by humans involves gauge interactions (e.g. electromagnetic interaction). By using the heat kernel method [25] we can express the Yang–Mills action in the form [26]:

$$\mathcal{L}_{\text{YM}}(F) \sim \lim_{t \rightarrow 0} \frac{\text{tr}(F^2 \exp(-t\mathcal{D}^2))}{\text{tr}(\exp(-t\mathcal{D}^2))},$$

where  $\mathcal{D}$  is the appropriate Dirac operator. Suppose that we have a one parameter ( $z$ ) family of differential structures and the corresponding family of Dirac operators  $\mathcal{D}(z)$ . The Duhamel's formula

$$\partial_z \left( e^{-t\mathcal{D}^2(z)} \right) = \int_0^t e^{-(t-s)\Delta(z)} \partial_z (\mathcal{D}^2(z)) e^{-s\Delta(z)} ds,$$

where  $\Delta$  is the scalar Laplacian, can be used to calculate the possible variation of  $\mathcal{L}_{\text{YM}}(F)$  with respect to  $z$ . Unfortunately our present knowledge of exoticness is too poor for performing such calculations. Nevertheless, we can try to estimate the possible effects in the following way. For an operator  $K$  with a smooth kernel we have the following asymptotic formula [27]:

$$\text{tr} \left( K e^{-t\mathcal{D}^2} \right) \sim \text{tr}(K) + \sum_{i=1}^{\infty} t^i a_i,$$

where the spectral coefficients  $a_i$  describe some important details of geometry of the underlying manifold [28,29]. So if  $F^2$  is smooth with respect to all differential structures (*e.g.* has compact support [17]) then the possible effects of exoticness might “decouple” (are negligible). This means that we are unlikely to discover exoticness by performing “local” experiments involving gauge interactions. If we consider only matter (fermions) coupled to gravity then the action can also be expressed in terms of the coefficients of the heat kernel expansion of the Dirac Laplacian,  $\mathcal{D}^2$ . In this case we may be able to detect the exoticness of differential structures only if the Dirac operator specifies it almost uniquely [7,27].

#### 4. Conclusions

If exotic smoothness has anything to do with the physical world it may be a source/explanation of various astrophysical and cosmological phenomena. Dark matter, vacuum energy substitutes and strange attracting/scattering centers are the most obvious among them. Exoticness of the spacetime might be responsible for the anomalies in supernovae properties or other high density objects [30–38]. Further research to distinguish exotic smoothness from other causes is necessary but it is a very nontrivial task.

#### REFERENCES

- [1] B. Riemann, Über die Hypothesen, welche der Geometrie zugrunde liegen, 1854.
- [2] M.H. Freedman, *J. Differential Geom.* **17**, 357 (1982).
- [3] S.K. Donaldson, *J. Differential Geom.* **18**, 279 (1983).
- [4] C. Brans, D. Randall, *Gen. Relativ. Gravitation* **25**, 205 (1993).
- [5] C. Brans, *Class. Quantum Grav.* **11**, 1785 (1994).
- [6] C. Brans, *J. Math. Phys.* **35**, 5494 (1994).
- [7] J. Śladkowski, *Acta Phys. Pol. B* **27**, 1649 (1996).
- [8] J. Król, *Found. Phys.* **34**, 843 (2004).
- [9] J. Król, *Found. Phys.* **34**, 361 (2004).
- [10] J. Król, *Found. Phys.* **36**, 1070 (2006).
- [11] J. Król, *Acta Phys. Pol. B* **32**, 3855 (2001).
- [12] J. Król, *Found. Phys.* **36**, 1778 (2006).
- [13] J. Śladkowski, *Acta Phys. Pol. B* **30**, 3495 (1999).
- [14] T. Asselmeyer-Maluga, J. Król, [hep-th/0904.1276](#).
- [15] Z. Bizaca, J. Etnyre, *Topology* **37**, 461 (1998).
- [16] Z. Bizaca, *J. Differential Geom.* **39**, 491 (1994).

- [17] C.H. Brans, T. Asselmeyer-Maluga, *Exotic Smoothness and Physics*, World Scientific Publ., Singapore 2007.
- [18] T. Asselmeyer, *Class. Quantum Grav.* **14**, 749 (1997).
- [19] T. Asselmeyer-Maluga, H. Rose, *Acta Phys. Pol. B* **38**, 3633 (2007).
- [20] J. Śladrkowski, *Int. J. Theor. Phys.* **33**, 2381 (1994).
- [21] J. Śladrkowski, *Int. J. Mod. Phys.* **D10**, 311 (2001).
- [22] A. Connes, *Noncommutative Geometry*, Academic Press, London 1994.
- [23] L.D. Landau, E.M. Lifshitz, *Classical Theory of Fields*, Pergamon Press, 1975.
- [24] J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, [gr-qc/9909085](#).
- [25] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer Verlag, Berlin 1991.
- [26] J.G. Várilly, J.M. Garcia-Bondía, *J. Geom. Phys.* **12**, 223 (1993).
- [27] P.B. Gilkey, *Invariance, the Heat Equation and the Atiyah-Singer Index Theorem*, Publish or Perish, Washington 1984.
- [28] R. Mańka, J. Śladrkowski, *Phys. Lett.* **B224**, 97 (1989).
- [29] R. Mańka, J. Śladrkowski, *Acta Phys. Pol. B* **21**, 509 (1990).
- [30] I. Bednarek, R. Mańka, *Int. J. Mod. Phys.* **D7**, 225 (1998).
- [31] R. Mańka, I. Bednarek, *Astron. Nachr.* **312**, 11 (1991).
- [32] I. Bednarek, R. Mańka, *J. Phys. G* **36**, 095201 (2009).
- [33] T. Asselmeyer-Maluga, C.H. Brans, *Gen. Relativ. Gravitation* **34**, 1767 (2002).
- [34] R. Mańka, I. Bednarek, *Europhys. Lett.* **78**, 32001 (2007).
- [35] R. Mańka, I. Bednarek, *Astrophys. Space Sci.* **176**, 325 (1991).
- [36] J. Król, *Int. J. Geom. Meth. Mod. Phys.* **4**, 297 (2007).
- [37] R. Mańka, I. Bednarek, G. Przybyła, *New J. Phys.* **4**, ARTN 14 (2002).
- [38] R. Mańka, I. Bednarek, *J. Phys. G* **27**, 1975 (2001).