THE BEHAVIOURS OF GURSEY INSTANTONS IN PHASE SPACE

Fatma Aydogmus[a†], Beyrul Canbaz[a], Cem Onem[b]
K. Gediz Akdeniz[a]

[a]Department of Physics, Istanbul University
34134 Vezneciler, Istanbul, Turkey
[b]Department of Physics, University of Trakya, Edirne, Turkey

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In our previous study, we have investigated the behaviours of spinor-type instantons in two-dimensional conformally invariant pure spinor Thirring model in phase space. In this paper, we study the role of the coupling constant in the evolution of the four-dimensional spinor-type instantons in phase space via the Heisenberg ansatz. For this purpose, we consider the Gursey model is a four-dimensional conformally invariant pure spinor model with nonlinear self-coupled spinor term. The model proposed in 1956 as a possible basis for a unitary description of elementary particles (Heisenberg–Bohr dream). This study will also lead us to investigate the dependence of the behaviours of spinor-type instantons in phase space on quantum fractional spinor number as well as dimensions.

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1. Introduction

It is known that instantons are classically topological solitons [1]. Instantons, characterized by a zero energy-momentum tensor as well as finite action, emerge as the solutions of coupled first order equations. In particle physics, they were described as tunnelling processes between vacua with different topological structure, reflecting their nonperturbative nature [2]. Especially, this property of instantons plays an important role in explaining the imprisonment of quarks in particles [3, 4].

† fatma.aydogmus@gmail.com; fatmaa@istanbul.edu.tr

On the other hand, the spinor-type instanton solutions are found in both the conformally invariant pure spinor two-dimensonal Thirring model (Thirring Instanton) with the nonlinear $\left(\overline{\psi}\,\psi\right)^2$ self-coupled term [5], and the four-dimensional conformally invariant pure spinor Gursey model (Gursey Instanton) with nonlinear $\left(\overline{\psi}\,\psi\right)^{\frac{4}{3}}$ self-coupled spinor term [6] by the spontaneous symmetry breaking of the conformal invariance of ψ spinor field, *i.e.* $\left\langle 0\left|\overline{\psi}\,\psi\right|0\right\rangle \neq 0$. It is also shown [6] that the Gursey Instanton is a special case of a class of exact solutions of the Gursey Model (the Kortel Solutions [7]) found via the Heisenberg anzats. Additionally, for the purpose of the progress of quantum field theories, many works have been done on the both models [8, 9].

Moreover, instantons were believed to play a vital role in various topics of both QCD and electroweak theory. Despite their undoubted importance for the theory of strong and electroweak interactions, experimental evidence for instanton-induced processes is still lacking today. As it has been mentioned in recent works [10], instanton-induced cross sections lie within measurable range, since an attentive analysis of the Large Hadron Collider (LHC) data in CERN might lead to the experimental confirmation of such processes.

In our previous paper [11], we investigated the stability of bifurcation points of two-dimensional Thirring instantons by the Euclidian configuration of the Heisenberg ansatz in phase space [12]. In this paper, we study the Gursey model to review stability analysis in behaviours of four-dimensional Gursey instantons and investigate the role of the coupling constant in the evolution of four-dimensional spinor-type instantons in phase space. It will be also remarkable to discuss similar behaviours between two-dimensional Thirring instantons and four-dimensional Gursey instantons to understand the dependence of the behaviours of spinor-type instantons in phase space on quantum fractional spinor number as well as dimensions.

2. The role of the Heisenberg ansatz in the Gursey wave equation

The Gursey model is four-dimensional conformally invariant pure spinor model. A direct proof of the conform-invariance of the Gursey wave equation was shown in Ref. [13]. The wave equation of the pure spinor Gursey model [13] with the positive coupling constant g is given as

$$i\,\gamma_\mu \partial_\mu\,\psi + g\,\left(\overline{\psi}\,\psi\right)^{\frac{1}{3}}\,\psi = 0\,, \tag{1}$$

where the spinor field ψ has $\frac{3}{2}$ scale dimension. Since we want to discuss the behaviours of its instanton-type solutions, we follow to well known extended idea for the spinor-type instanton solutions first time given by Akdeniz and Smailagic [5].

In Ref. [5], the $(\overline{\psi}\,\psi)$ of the spinor-type instanton solutions are also related to spontaneous symmetry breaking of the full conformal group and $(\overline{\psi}\,\psi)$ are then characterized by their being invariant under the transformations of a special subgroup [14] which, in turn, reflects the final symmetry properties of the ground state of the system as [5]

$$R_\mu = \frac{1}{2}\left(aP_\mu + \frac{1}{a}D_\mu\right), \tag{2}$$

where

$$R_\mu\left(\overline{\psi}\,\psi\right) = \frac{i}{a}\left[\frac{a^2-x^2}{2}\partial_\mu + (x\partial + 2d)x_\mu\right]\left(\overline{\psi}\,\psi\right) = 0\,, \tag{3}$$

and a is a parameter with the dimensions of a length, P_μ is momentum operator and D_μ is a conformal scale invariant operator in the four-dimensional Euclidean space-time. Then, one finds $\overline{\psi}\,\psi = \pm\frac{a}{g(a^2+x^2)}$ as a solution which is related with the special case (instanton) [5] of Euclidian configuration of the Heisenberg ansatz [12]

$$\psi = \left[i\,x_\mu\gamma_\mu\,\chi\,(s) + \varphi(s)\right]c\,, \tag{4}$$

where c is an arbitrary spinor constatnt, and $\chi(s)$ and $\varphi(s)$ are real functions of $s = x_\mu^2 = r^2 + t^2$ in the Euclidean space-time, *i.e.* $r^2 = x_1^2 + x_2^2 + x_3^2$. Inserting Eq. (4) into Eq. (1) with

$$i\,\gamma_\mu\partial_\mu\,\psi = \left[-4\chi(s) - 2s\frac{d\chi(s)}{ds} + 2ix_\mu\gamma_\mu\frac{d\varphi(s)}{ds}\right]\bar{c}c \tag{5}$$

and

$$\left(\overline{\psi}\,\psi\right)^{\frac{1}{3}} = \left(s\chi(s)^2 + \varphi(s)^2\right)\left(\bar{c}c\right)^{\frac{1}{3}}\,, \tag{6}$$

one can obtain the following nonlinear differential equation system from Eqs. (5) and (6)

$$4\chi(s) + 2s\frac{d\chi(s)}{ds} - g(\bar{c}c)^{\frac{1}{3}}\left[s\,\chi^2(s) + \varphi^2(s)\right]^{\frac{1}{3}}\varphi(s) = 0\,, \tag{7a}$$

$$2\frac{d\varphi(s)}{ds} + g(\bar{c}c)^{\frac{1}{3}}\left[s\,\chi^2(s) + \varphi^2(s)\right]^{\frac{1}{3}}\chi(s) = 0\,. \tag{7b}$$

Equations (7a) and (7b) turn into the following nonlinear differential equations for $g(\bar{c}c)^{\frac{1}{3}} = \alpha$

$$4\chi(s) + 2s\frac{d\chi(s)}{ds} - \alpha\left[s\,\chi^2(s) + \varphi^2(s)\right]^{\frac{1}{3}}\varphi(s) = 0\,, \tag{8a}$$

$$2\frac{d\varphi(s)}{ds} + \alpha\left[s\,\chi^2(s) + \varphi^2(s)\right]^{\frac{1}{3}}\chi(s) = 0\,. \tag{8b}$$

By the transformations given in Ref. [7], $\chi = A\,s^{-\sigma}F(z)$, $\varphi = B\,s^{-\tau}G(z)$, and $z = \ln s$ with $\sigma = \tau + \frac{1}{2}$, $\tau = \frac{3}{4}$, and $A^2 = B^2$, we achieve the dimensionless form of the nonlinear differential equation system (8) as

$$2\frac{dF}{dz} + \frac{3}{2}F - \alpha(AB)^{\frac{1}{3}}\left[F^2 + G^2\right]^{\frac{1}{3}}G = 0\,, \tag{9a}$$

$$2\frac{dG}{dz} - \frac{3}{2}G + \alpha(AB)^{\frac{1}{3}}\left[F^2 + G^2\right]^{\frac{1}{3}}F = 0\,, \tag{9b}$$

where F and G are dimensionless functions of z, A and B are constants. Long time ago, a class of exact solutions of this nonlinear equation system in the elliptic integration form was found by Kortel [7].

3. Stability analysis of the Gursey instantons

In order to find the Gursey instanton configuration in dimensionless Eqs. (9a) and (9b), we have to consider $\alpha(AB)^{\frac{1}{3}} = 1$ [6]

$$2\frac{dF}{dz} + \frac{3}{2}F - \left[F^2 + G^2\right]^{\frac{1}{3}}G = 0\,, \tag{10a}$$

$$2\frac{dG}{dz} - \frac{3}{2}G + \left[F^2 + G^2\right]^{\frac{1}{3}}F = 0\,. \tag{10b}$$

The fixed points of the nonlinear differential equations system (10) are as follows

$$F = G = 0,\ \pm\frac{3\sqrt{3}}{4}\,.$$

The stability of the fixed points $\left(-\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{3}}{4}\right)$, (0,0), $\left(\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}\right)$ in Eqs. (10a) and (10b) can be investigated by their fluctuations in the equations. Differentiating gives

$$F'' = -\frac{3}{4}F' + \frac{1}{2}\alpha\left[F^2 + G^2\right]^{\frac{1}{3}} G' + \frac{\alpha G\left[2FF' + 2GG'\right]}{6[F^2 + G^2]^{\frac{2}{3}}}, \tag{11a}$$

$$G'' = \frac{3}{4}G' - \frac{1}{2}\alpha\left[F^2 + G^2\right]^{\frac{1}{3}} F' + \frac{\alpha F\left[2FF' + 2GG'\right]}{6[F^2 + G^2]^{\frac{2}{3}}} \tag{11b}$$

which can be written as the matrix form

$$\begin{pmatrix} F'' \\ G'' \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} + \frac{\alpha FG}{3[F^2+G^2]^{\frac{2}{3}}} & \frac{\alpha G^2}{3[F^2+G^2]^{\frac{2}{3}}} + \frac{1}{2}\alpha(F^2+G^2)^{\frac{1}{3}} \\ -\frac{\alpha F^2}{3[F^2+G^2]^{\frac{2}{3}}} - \frac{1}{2}\alpha(F^2+G^2)^{\frac{1}{3}} & \frac{3}{4} - \frac{\alpha FG}{3[F^2+G^2]^{\frac{2}{3}}} \end{pmatrix} \begin{pmatrix} F' \\ G' \end{pmatrix}. \tag{12}$$

The system has a singularity point at (0,0). As it is known, eigenvalues can be used to determine whether a fixed point is stable or unstable. If we examine the stability of the $\left(-\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{3}}{4}\right)$ and $\left(\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}\right)$ points,

$$\lambda_{\pm\frac{3\sqrt{3}}{4}, \pm\frac{3\sqrt{3}}{4}} = \pm\frac{i\sqrt{3}}{2}$$

is found. So equilibrium points have pure imaginary eigenvalues, we can say that the characterization of $\left(\pm\frac{3\sqrt{3}}{4} \pm \frac{3\sqrt{3}}{4}\right)$ fixed points is elliptic. The elliptic fixed point corresponds to a stable circular orbit around the fix points.

Moreover, we can find all fixed points of the Gursey wave equation depend on $\alpha(AB)^{\frac{1}{3}}$ values. The fixed points are found, generally,

$$\left(\pm\frac{3\sqrt{\frac{3}{2}}}{8\left[\alpha\left(AB\right)^{\frac{1}{3}}\right]^{\frac{3}{2}}}, \pm\frac{3\sqrt{\frac{3}{2}}}{8\left[\alpha(AB)^{\frac{1}{3}}\right]^{\frac{3}{2}}}\right).$$

For this values,

$$\lambda_{\pm} = \pm\frac{1}{4}\sqrt{9 - \frac{16\alpha(AB)^{\frac{1}{3}}FG}{(F^2+G^2)^{\frac{2}{3}}} - \frac{80}{3}\alpha^2(AB)^{\frac{2}{3}}\left(F^2+G^2\right)^{\frac{2}{3}}} \tag{13}$$

is found. For $\alpha(AB)^{\frac{1}{3}} > 0$, we find always unreal eigenvalues. So Gursey instantons exhibit stability behaviours around the fixed points, as it can be seen in Figs. 1, 2, and 3.

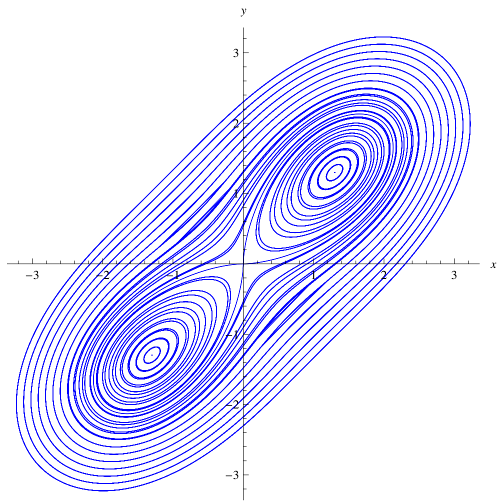

Fig. 1. Duffing-type stabile characterization of Gursey instantons for $\alpha(AB)^{\frac{1}{3}} = 1$ in the phase space; the equilibrium points are $\left(-\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{3}}{4}\right)$ and $\left(\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}\right)$.

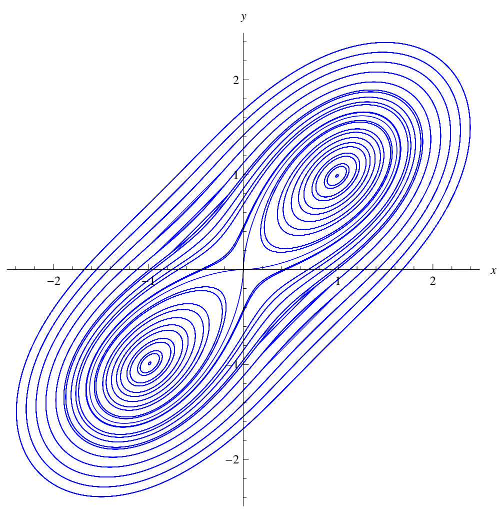

Fig. 2. Duffing-type stabile characterization of Gursey instantons for $\alpha(AB)^{\frac{1}{3}} = 1.2$ in the phase space; the equilibrium points are $\left(-\frac{5\sqrt{\frac{5}{2}}}{8}, -\frac{5\sqrt{\frac{5}{2}}}{8}\right)$ and $\left(\frac{5\sqrt{\frac{5}{2}}}{8}, \frac{5\sqrt{\frac{5}{2}}}{8}\right)$.

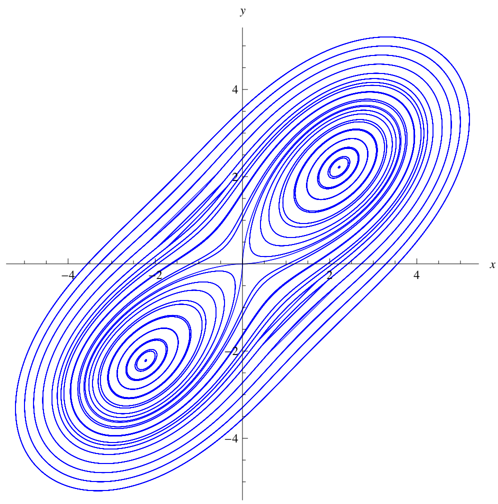

Fig. 3. Duffing-type stabile characterization of Gursey instantons for $\alpha(AB)^{\frac{1}{3}} = 0.7$ in the phase space; the equilibrium points are $\left(-\frac{15\sqrt{\frac{15}{14}}}{7}, -\frac{15\sqrt{\frac{15}{14}}}{7}\right)$ and $\left(\frac{15\sqrt{\frac{15}{14}}}{7}, \frac{15\sqrt{\frac{15}{14}}}{7}\right)$.

4. The behaviours of the Gursey instantons in phase space

If one carefully examines space parameter of the model, it will be understood that there is no difference for the properties of dynamical behaviours of the Gursey instantons in the phase space. Therefore, we particularly consider parameters which are analytically related: The dynamics behaviour of the Gursey instantons in Fig. 1 also shows that the equilibrium points $\left(-\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{3}}{4}\right)$ and $\left(\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}\right)$ fixed points have, as expected, Duffing-type stability characterization. One can also show that this result does not depend on whether the coupling constant changes. For example, the equilibrium points $\left(-\frac{5\sqrt{\frac{5}{2}}}{8}, -\frac{5\sqrt{\frac{5}{2}}}{8}\right)$ and $\left(\frac{5\sqrt{\frac{5}{2}}}{8}, \frac{5\sqrt{\frac{5}{2}}}{8}\right)$ fixed point for $\alpha(AB)^{\frac{1}{3}} = 1.2$ in Fig. 2, and equilibrium points $\left(-\frac{15\sqrt{\frac{15}{14}}}{7}, -\frac{15\sqrt{\frac{15}{14}}}{7}\right)$ and $\left(\frac{15\sqrt{\frac{15}{14}}}{7}, \frac{15\sqrt{\frac{15}{14}}}{7}\right)$ fixed point for $\alpha(AB)^{\frac{1}{3}} = 0.7$ in Fig. 3 are also stable with Duffing-type stability characterization in the phase space.

5. Conclusion

In this paper, we investigate the role of the coupling constant in the evolution of four-dimensional spinor-type instantons in the phase space via the Heisenberg ansatz and the characterization of equilibrium points of Gursey instantons. We show that they exhibit stability behaviours around the $\left(-\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{3}}{4}\right)$ and $\left(\frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}\right)$ equilibrium points. As it is realized in Figs. 1, 2, and 3, this stability of the behaviours of the Gursey instantons has unforced Duffing oscillator character stability behaviours with no coupling constant dependence.

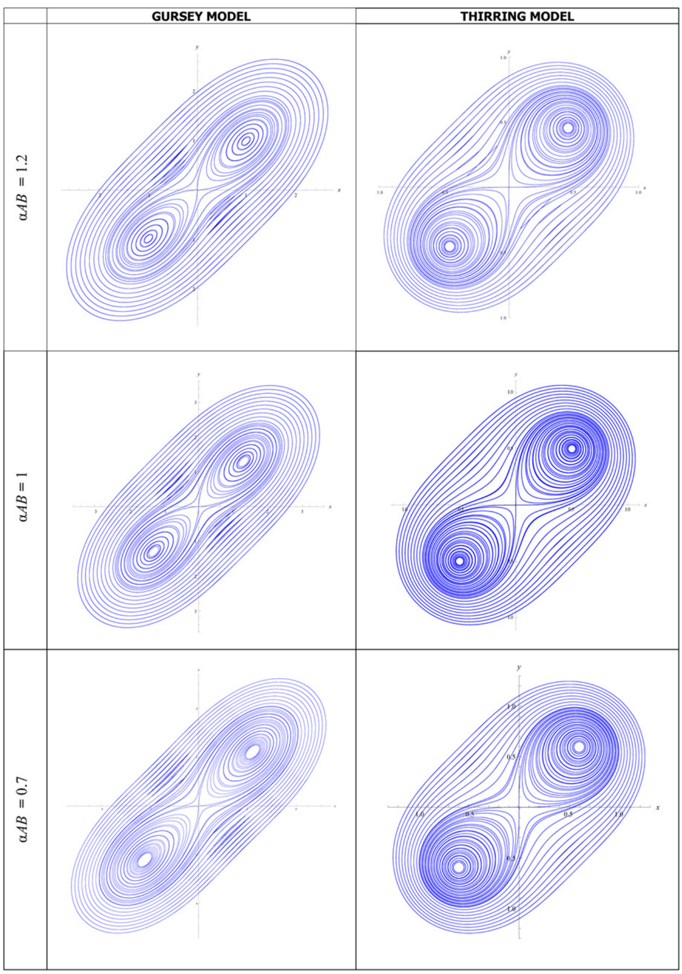

Fig. 4. Dimensionless behaviours of the Gursey and the Thirring instantons in the phase space.

It is remarkable that, as seen from Fig. 4, Duffing-type stabile characterization of the four-dimensional Gursey instantons is similar to Duffing-type stabile characterization of the two-dimensional Thirring instantons in phase space [11]. Both Gursey and Thirring instantons exhibit stability behaviours near fixed points for positive coupling constants in these models.

The above results can be interestingly concluded that the behaviours of spinor-type instantons in phase space are not dependent on the quantum fractional spinor number as well as dimensions. These results also corroborate the behaviours of spinor-type instantons as the quantum vacuum awareness of the all fractional spinor-type particles [11]. This approch also leads to define fractional spinor-type zero size instantons [15] in quantum field theory.

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