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# ON A GENERALIZATION OF THE ELLIPTIC LAW FOR RANDOM MATRICES*

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We consider the products of $m \geq 2$ independent large real random matrices with independent tuples $(X_{jk}^{(q)}, X_{kj}^{(q)})$, $1 \leq j < k \leq n$ of entries. The entries $X_{jk}^{(q)}, X_{kj}^{(q)}$ are standardized and correlated with correlation coefficient $\rho = \boldsymbol{E}[X_{jk}^{(q)} X_{kj}^{(q)}]$. The limit distribution of the empirical spectral distribution of the eigenvalues of such products does not depend on $\rho$ and is equal to the distribution of the $m^{\text{th}}$ power of a uniformly distributed random variable on the unit disc.

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## 1. Introduction

For any $m, n \geq 1$, we consider a family of real random variables $X_{j,k}^{(q)}$, $1 \leq j, k \leq n, q = 1, \ldots, m$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the following conditions **(C0)** are fulfilled:

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(a) random vectors $(X_{jk}^{(q)}, X_{kj}^{(q)})$ are mutually independent and $\mathbb{E}(X_{jk}^{(q)} X_{kj}^{(q)}) = \rho, |\rho| \le 1$, for $1 \le j < k \le n, q = 1, \dots, m$;

(b) $\mathbb{E}\, X_{jk}^{(q)} = 0$ and $\mathbb{E}(X_{jk}^{(q)})^2 = 1$ for any $1 \le j, k \le n, q = 1, \dots, m$;

(c) diagonal entries and off-diagonal entries are independent.

We say that the random variables $X_{j,k}^{(q)}$, $1 \le j, k \le n, q = 1, \dots, m$, satisfy the following uniform integrability condition **(UI)** if

$$\lim_{M\to\infty} \max_{q,j,k} \boldsymbol{E} \left|X_{jk}^{(q)}\right|^2 \boldsymbol{I}\Big\{\left|X_{jk}^{(q)}\right| > M\Big\} = 0\,. \tag{1}$$

Here and in what follows, $\boldsymbol{I}\{B\}$ denotes the indicator of the event $B$.

We introduce $m$ independent random matrices $\boldsymbol{X}^{(q)}, q = 1, \dots, m$, as follows

$$\boldsymbol{X}^{(q)} := \frac{1}{\sqrt{n}} \left[X_{jk}^{(q)}\right]_{j,k=1}^{n}\,.$$

Denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the matrix $\boldsymbol{W} := \prod_{q=1}^m \boldsymbol{X}^{(q)}$ and define the empirical spectral measure by

$$\mu_n(B) = \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{I}\{\lambda_k \in B\}\,, \qquad B \in \mathcal{B}(C)\,,$$

where $\mathcal{B}(C)$ is a Borel $\sigma$-algebra of $C$.

We say that the sequence of random probability measures $m_n(\cdot)$ converges weakly in probability to the probability measure $m(\cdot)$ (we will write $m_n \to m$) if for all continuous and bounded functions $f : \mathbb{C} \to \mathbb{C}$ and all $\varepsilon > 0$

$$\lim_{n\to\infty} \boldsymbol{P} \left( \left| \int_{\mathbb{C}} f(x) m_n(dz) - \int_{\mathbb{C}} f(x) m(dz) \right| > \varepsilon \right) = 0\,.$$

A fundamental problem in the theory of random matrices is to determine the limiting distribution of $\mu_n$ as the size of the random matrix tends to infinity. The following theorem gives the solution of this problem for matrices which satisfy (**C0**) and (**UI**).

**Theorem 1.1.** *Let $m \ge 2$ and $X_{jk}^{(q)}, j, k = 1, \dots n, q = 1, \dots, m$, satisfy* (**C0**) *with $|\rho| < 1$ and* (**UI**). *Then $\mu_n \to \mu$ in probability, and $\mu$ has the density $g$*

$$g(x, y) = \frac{1}{\pi m \left(x^2 + y^2\right)^{\frac{m-1}{m}}} \boldsymbol{I}\left\{x^2 + y^2 \le 1\right\}\,,$$

*which does not depend on $\rho$.*

Theorem 1.1 was announced in the talk of F. Götze "Spectral Distribution of Random Matrices and Free Probability" at the Advanced School and Workshop on Random Matrices and Growth Models, Trieste, Italy. Recently O'Rourke, Renfrew, Soshnikov and Vu, see [1], proved the result of Theorem 1.1 under additional assumptions on the moments of $X_{jk}^{(q)}$.

The case of $m = 1$ was considered in 1985 by Girko [2]. He showed that for $m = 1$, under the additional assumptions that the distribution of r.v.s, $X_{jk}^{(1)}$ has a density the limiting measure $\mu$ has a density of uniform distribution on the ellipse $\mathcal{E} = \{(x, y) : \frac{x^2}{(1-\rho)^2} + \frac{y^2}{(1+\rho)^2} \le 1\}$. This result was called the elliptic law. For the Gaussian matrices, the elliptic law was proved in [3]. Under the assumption of a finite fourth moment, the elliptic law was recently proved by Naumov in [4, 5]. Nguyen and O'Rourke in [6] and Götze, Naumov, Tikhomirov in [7] extended the elliptic law on the case when $X_{jk}^{(1)}$s have only finite second moments and non-identical distributions. For $m = 1$ and $\rho = 0$, we have the circular law, *i.e.* the limiting distribution $\mu$ is a uniform distribution on the unit disc. See, for example, the result of Götze, Tikhomirov [8] and Tao, Vu in [9].

In the case of $m > 1$, $\rho = 0$ and $X_{jk}^{(q)}$ and $X_{kj}^{(q)}$ are independent for $1 \le j < k \le n$, Theorem 1.1 was proved by Götze and Tikhomirov in [8]. See also the result of O'Rourke and Soshnikov [10].

## 2. Proof of the main result

In the following, we shall give the proof of Theorem 1.1. We skip almost all proofs which may be found in the extended version of this paper available at `arXiv.org`, see [11]. We shall use the logarithmic potential approach first suggested for the proof of the circular law by Götze and Tikhomirov in [12]. This approach was developed in many papers (see, for instance [8, 13] and [14]). We define the logarithmic potential of the empirical spectral measure of the matrix $\boldsymbol{W}$ by the formula

$$U_n(z) = -\int_{\mathbb{C}} \ln|w - z|\mu_n(dw)\,.$$

Let us denote by $s_1 \ge s_2 \ge \cdots \ge s_n$ the singular values of $\boldsymbol{W} - z\boldsymbol{I}$ and introduce the empirical spectral measure $\nu_n(\cdot, z)$ of squares of singular values. We can rewrite the logarithmic potential of $\mu_n$ via the logarithmic moments of the measure $\nu_n$ by

$$U_{\mu_n}(z) = -\int_{\mathbb{C}} \ln|z - w|\mu_n(dw) = -\tfrac{1}{2}\int_0^\infty \ln x \nu_n(dx)\,.$$

This allows us to consider the Hermitian matrices $(\boldsymbol{W} - z\boldsymbol{I})^*(\boldsymbol{W} - z\boldsymbol{I})$ instead of $\boldsymbol{W}$. To prove Theorem 1.1, we need the following lemma.

**Lemma 2.1.** *Suppose that for a.a. $z \in \mathbb{C}$ there exists a probability measure $\nu_z$ on $[0, \infty)$ such that*
*(a) $\nu_n \xrightarrow{weak} \nu_z$ as $n \to \infty$ in probability,*
*(b) ln is uniformly integrable in probability with respect to $\{\nu_n\}_{n \geq 1}$.*
*Then, there exists a probability measure $\mu$ such that*
*(a) $\mu_n \xrightarrow{weak} \mu$ as $n \to \infty$ in probability,*
*(b) for a.a. $z \in \mathbb{C}$*

$$U_\mu(z) = -\tfrac{1}{2} \int\limits_0^\infty \ln x \nu_z(dx) \,.$$

**Proof.** See [1, Lemma 4.3] for the proof. □

**Proof of Theorem 1.1.** From Lemma 2.1, it follows that to prove Theorem 1.1 it is enough to check conditions (a) and (b) and show that $\nu_z$ determines the logarithmic potential of the measure $\mu$. In Theorem 3.1, we find the limit distribution of the singular values of $\boldsymbol{W}(z) = \boldsymbol{W} - z\boldsymbol{I}$ (Section 3). The solution of this problem is divided into several steps. We start with a symmetrization of the one-sided distribution function. Then, we reduce the problem to the case of truncated random variables. Next, we show that the limit of empirical distribution of singular values of the product of matrices with truncated random variables is the same as the distribution of the product of matrices with Gaussian entries. Finally, we show that the limit of the expected distribution of the singular values of matrices with Gaussian entries exists and its Stieltjes transform $s(z)$ satisfies the following system of equations

$$\begin{aligned} 1 + ws(\alpha, z) + (-1)^{m+1} w^m s(\alpha, z)^{m+1} &= 0 \,, \\ (w - \alpha)^2 + (w - \alpha) - 4|z|^2 s(\alpha, z) &= 0 \,. \end{aligned}$$

From paper [13] we know that the measure with the Stieltjes transform $s(z)$ which satisfies this system of equations determines the logarithmic potential of the measure $\mu$.

In Section 4, Lemma 4.4, we show that $\ln(\cdot)$ is uniformly integrable in probability with respect to $\{\nu_n\}_{n \geq 1}$. □

## 3. Singular values of the shifted matrices

In this section, we prove that there exists the limit distribution for the empirical spectral distribution of $\boldsymbol{W} - z\boldsymbol{I}$. By $\mathcal{G}_n(x, z)$ we denote the empirical spectral distribution function of $(\boldsymbol{W}-z\boldsymbol{I})(\boldsymbol{W}-z\boldsymbol{I})^*$ (the distribution function of the uniform distribution on the squared singular values of $\boldsymbol{W}-z\boldsymbol{I}$). This distribution function corresponds to the measure $\nu_n(\cdot,z)$ introduced in the previous section. Let $G_n(x,z):=\mathbb{E}\,\mathcal{G}_n(x,z)$.

We say the entries $X_{j,k}^{(q)}$, $1\le j,k\le n, q=1,\dots,m$ of the matrices $\boldsymbol{X}^{(q)}$ satisfy Lindeberg's condition $(\mathbf{L})$ if

$$\text{for all } \tau>0,\ \ L_n(\tau):=\max_{q=1,\dots,n}\frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}\,X_{ij}^2\boldsymbol{I}\left(|X_{ij}|\ge\tau\sqrt{n}\right)\to 0 \text{ as } n\to\infty\,.$$

It easy to see that $(\mathbf{UI})\Rightarrow(\mathbf{L})$.

The main result of this section is the following Theorem.

**Theorem 3.1.** *Let $m\ge 2$ and $X_{jk}^{(q)}, j,k=1,\dots n, q=1,\dots,m$ satisfy $(\mathbf{C0})$ with $|\rho|<1$ and $(\mathbf{L})$. Then, there exists a distribution function $G(x,z)$ such that:*
*1. $G_n(x,z)\to G(x,z)$ as $n\to\infty$;*
*2. Stieltjes transform $s(\alpha,z)$ of the distribution function $G(x,z)$, defined by the equality $s(\alpha,z):=\int\frac{1}{x-\alpha}dG(x,z)$, satisfies the following system of equations:*

$$\begin{aligned} 1+ws(\alpha,z)+(-1)^{m+1}w^m s(\alpha,z)^{m+1} &= 0\,,\\ (w-\alpha)^2+(w-\alpha)-4|z|^2 s(\alpha,z) &= 0\,, \end{aligned}$$

*where* $\operatorname{Im}(w-\alpha)>0$ *for* $\operatorname{Im}\alpha>0$.

**Remark.** It is well-known that the distribution function with the Stieltjes transform satisfying the system exists and is unique. Moreover, this distribution is finitely supported and has a density. (See, for instance, [13].) In particular, if $G_n(x,z)$ converges to $G(x,z)$, then this convergence is uniformly in $x\in\mathbb{R}$, *i.e.*

$$\lim_{n\to\infty}\Delta_n(z)=\sup_x|G_n(x,z)-G(x,z)|\to 0\,. \tag{2}$$

**Remark.** It is easy to extend this theorem and show that $\mathcal{G}_n(x,z)$ weakly converges in probability to $G(x,z)$.

Introduce the following matrices

$$\begin{aligned} \boldsymbol{V} &= \begin{pmatrix}\boldsymbol{W} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{W}^*\end{pmatrix}, & \boldsymbol{J}(z) &= \begin{pmatrix}\boldsymbol{O} & z\boldsymbol{I}\\ \overline{z}\boldsymbol{I} & \boldsymbol{O}\end{pmatrix},\\ \boldsymbol{J} &= \boldsymbol{J}(1)\,, & \boldsymbol{V}(z) &= \boldsymbol{V}\boldsymbol{J}-\boldsymbol{J}(z)\,, \end{aligned} \tag{3}$$

where $\boldsymbol{I}$ denotes the unit matrix of the corresponding order and $\alpha = u + iv \in \mathbb{C}^+$ $(v > 0)$. Note that $\boldsymbol{V}(z)$ is a Hermitian matrix. The eigenvalues of the matrix $\boldsymbol{V}(z)$ are $-s_1, \dots, -s_n, s_n, \dots, s_1$. Note that the symmetrization of the distribution function $\mathcal{G}_n(x, z)$ is a function $\widetilde{\mathcal{G}}_n(x, z)$ which is the empirical distribution function of the eigenvalues of the matrix $\boldsymbol{V}(z)$. We get

$$\Delta_n(z) := \sup_x |\mathcal{G}_n(x, z) - G(x, z)| = 2 \sup_x |\widetilde{\mathcal{G}}_n(x, z) - \widetilde{G}(x, z)| =: 2\widetilde{\Delta}_n(z) .$$

We shall proof that $\lim_{n\to\infty} \widetilde{\Delta}_n(z) = 0$. In what follows, we shall consider symmetrized distribution functions only. We shall omit the symbol "$\widetilde{\cdot}$" in the corresponding notation.

We show that the limit distribution of the singular values of a product of random matrices satisfying the assumptions of Theorem 3.1 does not depend on the distribution of the matrix entries. Let $\boldsymbol{Y}^{(1)}, \dots, \boldsymbol{Y}^{(m)}$ be $n \times n$ independent random matrices with independent Gaussian entries $n^{-1/2} Y_{jk}^{(q)}$ such that $Y_{jk}^{(q)}, 1 \le j, k \le n, q = 1, \dots, m$ satisfy (**C0**). For any $\varphi \in [0, \frac{\pi}{2}]$ and any $\nu = 1, \dots, m$, introduce the following matrices

$$\boldsymbol{Z}^{(\nu)}(\varphi) = \boldsymbol{X}^{(\nu)} \sin\varphi + \boldsymbol{Y}^{(\nu)} \cos\varphi ,$$

where

$$\left[\boldsymbol{Z}^{(q)}(\varphi)\right]_{jk} = \frac{1}{\sqrt{n}} Z_{jk}^{(q)} = \frac{1}{\sqrt{n}} \left( X_{jk}^{(q)} \sin\varphi + Y_{jk}^{(q)} \cos\varphi \right) .$$

We define the matrices $\boldsymbol{W}(\varphi), \boldsymbol{V}(\varphi)$ and $\boldsymbol{V}(z, \varphi)$ similarly to (3).

In these notations, $\boldsymbol{W}(0), \boldsymbol{V}(0)$ and $\boldsymbol{V}(z, 0)$ are formed by $\boldsymbol{Y}^{(q)}$, $q = 1, \dots, m$, and $\boldsymbol{W}(\frac{\pi}{2}), \boldsymbol{V}(\frac{\pi}{2})$ and $\boldsymbol{V}(z, \frac{\pi}{2})$ are formed by $\boldsymbol{X}^{(q)}$, $q = 1, \dots, m$. Let $s_n(\alpha, z, \varphi)$ denote the Stieltjes transform of the symmetrized expected distribution function of singular values of $\boldsymbol{W}(\varphi) - z\boldsymbol{I}$. Then, $s_n(\alpha, z, \pi/2) = s_n(\alpha, z)$ denotes the Stieltjes transform of the distribution function $G_n(x, z)$ and $s_n(\alpha, z, 0)$ denotes the Stieltjes transform of the symmetrized expected distribution function of the singular values of the matrix $\boldsymbol{W}(0) - z\boldsymbol{I}$. We prove the following lemma.

**Lemma 3.2.** *Under the assumptions of Theorem 1.1, the following holds: for any $\delta > 0$,*

$$\left| s_n\left(\alpha, z, \frac{\pi}{2}\right) - s_n(\alpha, z, 0) \right| \to 0 \quad as \quad n \to \infty$$

*uniformly in $\alpha = u + iv$ with $v \ge \delta$.*

Due to Lemma 3.2, we may consider the Gaussian case only. We will omit the argument $\varphi = 0$ from the notation of the Stieltjes transform. We prove the following statement.

**Lemma 3.3.** *Let r.v.s $Y_{jk}^{(q)}$, $q = 1, \dots, m$, $j, k = 1, \dots n$ be Gaussian r.v.s satisfying the conditions* (**C0**). *Then, the following limit exists*

$$g = g(\alpha, z) = \lim_{n\to\infty} s_n(\alpha, z)\,,$$

*and satisfy the system of equations*

$$\begin{aligned} 1 + wg + (-1)^{m+1} w^{m-1} g^{m+1} &= 0\,, \\ g(w-\alpha)^2 + (w-\alpha) - g|z|^2 &= 0\,, \end{aligned} \tag{4}$$

*with a function $w = w(\alpha, z)$ such that* $\mathrm{Im}(w - \alpha) > 0$.

## 4. The minimal singular value of the matrix $\boldsymbol{V(z)}$

We shall use the following theorem which was proved in [7].

**Theorem 4.1.** *Assume that $X_{jk}$, $1 \le j, k \le n$ satisfy the conditions* **(C0)** *and* **(UI)**. *Let $\boldsymbol{X} = [X_{jk}]_{j,k=1}^n$ and $\boldsymbol{M}_n$ denote a non-random matrix with $\|\boldsymbol{M}_n\| \le Kn^Q =: K_n$ for some $K > 0$ and $Q \ge 0$. Then, there exist constants $C, A, B > 0$ depending on $K, Q$ and $\rho$ such that*

$$\mathbb{P}\left(s_n\left(\boldsymbol{X} + \boldsymbol{M}_n\right) \le n^{-B}\right) \le Cn^{-A}\,, \tag{5}$$

*where $s_n(\boldsymbol{X} + \boldsymbol{M}_n)$ is the smallest singular value of $\boldsymbol{X} + \boldsymbol{M}_n$.*

**Lemma 4.2.** *Under the conditions of Theorem 1.1, there exists a constant $C$ such that for any $k \le n(1 - C\Delta_n^{\frac{1}{m+1}}(z))$,*

$$\mathbb{P}\{s_k \le \Delta_n(z)\} \le C\Delta_n^{\frac{1}{m+1}}(z)\,.$$

**Lemma 4.3.** *Let $n_1 := [n - n\delta_n] + 1$ and $n_2 := [n - n^\gamma]$ for any sequence $\delta_n \to 0$, and some $0 < \gamma < 1$. Under the conditions of Theorem 1.1, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln s_j\left(\boldsymbol{X}^{(q)}\right) = 0\,, \quad \textit{for} \quad q = 1, \dots, m-1\,,$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln s_j\left(\boldsymbol{X}^{(m)} + \boldsymbol{M}_n\right) = 0\,,$$

*where $||\boldsymbol{M}_n|| \le n^Q$ for some $Q > 0$.*

**Lemma 4.4.** *Assume that the assumptions of Theorem 1.1 hold, then* $\ln(\cdot)$ *is uniformly integrable in probability with respect to* $\{\nu_n\}_{n\geq 1}$.

**Proof of Lemma 4.4.** It is enough to check that

$$\lim_{t\to\infty}\varlimsup_{n\to\infty}\mathbb{P}\left(\int_0^\infty|\ln x|\nu_n(dx)>t\right)=0\,. \tag{6}$$

Let $k_0=[n(1-C\varDelta_n^{\frac{1}{m+1}}(z))]$. We introduce the event

$$\begin{aligned}\varOmega_0:=\varOmega_{0,n}\ :=\ &\Big\{\omega\in\varOmega: s_n\left(\boldsymbol{X}^{(q)}\right)\geq n^{-b}, q=1,\ldots,m-1,\\ &s_n\left(\boldsymbol{X}^{(m)}+\boldsymbol{M}_n\right)\geq n^{-b}, s_{k_0}\geq\varDelta_n(z)\Big\}\end{aligned}$$

for some $b>0$ which will be chosen later and $\boldsymbol{M}_n=-z(\prod_{i=1}^{m-1}\boldsymbol{X}^{(q)})^{-1}$. Note that the matrices $\boldsymbol{X}^{(m)}$ and $\boldsymbol{M}_n$ are independent and it follows from Theorem 4.1 that $\|\boldsymbol{M}_n\|_2\leq n^Q$ for some $Q>0$ with probability close to one. From Theorem 4.1 and Lemma 4.2, we conclude that $\varlimsup_{n\to\infty}\mathbb{P}(\varOmega_0^c)=0$. It follows that it is enough to prove that

$$\lim_{t\to\infty}\varlimsup_{n\to\infty}\mathbb{P}\left(\int_0^\infty|\ln x|\nu_n(dx)>t,\varOmega_0\right)=0\,.$$

We may split the integral $\int_0^\infty|\ln x|\nu_n(dx)$ into three terms

$$T_1:=-\int_0^{\varDelta_n}\ln x\nu_n(dx,z)\,,\qquad T_2:=\int_{\varDelta_n}^{\varDelta_n^{-1}}|\ln x|\nu_n(dx,z)\,,$$

$$T_3:=\int_{\varDelta_n^{-1}}^\infty\ln x\nu_n(dx,z)\,.$$

We set $n':=k_0+1$ and $n'':=[n-n^{1-\gamma}]$. Applying Lemma 4.3, we may show that $T_1=o(1)$. For the term $T_3$, we may write the bound

$$T_3\leq\varDelta_n|\ln\varDelta_n|\int_0^\infty x^2\nu_n(dx,z)\to 0\quad\text{as}\quad n\to\infty\,,$$

where we have used the fact that $x^{-2}\ln x$ is a decreasing function for $x \geq \sqrt{e}$. It remains to estimate $T_2$. Integrating by parts and applying (2), we write

$$\mathbb{E}\, T_2 \leq C\varDelta_n|\ln \varDelta_n| + \int_{\varDelta_n}^{\varDelta_n^{-1}} |\ln x| dG(x,z) < \infty\,.$$

These facts and Markov's inequality finish the proof of lemma. □

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