

UNITARY CAUSAL QUANTUM STOCHASTIC DOUBLE PRODUCTS AS UNIVERSAL INTERACTIONS I*

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After reviewing the theory of triangular (causal) and rectangular quantum stochastic double product integrals, we consider examples when these consist of unitary operators. We find an explicit form for all such rectangular product integrals which can be described as second quantizations. Causal products are proposed as paradigm limits of large random matrices in which the randomness is explicitly quantum or noncommutative in character.

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1. Introduction: from discrete to continuous double products

Double products of *triangular* (or causal) and *rectangular* forms

$$\prod_{1 \leq j < k \leq N} x_{j,k}, \quad \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} x_{j,k} \quad (1)$$

are problematical when the elements $x_{j,k}$ do not commute with each other because there is no preferred ordering for them. Thus, while $\prod_{1 \leq j \leq N} x_j$ is naturally defined as $x_1 x_2 \dots x_N$, respecting the natural ordering of the

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integers, one might define the triangular product in (1) either as

$$\prod_{1 \leq j < N} \left\{ \prod_{j < k \leq N} x_{j,k} \right\} \quad \text{or as} \quad \prod_{1 < k \leq N} \left\{ \prod_{1 \leq j < k} x_{j,k} \right\}. \tag{2}$$

Similarly, one might take either

$$\prod_{1 \leq j \leq m} \left\{ \prod_{1 \leq k \leq n} x_{j,k} \right\} \quad \text{or} \quad \prod_{1 \leq k \leq n} \left\{ \prod_{1 \leq j \leq m} x_{j,k} \right\} \tag{3}$$

to define the rectangular product in (1).

These ambiguities are resolved if the quantities $x_{j,k}$ have the property which we call *weak commutativity*, that $x_{j,k}$ commutes with $x_{j',k'}$ whenever both $j \neq j'$ and $k \neq k'$ (but not necessarily when $j = j'$ but $k \neq k'$ or *vice versa*). Under the assumption of weak commutativity, the two alternatives (2) are equal as are the two alternatives (3). More generally, weak commutativity implies the equality of all triangular products of the form

$$\prod_{1 \leq r \leq \frac{1}{2}N(N-1)} x_{j_r, k_r}, \quad \text{where} \quad \left((j_1, k_1), (j_2, k_2), \dots, \left(j_{\frac{1}{2}N(N-1)}, k_{\frac{1}{2}N(N-1)} \right) \right)$$

is any ordering of the $\frac{1}{2}N(N-1)$ indices $\{(1, 2), (1, 3), \dots, (N-1, N)\}$ with the property that (j_r, k_r) always precedes (j_s, k_s) whenever *both* $j_r \leq j_s$ and $k_r \leq k_s$. Similarly, given weak commutativity, all rectangular products $\prod_{1 \leq r \leq mn} x_{j_r, k_r}$ have the same value for any ordering of the mn indices $\{(1, 1), \dots, (m, 1), (1, 2), \dots, (m, 2), \dots, (1, n), \dots, (m, n)\}$ having this same property.

An example where weak commutativity arises is when each $x_{j,k}$ is obtained by embedding a 2×2 matrix, in the triangular case at the intersections of the j^{th} and k^{th} rows and columns of an $N \times N$ matrix, and in the rectangular case in the intersections of the j^{th} and $(m+k)^{\text{th}}$ rows and columns of an $(m+n) \times (m+n)$ matrix, and then completing by adding 1s and 0s in the remaining diagonal and nondiagonal places respectively. A second example is when, in the triangular case, each $x_{j,k}$ is the ampliation to the N -fold tensor product $\otimes^N \mathcal{H}$ of a Hilbert space \mathcal{H} with itself, of an operator on the 2-fold tensor product $\mathcal{H} \otimes \mathcal{H}$ embedded in the j^{th} and k^{th} copies of \mathcal{H} within $\otimes^N \mathcal{H}$, and similarly in the rectangular case if it is the ampliation to $\otimes^{m+n} \mathcal{H}$ of such an operator embedded in the j^{th} and $(m+k)^{\text{th}}$ copies of \mathcal{H} within $\otimes^{m+n} \mathcal{H} = (\otimes^m \mathcal{H}) \otimes (\otimes^n \mathcal{H})$.

This paper concerns double product integrals, denoted

$$\prod_{<_{[a,b]} } (1 + dr), \quad \prod_{[a,b] \times [c,d]} (1 + dr), \tag{4}$$

which are continuous analogs of such discrete triangular and rectangular double products defined over either an interval $[a, b[$ of the real line or the Cartesian product $[a, b[\times [c, d[$ of two such intervals. Here, we denote by $\llcorner_{[a,b[}$ the set $\{(x, y) \in \mathbb{R}^2 : a \leq x < y < b\}$. In analogy with and by extension of the second example above, they consist of operators in continuous tensor products of Hilbert spaces such as Fock spaces. Each such product is characterized by a stochastic differential generator dr . This is an element of $\mathcal{I} \otimes \mathcal{I}$, where $\mathcal{I} = \mathbb{C}\langle dP, dQ, dT \rangle$ is the complex vector space of linear combinations of the stochastic differentials of the components of a quantum planar Brownian motion (P, Q) and of the time process T .

The quantum Itô product formula equips the space \mathcal{I} , and hence also $\mathcal{I} \otimes \mathcal{I}$, with an associative multiplication so that both become \dagger -algebras under natural involutions in which dP, dQ and dT are all self-adjoint.

Alternative notations which we will find useful are

$$\begin{aligned} \prod_{\llcorner_{[a,b[}} (1 + dr) &= \prod_{a \leq x < y < b} (1 + dr(x, y)) , \\ \prod_{[a,b[\times [c,d[} (1 + dr) &= \prod_{a \leq x < b, c \leq y < d} (1 + dr(x, y)) . \end{aligned}$$

We also write, for example, when

$$dr = dP \otimes dQ - dQ \otimes dP , \tag{5}$$

$$\begin{aligned} \prod_{\llcorner_{[a,b[}} (1 + dr) &= \prod_{a \leq x < y < b} (1 + dP(x) dQ(y) - dQ(x) dP(y)) , \\ \prod_{[a,b[\times [c,d[} (1 + dr) &= \prod_{a \leq x < b, c \leq y < d} (1 + dP(x) \otimes dQ(y) - dQ(x) \otimes dP(y)) . \end{aligned}$$

The latter rectangular product integral lives in a double continuous tensor product, namely the tensor product of two Fock spaces.

In both cases, the product integrals (4) can be approximated heuristically by partitioning the underlying intervals and replacing each differential in \mathcal{I} contributing to dr by increments of the corresponding process over the subintervals of the partition, so that

$$\begin{aligned} \prod_{\llcorner_{[a,b[}} (1 + dr) &\simeq \prod_{1 \leq j < k \leq N} (1 + \delta_{j,k} r) , \\ \prod_{[a,b[\times [c,d[} (1 + dr) &\simeq \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} (1 + \delta_{j,m+k} r) , \end{aligned}$$

where, for example, in the case (5), with the partition

$$\begin{aligned}
 [a, b[&= \bigcup_{j=1}^N [x_{j-1}, x_j[\quad \text{with} \quad x_j = a + \frac{j}{N} (b - a) , \\
 \delta_{j,kr} &= (P(x_j) - P(x_{j-1})) (Q(x_k) - Q(x_{k-1})) \\
 &\quad - (Q(x_j) - Q(x_{j-1})) (P(x_k) - P(x_{k-1})) .
 \end{aligned}$$

Because of the commutation relations

$$[P(s), Q(t)] = -2i \min \{s, t\} I, [P(s), P(t)] = [Q(s), Q(t)] = 0 \quad (6)$$

satisfied by the processes P and Q , we have weak commutativity of the $\delta_{j,kr}$ and the approximating discrete double products are thus well-defined in all cases.

In some cases, this form of approximation can be used to explicitly evaluate the corresponding continuous double product as a limit. Indeed, if one follows the original philosophy of Volterra [1], that product integrals are direct multiplicative analogues of additive integrals such as those of Riemann, Lebesgue and Itô [2], then our double products should properly be defined as such limits. However, there are formidable technical difficulties in making such a definition general and rigorous in the triple generalization involved in *quantum, stochastic, double* product integrals. So we follow an alternative definition.

We define our double product integrals as solutions of quantum stochastic differential equations (QSDEs) which are themselves driven by solutions of QSDEs, using quantum stochastic calculus. There are two definitions in the rectangular case, corresponding to the two equivalent forms (3) of the approximations, namely

$$\prod_{[a,b[\times[c,d[} (1 + dr) = \frac{b}{a} \prod \left(1 + \widehat{\prod}_c^d (1 + dr) \right) \quad (7)$$

$$= \prod \frac{d}{c} \left(1 + \frac{b}{a} \widehat{\prod} (1 + dr) \right) . \quad (8)$$

Here, the *single* product integrals are all defined as solutions of quantum stochastic differential equations (QSDEs). For example, $\widehat{\prod}_c^d (1 + dr)$ is the value $X(d)$ at d of the solution of the QSDE

$$dX = (X + 1) dr, \quad X(c) = 0$$

in which the second copy of \mathcal{I} in $\mathcal{I} \otimes \mathcal{I}$ is operative in the QSDE, the first copy being an initial system algebra in which the basis elements dP, dQ and

dT are multiplied according to the quantum Itô product rule. Note that the algebra \mathcal{I} is nonunital so that an initial condition $X(c) = 1$ is meaningless; the 1 in the QSDE is the identity operator in the Fock space. $\widehat{\prod}_c^d(1 + dr)$ is thus of the form

$$X(d) = dP \otimes J_c^d(\alpha) + dQ \otimes J_c^d(\beta) + dT \otimes J_c^d(\gamma) ,$$

where the operators $J_c^d(\alpha)$, $J_c^d(\beta)$ and $J_c^d(\gamma)$ are finite sums of iterated quantum stochastic integrals over $[c,d]$ of at most second order because $\mathcal{I}^3 = 0$. The right-hand side of (7) becomes

$${}_a^b \prod \left(1 + \widehat{\prod}_c^d(1 + dr) \right) = {}_a^b \prod \left(1 + dP \otimes J_c^d(\alpha) + dQ \otimes J_c^d(\beta) + dT \otimes J_c^d(\gamma) \right)$$

which is, by definition, the value $Y(b)$ at b of the solution of the QSDE

$$dY = Y \left(dP \otimes J_c^d(\alpha) + dQ \otimes J_c^d(\beta) + dT \otimes J_c^d(\gamma) \right) , \quad Y(a) = I .$$

Here, the system algebra consists of finite sums of iterated stochastic integrals over $[c,d]$ and is on the right, so that the solution Y consists of operators on the tensor product of two Fock spaces. Existence and uniqueness follow using results of Fagnola [3].

More details of the construction of both rectangular and triangular double product integrals, in particular proof of the equivalence of the two forms (7) and (8), can be found in [4]. It is also proved in [4] that a necessary and sufficient condition for either a rectangular or triangular double product integral to be unitary valued is that the differential generator dr satisfy the condition

$$dr + dr^\dagger + drdr^\dagger = 0 . \tag{9}$$

In the present paper, we analyze some examples of generators dr satisfying the unitarity condition (9) and show how the corresponding unitary double product integrals can be constructed as second quantizations of unitary operators which can be determined explicitly as limits of discrete products. Finally, we discuss the possibility that such product integrals provide alternative paradigms to models of random evolutions in terms of large random matrices.

2. Analysis of the unitarity condition

The components P and Q of quantum planar Brownian motion [5] (P, Q) are individually classical one-dimensional Brownian motions, so that for example, for each $t > 0$, $P(t)$ is a normally distributed random variable of zero

mean and variance t . But P and Q do not commute with each other, instead they satisfy the Heisenberg commutation relations (6) in the rigorous Weyl sense. The pair (P, Q) is realised [5] as processes of self-adjoint operators in the Fock space $\mathcal{F} = \Gamma(L^2[0, \infty[)$ over the Hilbert space $L^2[0, \infty[$ (some definitions are given in Section 4 below). Probabilities are determined in the usual quantum probabilistic way [6, 7] using the vacuum as state vector. Corresponding to each natural direct sum decomposition such as

$$L^2[0, \infty[= L^2[0, s_1[\oplus \bigoplus_{j=1}^{N-1} L^2[s_j, s_{j+1}[\oplus L^2[s_N, \infty[,$$

the Fock space \mathcal{F} decomposes into a tensor product

$$\mathcal{F} = \mathcal{F}_0^{s_1} \otimes \bigotimes_{j=1}^{N-1} \mathcal{F}_{s_j}^{s_{j+1}} \otimes \mathcal{F}_{s_N}^\infty$$

in such a way that each exponential vector is a product vector

$$e(f) = e(f_0^{s_1}) \otimes e(f_{s_1}^{s_2}) \otimes e(f_{s_2}^{s_3}) \otimes \dots \otimes e(f_{s_{N-1}}^{s_N}) \otimes e(f_N^\infty)$$

formed from the restrictions of the function $f \in L^2[0, \infty[$ to the intervals $[0, s_1[, [s_1, s_2[, [s_2, s_3[, \dots, [s_{N-1}, s_N[, [s_N, \infty[$.

The quantum Itô product table for planar Brownian motion (P, Q) enables quantum stochastic integrals [8] to be multiplied according to the Leibniz–Itô formula for the stochastic differential of their product

$$d(MN) = (dM)N + MdN + dMdN,$$

where the product $dMdN$ is evaluated from the Itô multiplication rule for the basic differentials, namely

	dP	dQ	dT
dP	dT	$-idT$	0
dQ	idT	dt	0
dT	0	0	0

Here, the time process T consists of multiples of the identity operator I , $T(t) = tI$.

The Itô algebra \mathcal{I} is nilpotent, $\mathcal{I}^3 = 0$, and $\mathcal{I}^2 = \mathbb{C}dT$ is one-dimensional. It follows that all products in $\mathcal{I} \otimes \mathcal{I}$ are scalar multiples $\eta dT \otimes dT$, where the scalars η for products of the basis elements $dr_{j,k} = dR_j \otimes dR_k$, $j, k = 1, 2, 3$ where $(dR_1, dR_2, dR_3) = (dP_\sigma, dQ_\sigma, dT)$, are given by the table for

η	dr_{11}	dr_{12}	dr_{21}	dr_{22}
dr_{11}	1	$-i$	$-i$	-1
dr_{12}	i	1	1	$-i$
dr_{21}	i	1	1	$-i$
dr_{22}	-1	i	i	1

$j, k \in \{1, 2\}$, while all such products involving $j = 3$ vanish. Let a general element of $\mathcal{I} \otimes \mathcal{I}$ be expanded in the form

$$dr = i \sum_{j,k=1}^3 \rho_{j,k} dr_{j,k}$$

for complex scalars $\rho_{j,k}$.

Theorem 1. *The unitarity condition (9) holds if and only if $\rho_{j,k}$ is real for all $(j, k) \neq (3, 3)$, and*

$$i(\rho_{33} - \bar{\rho}_{33}) + (\rho_{11} - \rho_{22})^2 + (\rho_{21} + \rho_{12})^2 = 0. \tag{10}$$

Proof. We have $dr^\dagger = -i \sum_{u,v=1}^3 \bar{\rho}_{u,v} dr_{u,v}$ and so, using the table,

$$\begin{aligned} dr dr^\dagger &= \left\{ \left(|\rho_{11}|^2 + |\rho_{12}|^2 + |\rho_{21}|^2 + |\rho_{22}|^2 \right) \right. \\ &\quad + 2i \operatorname{Im} \left((\rho_{12} + \rho_{21}) (\bar{\rho}_{11} - \bar{\rho}_{22}) \right) \\ &\quad \left. + 2 \operatorname{Re} (\rho_{12} \bar{\rho}_{21} - \rho_{22} \bar{\rho}_{11}) \right\} dT \otimes dT. \end{aligned} \tag{11}$$

Thus, for the condition (9) to hold, we require firstly that $\rho_{uv} - \bar{\rho}_{uv} = 0$ for all $(u, v) \neq (3, 3)$, so that these ρ_{uv} must all be real numbers. When this is the case, (11) becomes

$$dr dr^\dagger = (\rho_{11} - \rho_{22})^2 + (\rho_{21} + \rho_{12})^2 .$$

Thus for (9) to hold, we must have (10). Conversely these conditions imply (9). □

Thus, the unitary generators without any time terms, that is, with $\rho_{uv} = 0$ if either u or $v = 3$, so that in particular $\rho_{33} = 0$, are of the form

$$dr_{\lambda,\mu} = i(\lambda(dP \otimes dQ - dQ \otimes dP) + \mu(dP \otimes dP + dQ \otimes dQ)) , \tag{12}$$

where $\lambda = \rho_{12} = -\rho_{21}$ and $\mu = \rho_{11} = \rho_{22}$ are real parameters.

3. Approximation by discrete double products

We approximate the rectangular and triangular double product integrals $\prod_{[a,b[\times[c,d[} (1+dr_{\lambda,\mu})$ and $\prod_{<[a,b[} (1+dr_{\lambda,\mu})$, where $dr_{\lambda,\mu}$ is the generator (12) as follows. We partition the intervals $[a, b[$ and $[c, d[$ into m and n equal sized subintervals,

$$[a, b[= \bigsqcup_{j=1}^m [x_{j-1}, x_j[, \quad [c, d[= \bigsqcup_{k=1}^n [y_{k-1}, y_k[, \tag{13}$$

where $x_j = a + \frac{j}{m}(b - a)$, $y_k = c + \frac{k}{n}(d - c)$. Writing the right-hand side of (12) as

$$i(\lambda(dP \otimes dQ - dQ \otimes dP) + \mu(dP \otimes dP + dQ \otimes dQ)) = i \sum_{u,v=1}^2 \rho_{uv} dR_u \otimes dR_v$$

we then make the approximation

$$\begin{aligned} \prod_{[a,b[\times[c,d[} (1 + dr_{\lambda,\mu}) &= \prod_{[a,b[\times[c,d[} \left(1 + i \sum_{u,v=1}^2 \rho_{uv} dR_u \otimes dR_v \right) \\ &\simeq \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \left(1 + i \left(\sum_{u,v=1}^2 \rho_{uv} \delta_j(R_u) \otimes \delta'_k(R_v) \right) \right), \end{aligned} \tag{14}$$

where

$$\delta_j(R_u) = R_u(x_j) - R_u(x_{j-1}), \quad \delta'_k(R_v) = R_v(y_k) - R_v(y_{k-1}).$$

Because of the commutation relation (6) each of the pairs $(R_1(x_j) - R_1(x_{j-1}), R_2(x_j) - R_2(x_{j-1}))$ satisfies the commutation relation

$$[R_1(x_j) - R_1(x_{j-1}), R_2(x_j) - R_2(x_{j-1})] = -2i(x_j - x_{j-1}) = -2i \frac{b-a}{m}$$

and each of $R_1(x_j) - R_1(x_{j-1})$ and $R_2(x_j) - R_2(x_{j-1})$ commutes with each of $R_1(x_k) - R_1(x_{k-1})$ and $R_2(x_k) - R_2(x_{k-1})$. Hence, the operators $(p_j, q_j)_{j=1}^m$ defined by

$$\begin{aligned} p_j &= \sqrt{\frac{m}{b-a}} (R_1(x_j) - R_1(x_{j-1})), \\ q_j &= \sqrt{\frac{m}{b-a}} (R_2(x_j) - R_2(x_{j-1})) \end{aligned}$$

satisfy the standard¹ canonical commutation relations

$$[p_j, q_k] = -2i\delta_{jk}, [p_j, p_k] = [q_j, q_k] = 0, \quad j, k = 1, 2, \dots, m.$$

Similarly, the operators $(p'_k, q'_k)_{k=1}^m$ defined by

$$p'_k = \sqrt{\frac{n}{d-c}} (R_1(y_k) - R_1(y_{k-1})),$$

$$q'_k = \sqrt{\frac{n}{d-c}} (R_2(y_k) - R_2(y_{k-1}))$$

satisfy the standard canonical commutation relations. The approximation (14) becomes

$$\prod_{[a,b] \times [c,d]} (1 + dr_{\lambda,\mu})$$

$$\simeq \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \left(1 + i\theta_{mn} (\lambda (p_j \otimes q'_k - q'_j \otimes p_k) + \mu (p_j \otimes p'_k + q_j \otimes q'_k)) \right), \tag{15}$$

where $\theta_{mn} = \sqrt{\frac{(b-a)(d-c)}{mn}}$.

4. A one-parameter unitary group in two degrees of freedom

We consider the self-adjoint operator

$$L(\lambda, \mu) = \lambda (pq' - qp') + \mu (pp' + qq'), \tag{16}$$

where (p, q) and (p', q') are mutually commuting standard canonical pairs, together with the one-parameter unitary group $(e^{ixL(\lambda,\mu)})_{x \in \mathbb{R}}$ of which $L(\lambda, \mu)$ is the infinitesimal generator. It is convenient to make use of two different unitarily equivalent standard representations of the quantum system of two degrees of freedom described by (p, q) , (p', q') .

The *Schrödinger* representation is in the Hilbert space $L^2(\mathbb{R}^2)$ and is given informally by

$$p = -\sqrt{2}i \frac{\partial}{\partial s}, \quad q = \sqrt{2}s, \quad p' = -\sqrt{2}i \frac{\partial}{\partial t}, \quad q' = \sqrt{2}t$$

¹ In quantum probability, it is most convenient to normalise Planck's constant to the value 4π .

in terms of the components of the vector² $(s, t)^\tau$ in \mathbb{R}^2 . More rigorously, the corresponding one-parameter unitary groups $(e^{ixp})_{x \in \mathbb{R}}$, $(e^{ixq})_{x \in \mathbb{R}}$, $(e^{ixp'})_{x \in \mathbb{R}}$ and $(e^{ixq'})_{x \in \mathbb{R}}$ are given by the actions:

$$\begin{aligned} (e^{ixp} f) ((s, t)^\tau) &= f \left((s + \sqrt{2}x, t)^\tau \right), \\ (e^{ixq} f) ((s, t)^\tau) &= e^{i\sqrt{2}xs} f ((s, t)^\tau), \\ (e^{ixp'} f) ((s, t)^\tau) &= f \left((s, t + \sqrt{2}x)^\tau \right), \\ (e^{ixq'} f) ((s, t)^\tau) &= e^{i\sqrt{2}xt} f ((s, t)^\tau). \end{aligned}$$

By contrast, the *Fock* representation, which we will now describe, is in the Fock space $\Gamma(\mathbb{C}^2)$ over the Hilbert space \mathbb{C}^2 . It is convenient to define the Fock space $\Gamma(\mathcal{H})$ over an arbitrary Hilbert space \mathcal{H} abstractly as another Hilbert space generated by a family $(e(f))_{f \in \mathcal{H}}$ of so-called *exponential* vectors satisfying

$$\langle e(f), e(g) \rangle = e^{\langle f, g \rangle}$$

for arbitrary $f, g \in \mathcal{H}$. Any two candidate Fock spaces are then unitarily equivalent under a unique unitary operator which exchanges the families of exponential vectors, and can thus be identified. If R is a unitary operator on \mathcal{H} , then there is a unique unitary operator $\Gamma(R)$ on $\Gamma(\mathcal{H})$, called the *second quantization* of R , such that for each $f \in \mathcal{H}$, $\Gamma(R)e(f) = e(Rf)$. If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is a direct sum, we identify $\Gamma(\mathcal{H})$ with the tensor product $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$ with $e(f_1 \oplus f_2) = e(f) \otimes e(f_2)$.

The family of *Weyl operators* $(W(f))_{f \in \mathcal{H}}$ consists of the unitary operators for which, for arbitrary $f, g \in \mathcal{H}$,

$$W(f) e(g) = e^{-\langle f, g \rangle - \frac{1}{2} \|f\|^2} e(f + g).$$

They satisfy the *Weyl relation*

$$W(f) W(g) = e^{-i\text{Im}\langle f, g \rangle} W(f + g).$$

For arbitrary unitary R on \mathcal{H} and $f \in \mathcal{H}$, we have

$$\Gamma(R) W(f) \Gamma(R)^{-1} = W(Rf). \tag{17}$$

We realise (p, q) and (p', q') in $\Gamma(\mathbb{C}^2)$ in terms of Weyl operators by defining

$$\begin{aligned} e^{ixp} &= W((x, 0)^\tau), & e^{ixq} &= W((-ix, 0)^\tau), \\ e^{ixp'} &= W((0, x)^\tau), & e^{ixq'} &= W((0, -ix)^\tau). \end{aligned}$$

² τ denotes transposition.

These satisfy the canonical commutation relations because, for example, the Weyl relation implies that

$$e^{ixp} e^{iyq} = e^{2ixy} e^{iyq} e^{ixp}$$

which is the rigorous form of the commutation relation

$$[p, q] = -2i.$$

To establish the unitary equivalence with the Schrödinger realisation, we first make the chain of identifications

$$\Gamma(\mathbb{C}^2) = \Gamma(\mathbb{C} \oplus \mathbb{C}) = \Gamma(\mathbb{C}) \otimes \Gamma(\mathbb{C}) = l^2 \otimes l^2,$$

where $\Gamma(\mathbb{C})$ is identified with the standard Hilbert space l^2 of square-sumable sequences with

$$e(z) = \left(1, \frac{z}{\sqrt{1!}}, \frac{z^2}{\sqrt{2!}}, \dots \right).$$

Next, we map the space l^2 onto $L^2(\mathbb{R})$ by mapping the sequence $(z_n)_{n=0}^\infty \in l^2$ to $\sum_{n=0}^\infty z_n h_n$, where $(h_n)_{n=0}^\infty$ is the orthonormal basis of Hermite functions in $L^2(\mathbb{R})$. Finally, we identify $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ with $L^2(\mathbb{R}^2)$ by extending the identification $(f \otimes g)((s, t)^\tau) = f(s)g(t)$, and thereby map $\Gamma(\mathbb{C}^2) = l^2 \otimes l^2$ onto $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L(\mathbb{R}^2)$. The Fock vacuum vector $e(0_{\mathbb{C}^2}) = e(0) \otimes e(0)$ is mapped to $h_0 \otimes h_0$ which is the Gaussian function

$$G(s, t) = \frac{1}{\pi} e^{-\frac{1}{2}(s^2+t^2)}.$$

Note that in the Schrödinger representation,

$$L(\lambda, \mu)(h_0 \otimes h'_0) = 0 \tag{18}$$

since

$$\begin{aligned} ((pq' - qp')G)(s, t) &= -\frac{2i}{\pi} \left(\frac{\partial}{\partial s} t - s \frac{\partial}{\partial t} \right) e^{-\frac{1}{2}(s^2+t^2)} = 0, \\ ((pp' + qq')G)(s, t) &= \frac{2}{\pi} \left(-\frac{\partial^2}{\partial s \partial t} + st \right) e^{-\frac{1}{2}(s^2+t^2)} = 0. \end{aligned}$$

Let us now realise $e^{ixL(\lambda, \mu)}$ in the Fock representation as the second quantization of a unitary operator given on \mathbb{C}^2 by left multiplication by a unitary 2×2 matrix. We introduce the notation

$$\nu = \lambda + i\mu = e^{i\phi} |\nu|.$$

Theorem 2. *In the Fock representation,*

$$e^{ixL(\lambda,\mu)} = \Gamma \left(\begin{bmatrix} \cos(2x|v|) & -e^{-i\phi} \sin(2x|v|) \\ e^{i\phi} \sin(2x|v|) & \cos(2x|v|) \end{bmatrix} \right). \tag{19}$$

Proof. We use the notation

$$\xi_{(\lambda,\mu)}(x) = \begin{bmatrix} \cos(2x|v|) & -e^{-i\phi} \sin(2x|v|) \\ e^{i\phi} \sin(2x|v|) & \cos(2x|v|) \end{bmatrix}.$$

Note first that $(\xi_{(\lambda,\mu)}(x))_{x \in \mathbb{R}}$ is indeed a one-parameter unitary group. Now, consider

$$\begin{aligned} & \Gamma(\xi_{(\lambda,\mu)}(x)) e^{iy p} \Gamma(\xi_{(\lambda,\mu)}(x))^{-1} \\ &= \Gamma(\xi_{(\lambda,\mu)}(x)) W((y, 0)^\tau) \Gamma(\xi_{(\lambda,\mu)}(x))^{-1} \\ &= W(\xi_{(\lambda,\mu)}(x) (y, 0)^\tau) \\ &= W\left(\left(\cos(2x|v|) y, e^{i\phi} \sin(2x|v|) y\right)^\tau\right) \\ &= W\left(\left(\cos(2x|v|) y, 0\right)^\tau\right) W\left(\left(0, e^{i\phi} \sin(2x|v|) y\right)^\tau\right). \end{aligned}$$

Since

$$\begin{aligned} & W\left(\left(0, e^{i\phi} \sin(x|v|) y\right)^\tau\right) \\ &= W\left(\left(0, \cos \phi \sin(2x|v|) y + i \sin \phi \sin(2x|v|) y\right)^\tau\right) \\ &= e^{i \cos \phi \sin \phi \sin(2x|v|)^2 y^2} W\left(\left(0, \cos \phi \sin(2x|v|) y\right)^\tau\right) W\left(\left(0, i \sin \phi \sin(2x|v|) y\right)^\tau\right) \\ &= e^{i \cos \phi \sin \phi \sin(2x|v|)^2 y^2} e^{iy \cos \phi \sin(2x|v|) p'} e^{-iy \sin \phi \sin(2x|v|) q'} \\ &= e^{i \sin(2x|v|) y (\cos \phi p' - \sin \phi q')}, \end{aligned}$$

we get

$$\Gamma(\xi_{(\lambda,\mu)}(x)) e^{iy p} \Gamma(\xi_{(\lambda,\mu)}(x))^{-1} = e^{iy(\cos(2x|v|)p + \sin(2x|v|)(\cos \phi p' - \sin \phi q'))}.$$

Forming $-i \frac{d}{dy} \Big|_{y=0}$, we deduce that

$$\Gamma(\xi_{(\lambda,\mu)}(x)) p \Gamma(\xi_{(\lambda,\mu)}(x))^{-1} = \cos(2x|v|) p + \sin(2x|v|) (\cos \phi p' - \sin \phi q').$$

Forming $\frac{d}{dx} \Big|_{x=0}$, in turn, we get

$$[iK, p] = 2|v| (\cos \phi p' - \sin \phi q') = 2(\lambda p' - \mu q') = [iL(\lambda, \mu), p],$$

where K is the self-adjoint infinitesimal generator of the one-parameter unitary group $(\Gamma(\xi_{(\lambda,\mu)}(x)))_{x \in \mathbb{R}}$. Similarly, we find that

$$\begin{aligned} [iK, q] &= [iL(\lambda, \mu), q], & [iK, p'] &= [iL(\lambda, \mu), p'], \\ [iK, q'] &= [iL(\lambda, \mu), q']. \end{aligned}$$

Since the Fock representation is irreducible, we deduce that K differs from $L(\lambda, \mu)$ by at most a multiple of the identity. But the second quantizations $\Gamma(\xi_{(\lambda,\mu)}(x))$ all map the vacuum vector $e(0_{\mathbb{C}^2}) = e(0) \otimes e(0)$ to itself, hence K annihilates $e(0) \otimes e(0)$. Moreover, $L(\lambda, \mu)$ also annihilates $e(0) \otimes e(0)$, since by (18), its Schrödinger equivalent annihilates the equivalent vector in the Schrödinger representation. It follows that $K = L(\lambda, \mu)$. \square

5. Realisation of rectangular double products as second quantizations

Since $\theta_{mn} = \sqrt{\frac{(b-a)(d-c)}{mn}}$, for large m and n , we can make the approximation

$$1 + i\theta_{mn}L(\lambda, \mu) \simeq \exp(i\theta_{mn}L(\lambda, \mu)),$$

for mutually commuting pairs (p, q) and (p', q') , to obtain from (15) the further approximation

$$\prod_{[a,b[\times [c,d[} (1 + d\tau_{\lambda,\mu}) \simeq \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \exp(i\theta_{mn}(\lambda(p_j q'_k - q_j p'_k) + \mu(p_j p'_k + q_j q'_k))). \tag{20}$$

We embed the Hilbert space $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ into $L^2([a, b[) \oplus L^2([c, d[)$ by mapping each element ε_i of the canonical orthonormal basis of \mathbb{C}^{m+n} to the vector χ_i , where

$$\chi_i = \begin{cases} \sqrt{\frac{m}{b-a}} \chi_{[a + \frac{i-1}{m}(b-a), a + \frac{i}{m}(b-a)[} \oplus 0 & \text{if } i = 1, 2, \dots, m \\ 0 \oplus \sqrt{\frac{n}{d-c}} \chi_{[c + \frac{i-m-1}{n}(d-c), c + \frac{i-m}{n}(d-c)[} & \text{if } i = m + 1, m + 2, \dots, m + n \end{cases}.$$

We regard each complex $(m + n) \times (m + n)$ matrix $M = [M_{i,l}]_{i,l \in \mathbb{N}_{m+n}}$ acting on \mathbb{C}^{m+n} by left multiplication as an operator on $L^2([a, b[) \oplus L^2([c, d[)$ given by

$$\sum_{i,l=1}^{m+n} M_{i,l} |\chi_i\rangle \langle \chi_l|.$$

Then using (19), the approximation (20) becomes

$$\prod_{[a,b] \times [c,d]} (1 + dr_{\lambda,\mu}) \simeq \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \Gamma \left(\begin{bmatrix} 1 & \dots & \binom{j}{0} & \dots & \binom{m+k}{0} & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ \binom{j}{0} 0 & \dots & \cos(2\theta_{mn} |v|) & \dots & -e^{-i\phi} \sin(2\theta_{mn} |v|) & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots & \dots & \vdots \\ \binom{m+k}{0} 0 & \dots & e^{i\phi} \sin(2\theta_{mn} |v|) & \dots & \cos(2\theta_{mn} |v|) & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \right)$$

which is equal to the second quantization of the matrix operator

$$\prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \begin{bmatrix} 1 & \dots & \binom{j}{0} & \dots & \binom{m+k}{0} & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ \binom{j}{0} 0 & \dots & \cos(2\theta_{mn} |v|) & \dots & -e^{-i\phi} \sin(2\theta_{mn} |v|) & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots & \dots & \vdots \\ \binom{k}{0} 0 & \dots & e^{i\phi} \sin(2\theta_{mn} |v|) & \dots & \cos(2\theta_{mn} |v|) & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} .$$

We denote by this matrix operator by $W_{m,n}$. The limit $\lim_{m,n \rightarrow \infty} W_{m,n}$ may then be found in two stages as follows. First, consider $W_{m,1}$ and $W_{1,n}$, which are respectively given by

$$\prod_{j=1}^m \begin{bmatrix} 1 & 0 & \dots & \binom{j}{0} & 0 & \dots & 0 & \binom{m+1}{0} \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \binom{j}{0} 0 & 0 & \dots & \alpha & 0 & \dots & 0 & \beta \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \\ \binom{m+1}{0} 0 & 0 & \dots & \gamma & 0 & \dots & 0 & \delta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & \beta\gamma & \beta\delta\gamma & \dots & \beta\delta^{m-3}\gamma & \beta\delta^{m-2}\gamma & \beta\delta^{m-1} \\ 0 & \alpha & \beta\gamma & \dots & \beta\delta^{m-4}\gamma & \beta\delta^{m-3}\gamma & \beta\delta^{m-2} \\ 0 & 0 & \alpha & \dots & \beta\delta^{m-5}\gamma & \beta\delta^{m-4}\gamma & \beta\delta^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & \beta\gamma & \beta\delta \\ 0 & 0 & 0 & \dots & 0 & \alpha & \beta \\ \gamma & \delta\gamma & \delta^2\gamma & \dots & \delta^{m-2}\gamma & \delta^{m-1}\gamma & \delta^m \end{bmatrix} \tag{21}$$

and

$$\prod_{k=1}^n \begin{bmatrix} \alpha & 0 & \dots & \beta^{k+1} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (k+1)\gamma & 0 & \dots & \delta & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^n & \beta & \alpha\beta & \alpha^2\beta & \dots & \alpha^{n-2}\beta & \alpha^{n-1}\beta \\ \gamma\alpha^{n-1} & \delta & \gamma\beta & \gamma\delta\beta & \dots & \gamma\alpha^{n-3}\beta & \gamma\alpha^{n-2}\beta \\ \gamma\alpha^{n-2} & 0 & \delta & \gamma\beta & \dots & \gamma\alpha^{n-4}\beta & \gamma\alpha^{n-3}\beta \\ \gamma\alpha^{n-3} & 0 & 0 & \delta & \dots & \gamma\alpha^{n-5}\beta & \gamma\alpha^{n-4}\beta \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma\alpha & 0 & 0 & 0 & \dots & \delta & \gamma\beta \\ \gamma & 0 & 0 & 0 & \dots & 0 & \delta \end{bmatrix}, \tag{22}$$

where

$$\alpha = \delta = \cos \left(2\sqrt{\frac{(b-a)(d-c)}{mn}} |v| \right)$$

and

$$\beta = -\bar{\gamma} = -e^{-i\phi} \sin \left(2\sqrt{\frac{(b-a)(d-c)}{mn}} |v| \right).$$

Using the limits

$$\begin{aligned} \lim_{m \rightarrow \infty} \delta^m &= \lim_{m \rightarrow \infty} \left(1 - \frac{1}{2} \left(2\sqrt{\frac{(b-a)(d-c)}{mn}} |v| \right)^2 \right)^m \\ &= e^{-2(b-a)(d-c)|v|^2/n}, \end{aligned}$$

the limit of the operator (21) is the matrix operator on $L^2([a, b]) \oplus \mathbb{C}$

$$\lim_{m \rightarrow \infty} W_{m,1} = \begin{bmatrix} I + A_n & -e^{-i\phi} \left| f_{[a,b]}^{(d-c)/n} \right\rangle \\ e^{i\phi} \left\langle g_{[a,b]}^{(d-c)/n} \right| & e^{-2(b-a)(d-c)|\nu|^2/n} \end{bmatrix}. \tag{23}$$

Here, $\left| f_{[a,b]}^{(d-c)/n} \right\rangle$ and $\left\langle g_{[a,b]}^{(d-c)/n} \right|$ denote respectively the map $\mathbb{C} \rightarrow L^2([a, b])$, $z \mapsto z f_{[a,b]}^{(d-c)/n}$ and the covector $L^2([a, b]) \rightarrow \mathbb{C}$, $h \mapsto \left\langle g_{[a,b]}^{(d-c)/n}, h \right\rangle$, where for $s \in [a, b]$

$$f_{[a,b]}^{(d-c)/n}(s) = e^{-2(d-c)|\nu|^2(b-s)/n}, \quad g_{[a,b]}^{(d-c)/n}(s) = e^{-2(d-c)|\nu|^2(s-a)/n},$$

and A_n is the integral operator on $L^2([a, b])$ whose kernel is

$$(s, t) \mapsto -2|\nu|^2 \chi_{<[a,b]} e^{-2|\nu|^2(d-c)(t-s)/n}.$$

Similarly, since

$$\lim_{n \rightarrow \infty} \alpha^n = e^{-2(b-a)(d-c)|\nu|^2/m},$$

the limit of the operator (22) is the matrix operator on $\mathbb{C} \oplus L^2([c, d])$

$$\lim_{n \rightarrow \infty} W_{1,n} = \begin{bmatrix} e^{-2(b-a)(d-c)|\nu|^2/m} & \left\langle g_{[c,d]}^{(b-a)/m} \right| \\ \left| f_{[c,d]}^{(b-a)/m} \right\rangle & I + D_m \end{bmatrix}. \tag{24}$$

The matrix product rules (21) and (22) continue to hold even if the scalars α, β, γ and δ are replaced by a scalar, a covector, a vector and an operator on $L^2(\mathbb{R})$ in the first case, and by an operator, a vector, a covector and a scalar, respectively, in the second. It is thus possible to evaluate $\lim_{m \rightarrow \infty} \left(\lim_{n \rightarrow \infty} W_{m,n} \right)$ and $\lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} W_{m,n} \right)$ by substituting into (21) and (22) respectively the four entries in the matrix (24) and in (23). Denoting by $V_{[a,b]}$ the integral operator on $L^2([a, b])$

$$(V_{[a,b]}f)(s) = \int_s^b f(t) dt, \quad s \in [a, b]$$

and using the limits

$$\lim_{m \rightarrow \infty} (I + D_m)^m = e^{-|\nu|^2(b-a)V_{[c,d]}}, \quad \lim_{n \rightarrow \infty} (I + A_n)^n = e^{-|\nu|^2(d-c)V_{[a,b]}},$$

one finds after some computation that the two iterated limits are equal, and that their common value is

$$\lim_{m,n \rightarrow \infty} W_{m,n} = \begin{bmatrix} I + A & B \\ C & I + D \end{bmatrix}, \tag{25}$$

where A, B, C and D are respectively integral operators from $L^2([a, b])$ to itself, from $L^2([c, d])$ to $L^2([a, b])$ and *vice versa*, and from $L^2([c, d])$ to itself, whose kernels are given respectively by

$$\begin{aligned} \text{Ker}A(s, t) &= \chi_{<[a,b]}(s, t) \sum_{N=0}^{\infty} \frac{(t-s)^N \left(-|v|^2(d-c)\right)^{N+1}}{N!(N+1)!}, \\ \text{Ker}B(s, t) &= \nu \chi_{[a,b] \times [c,d]} \sum_{N=0}^{\infty} \frac{\left(-|v|^2(b-s)(t-c)\right)^N}{(N!)^2}, \\ \text{Ker}C(s, t) &= -\bar{\nu} \chi_{[c,d] \times [a,b]} \sum_{N=0}^{\infty} \frac{\left(-|v|^2(d-s)(t-a)\right)^N}{(N!)^2}, \\ \text{Ker}D(s, t) &= \chi_{<[c,d]}(s, t) \sum_{N=0}^{\infty} \frac{(t-s)^N \left(-|v|^2(b-a)\right)^{N+1}}{N!(N+1)!}. \end{aligned}$$

See [9] for more details of essentially this argument in the case of $\mu = 0$ and [10] for an alternative derivation where also it is shown rigorously that the limit (25) is indeed a unitary operator and that these heuristics are rigorously justified. Namely,

$$\Gamma \left(\begin{bmatrix} I + A & B \\ C & I + D \end{bmatrix} \right) = \prod_{[a,b] \times [c,d]} (1 + i\lambda(dP \otimes dQ - dQ \otimes dP))$$

in so far as the relevant QSDEs are satisfied.

An explicit form similar to (25) for the corresponding unitary operator $W_{<[a,b]}$ such that

$$\Gamma \left(W_{<[a,b]} \right) = \prod_{<[a,b]} (1 + i(\lambda(dP \otimes dQ - dQ \otimes dP) + \mu(dP \otimes dP - dP \otimes dP)))$$

is more difficult to obtain. However, it has been found provisionally and verified to be unitary [11]. At present, it remains to be verified that $\Gamma \left(W_{<[a,b]} \right)$ satisfies the defining QSDE.

6. Universality

The unitary generators of form $dr_{\lambda,\mu}$ of form (12) can be characterized among all such generators by their invariance under rotational automorphisms of the form

$$(dP, dQ) \mapsto (\cos \theta dP - \sin \theta dQ, \sin \theta dP + \cos \theta dQ)$$

of the Itô algebra \mathcal{I} . The corresponding double products are likewise invariant under gauge transformations

$$(P, Q) \mapsto (\cos \theta P - \sin \theta Q, \sin \theta P + \cos \theta Q)$$

of the underlying quantum planar Brownian motion.

The generators $dr_{\lambda,\mu}$ are complemented by a second real two-parameter family

$$dr'_{\lambda,\mu} = i(\lambda(dP \otimes dQ + dQ \otimes dP) + \mu(dP \otimes dP - dQ \otimes dQ)) + \eta_{\lambda,\mu} dT \otimes dT$$

of unitary generators which require the additional non-zero time term of the form of $\eta_{\lambda,\mu} dT \otimes dT$ to satisfy the unitarity condition (9). The corresponding double products are no longer given by second quantizations. However, they can be characterized as unitary implementors of explicit Bogolubov transformations, that is, invertible real-linear transformations of the complex Hilbert space $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ which preserve the imaginary part of the inner product. See [12] for the case when $\mu = 0$.

It is expected that the full 4-dimensional manifold of unitary double products, with generators of the form of

$$d\rho_{\lambda,\mu,\lambda',\mu'} = dr_{\lambda,\mu} + dr'_{\lambda',\mu'} + \eta_{\lambda,\mu,\lambda',\mu'} dT \otimes dT,$$

admits a similar characterization as unitary implementors of explicit Bogolubov transformations, but this has yet to be established in full generality.

We propose that triangular (or causal) double products of the form of

$$\prod_{<[a,b[} (1 + d\rho_{\lambda,\mu,\lambda',\mu'})$$

provide a randomized model for unitary time evolution of complex non-relativistic quantum systems, and conjecture that these products offer paradigm universal limits for large random unitary matrices.

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