N-BY-N RANDOM MATRIX THEORY WITH MATRIX REPRESENTATIONS OF OCTONIONS

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The eigenvalue statistics of real adjoints of $N \times N$ Hermitian octonion random matrices are studied numerically. By allowing various matrix elements to be turned OFF or ON, we are able to observe eigenvalue statistics that are described by the three Gaussian ensembles of classical random matrix theory. In certain cases, we have also observed eigenvalues that appear to be a superposition of two independent spectra, each of which is described by statistics of the Gaussian symplectic ensemble.

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Random matrix theory (RMT) is a vast subject which has been successfully applied to many fields of study over the years (see, for example, Refs. [1–14]). As of 2012, a journal [15] dedicated to the theory and applications of random matrices has been active, indicating just how large the subject has become. Classical RMT is a usual starting point for novice researchers; it consists of three ensembles known as the Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles. The matrices that are used to construct the GOE, GUE, and GSE are real symmetric, complex Hermitian, and quaternion real, respectively, with appropriately defined matrix elements — Refs. [16, 17] provide edifying discussions on the Gaussian ensembles. A natural question for the novice to ask is: Why is there not a classical ensemble associated with the octonions [18–20], which are the largest normed division algebra? The answer: It is well-understood that octonions cannot be represented as a matrix algebra and, for this reason, there is not a place for them in RMT. However, a recent study [21] has shown that some pseudo-real matrix representations of octonions [22] can be easily brought into RMT and that, at least for 2×2 matrices, some interesting results can be obtained. For example, a random matrix model that was tunable to produce level repulsion from linear to octic was introduced.

Unfortunately, the work of Ref. [21] did not go beyond 2×2 matrices and it was left as an open question as to what would happen if the model was extended to $N \times N$ matrices, where N is large. The purpose of the current study is to provide some answers to this question, through numerical experiments. (It should be noted that the idea of bringing octonions into RMT is far from novel — Dyson himself considered this in footnote 10 of his famous paper on the "threefold way" [23].)

We begin with a review of some terminology. Let $\mathbb{O}$ denote the octonion algebra over the real number field $\mathbb{R}$ and let $a \in \mathbb{O}$ — it is known that, based on an extension of the Cayley–Dickson process and results from real matrix representations of quaternions, a (pseudo)-real matrix representation of a is [22]

$$\omega(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{pmatrix}, \tag{1}$$

where $a_k \in \mathbb{R}$ $(k = 0, 1, \ldots, 7)$. The matrix $\omega(a)$ is known as the left matrix representation of a over $\mathbb{R}$. Next, consider the following 2×2 Hermitian octonion matrix

$$A = \begin{pmatrix} b & a \\ \bar{a} & c \end{pmatrix}, \tag{2}$$

where $b, c \in \mathbb{R}$, and its real adjoint [22]

$$\omega(A) = \begin{pmatrix} bI_8 & \omega(a) \\ \omega^T(a) & cI_8 \end{pmatrix}, \tag{3}$$

where I_8 is an 8×8 identity matrix. It is known that $\omega(A)$ (and hence A) has two real eigenvalues, each of which is eightfold degenerate. In Ref. [21], an ensemble of random matrices given by $\omega(A)$ was studied — the variables b and c were taken to be zero-centered Gaussian distributed with a variance of two and the variables a_k $(k = 0, 1, \ldots, 7)$ were zero-centered Gaussian distributed with a variance of one. It was shown that the nearest-neighbour spacing distribution (NNSD) of an ensemble of such matrices is given by $P_{\mathrm{W}}(S; \beta = 8)$, where (see, for example, Ref. [24])

$$P_{\mathrm{W}}(S; \beta) = \mathcal{A}(\beta) S^{\beta} \exp\left[-\mathcal{B}(\beta) S^2\right], \tag{4a}$$

$$\mathcal{A}(\beta) = 2\frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{\beta+1}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{\beta+2}}, \qquad \text{and} \qquad \mathcal{B}(\beta) = \frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{2}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{2}}. \tag{4b}$$

(Note that S represents normalized spacings, such that $\int_0^\infty S P_{\mathrm{W}}(S;\beta)\,\mathrm{d}S = 1$, and that $\int_0^\infty P_{\mathrm{W}}(S;\beta)\,\mathrm{d}S = 1$.) $P_{\mathrm{W}}(S;\beta)$ are the Wigner surmises of RMT and β is referred to as the "level repulsion" parameter — $\beta = 1, 2,$ and 4, for the GOE, GUE, and GSE, respectively. The Wigner surmises are exact for 2×2 random matrices and serve as excellent approximations for $N \times N$ random matrices. Given that $P_{\mathrm{W}}(S;\beta = 8)$ describes the NNSD for an ensemble of 2×2 random matrices defined by $\omega(A)$, octic level repulsion is present. What of $N \times N$ random matrices?

Consider the $N \times N$ Hermitian octonion matrix (our notation is an extension of that given in Ref. [22] for 3×3 Hermitian octonion matrices)

$$A_{N\times N} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \overline{a}_{12} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{1N} & \overline{a}_{2N} & \dots & a_{NN} \end{pmatrix}, \tag{5}$$

where $a_{11}, a_{22}, ..., a_{NN} \in \mathbb{R}$. The real adjoint of $A_{N\times N}$ is

$$\omega(A_{N\times N}) = \begin{pmatrix} \omega(a_{11}) & \omega(a_{12}) & \dots & \omega(a_{1N}) \\ \omega^T(a_{12}) & \omega(a_{22}) & \dots & \omega(a_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^T(a_{1N}) & \omega^T(a_{2N}) & \dots & \omega(a_{NN}) \end{pmatrix}, \tag{6}$$

and it is understood that $\omega(a_{ii}) = a_{ii} I_8$ (where $i = 1, 2, \dots, N$). Note that $\omega(a_{ij})$ (where $i, j = 1, 2, \dots, N$, $i \neq j$) are real matrix representations of $a_{ij} \in \mathbb{O}$ and have the matrix form given by

$$\omega(a_{ij}) = \begin{pmatrix} a_{ij_0} & -a_{ij_1} & -a_{ij_2} & -a_{ij_3} & -a_{ij_4} & -a_{ij_5} & -a_{ij_6} & -a_{ij_7} \\ a_{ij_1} & a_{ij_0} & -a_{ij_3} & a_{ij_2} & -a_{ij_5} & a_{ij_4} & a_{ij_7} & -a_{ij_6} \\ a_{ij_2} & a_{ij_3} & a_{ij_0} & -a_{ij_1} & -a_{ij_6} & -a_{ij_7} & a_{ij_4} & a_{ij_5} \\ a_{ij_3} & -a_{ij_2} & a_{ij_1} & a_{ij_0} & -a_{ij_7} & a_{ij_6} & -a_{ij_5} & a_{ij_4} \\ a_{ij_4} & a_{ij_5} & a_{ij_6} & a_{ij_7} & a_{ij_0} & -a_{ij_1} & -a_{ij_2} & -a_{ij_3} \\ a_{ij_5} & -a_{ij_4} & a_{ij_7} & -a_{ij_6} & a_{ij_1} & a_{ij_0} & a_{ij_3} & -a_{ij_2} \\ a_{ij_6} & -a_{ij_7} & -a_{ij_4} & a_{ij_5} & a_{ij_2} & -a_{ij_3} & a_{ij_0} & a_{ij_1} \\ a_{ij_7} & a_{ij_6} & -a_{ij_5} & -a_{ij_4} & a_{ij_3} & a_{ij_2} & -a_{ij_1} & a_{ij_0} \end{pmatrix}. \tag{7}$$

In our first study, we begin with the matrix $\omega(A_{N\times N})$ and set $N = 2000$. Next, the independent variables a_{ii} are taken to be zero-centered Gaussian distributed with a variance of two and the variables a_{ij_k} ($i \neq j$, k = 0,1,...,7) are taken to be zero-centered Gaussian distributed with a variance of one. The eigenvalues, E_i (where $i = 1, 2, \dots, N$), of this matrix are then determined numerically and we can begin to make some observations. The first thing we note is that the eigenvalues are not eightfold degenerate, as was the case for 2×2 matrices, and that $8N$ distinct eigenvalues exist. (It should be noted that for $N = 3$, the eigenvalues were found to be fourfold degenerate, in agreement with Refs. [22, 25, 26]. However, when $N > 3$, we noted that $8N$ distinct eigenvalues were always present, in contrast to Ref. [22] where it was stated that the multiplicity [degeneracy] was two for those cases.)

To study details of the eigenvalues, we transform them as follows:

$$x_i = E_i \Big/ \sqrt{\beta' v^2 N}\,, \tag{8}$$

where v^2 is the variance of the variables a_{ij_k} ($i \neq j$) — in our case, $v^2 = 1$. Shown in Fig. 1, as a histogram, is the density (where the area has been normalized to unity) of the transformed eigenvalues with $\beta' = 8$ in Eq. (8). The solid curve is given by

$$\rho(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}\,, & |x| < 2\,, \\ 0\,, & |x| > 2\,, \end{cases} \tag{9}$$

and $\int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x = 1$. In classical RMT $\rho(x)$, which is known as the Wigner semicircle law, holds (in the large-N limit) for the GOE, GUE, and GSE,

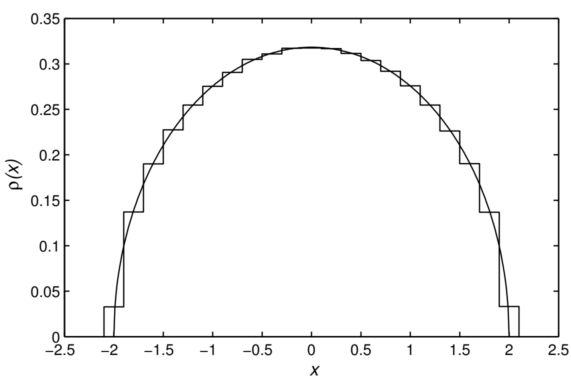

Fig. 1. Numerical study (histogram) of the eigenvalue density of a random matrix defined by $\omega(A_{N\times N})$, with $N = 2000$ (see the text for details). The solid curve is the Wigner semicircle law, *i.e.* Eq. (9), with $\beta' = 8$.

and in those cases, $\beta' = \beta = 1, 2$, or 4, respectively. What we have just shown, at least numerically, is that the Wigner semicircle law also holds for the random matrix given by $\omega(A_{N\times N})$, if $\beta' = 8$.

Next, we study the NNSD of the unfolded [1] eigenvalues, where $\rho(x)$ can be used to perform the unfolding. The result of this is given as the histogram in Fig. 2 — note that the area under the histogram has been normalized to be unity, as has the mean spacing. The solid curve represents $P_{\mathrm{W}}(S; \beta = 1)$, which is the Wigner surmise for the GOE. We have, therefore, shown that the NNSD of an $N \times N$ random matrix given by $\omega(A_{N\times N})$ is well-described by GOE statistics and hence octic repulsion is not present as it was in the 2×2 case.

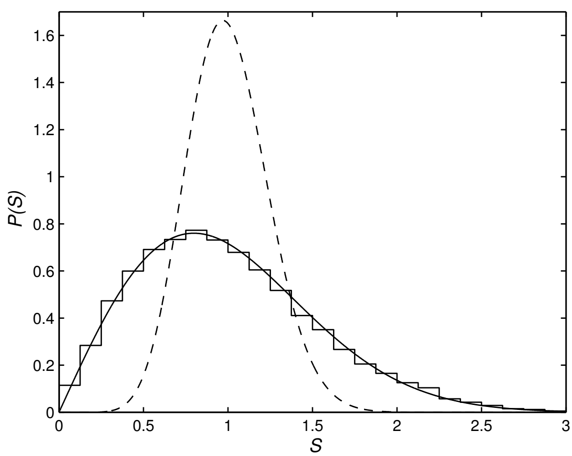

Fig. 2. Numerical study (histogram) of the NNSD of unfolded eigenvalues of a random matrix defined by $\omega(A_{N\times N})$, with $N = 2000$ (see the text for details). The solid curve is the Wigner surmise for the GOE, $P_{\mathrm{W}}(S; \beta = 1)$, and the dashed curve is $P_{\mathrm{W}}(S; \beta = 8)$.

To examine long-range properties of the unfolded eigenvalues of $\omega(A_{N\times N})$, we have chosen to calculate the number variance, $\Sigma^2(L)$, for a given interval length L. (Note that, when calculating $\Sigma^2(L)$, the intervals are allowed to overlap — the start position of each new interval in our study was arbitrarily taken to be 0.2 units from the start position of the previous interval.) For our set of unfolded eigenvalues, we get what is shown in Fig. 3. It is known that the number variance for the GUE, GOE, and GSE are (see, for example, [16] or [1]), respectively,

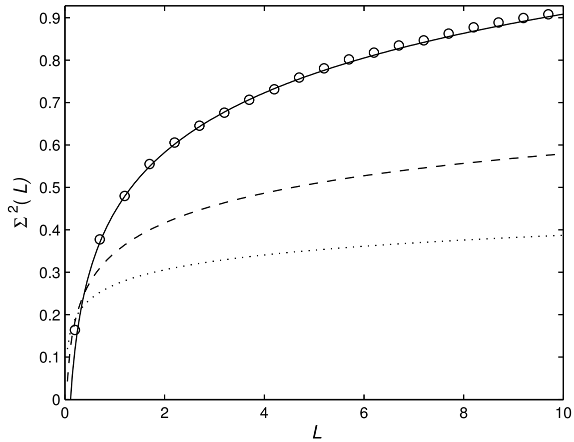

Fig. 3. Numerical study (open circles) of the number variance of unfolded eigenvalues of a random matrix defined by $\omega(A_{N\times N})$, with $N = 2000$ (see the text for details). The solid, dashed, and dotted curves correspond to the large-L analytical forms of $\Sigma^2(L)$ for the GOE, GUE, and GSE, respectively.

$$\begin{aligned}\Sigma^2_{\beta=2}(L) &= \frac{1}{\pi^2}\left[\ln(2\pi L)+\gamma+1-\cos(2\pi L)-\mathrm{Ci}(2\pi L)\right] \\ &\quad +L\left[1-\frac{2}{\pi}\mathrm{Si}(2\pi L)\right], \qquad (10)\\ \Sigma^2_{\beta=1}(L) &= 2\Sigma^2_{\beta=2}(L)+\left[\mathrm{Si}(\pi L)/\pi\right]^2-\mathrm{Si}(\pi L)/\pi\,, \qquad (11)\end{aligned}$$

and

$$\Sigma^2_{\beta=4}(L) = (1/2)\Sigma^2_{\beta=2}(2L)+\left[\mathrm{Si}(2\pi L)/2\pi\right]^2\,, \qquad (12)$$

where

$$\mathrm{Si}(x)=\int_0^x \frac{\sin y}{y}\,\mathrm{d}y \qquad \text{and} \qquad \mathrm{Ci}(x)=\gamma+\ln x+\int_0^x\frac{\cos y-1}{y}\,\mathrm{d}y\,, \qquad (13)$$

and $\gamma = 0.5772\ldots$ is Euler's constant. For large L, it is often convenient to work with the following forms [1] (which are also shown in Fig. 3, along with our numerical study)

$$\begin{aligned}\Sigma^2_{\beta=1}(L) &= \frac{2}{\pi^2}\left[\ln(2\pi L)+\gamma+1-\frac{\pi^2}{8}\right]+O\left(L^{-1}\right)\,, \qquad (14)\\ \Sigma^2_{\beta=2}(L) &= \frac{1}{\pi^2}\left[\ln(2\pi L)+\gamma+1\right]+O\left(L^{-1}\right)\,, \qquad (15)\end{aligned}$$

and

$$\Sigma^2_{\beta=4}(L) = \frac{1}{2\pi^2}\left[\ln(4\pi L) + \gamma + 1 + \frac{\pi^2}{8}\right] + O\left(L^{-1}\right) . \tag{16}$$

It is apparent from Fig. 3 that the number variance of the eigenvalues of $\omega(A_{N\times N})$, as currently defined, follows that of the GOE. Therefore, what we have shown is that a random matrix with elements taken to be pseudo-real matrix representations of octonions just follow GOE statistics and, so far, nothing very interesting has been learned.

In the hope of finding more interesting results, we now move on to our second study. Following concepts presented in Ref. [21], we next modify $\omega(a_{ij})$ $(i \neq j)$ to include eight parameters α_k (where $k = 0, 1, \ldots, 7$) as follows:

$$\omega(a_{ij};\alpha) = \begin{pmatrix} \alpha_0 a_{ij_0} & -\alpha_1 a_{ij_1} & \cdots & -\alpha_6 a_{ij_6} & -\alpha_7 a_{ij_7} \\ \alpha_1 a_{ij_1} & \alpha_0 a_{ij_0} & \cdots & \alpha_7 a_{ij_7} & -\alpha_6 a_{ij_6} \\ \alpha_2 a_{ij_2} & \alpha_3 a_{ij_3} & \cdots & \alpha_4 a_{ij_4} & \alpha_5 a_{ij_5} \\ \alpha_3 a_{ij_3} & -\alpha_2 a_{ij_2} & \cdots & -\alpha_5 a_{ij_5} & \alpha_4 a_{ij_4} \\ \alpha_4 a_{ij_4} & \alpha_5 a_{ij_5} & \cdots & -\alpha_2 a_{ij_2} & -\alpha_3 a_{ij_3} \\ \alpha_5 a_{ij_5} & -\alpha_4 a_{ij_4} & \cdots & \alpha_3 a_{ij_3} & -\alpha_2 a_{ij_2} \\ \alpha_6 a_{ij_6} & -\alpha_7 a_{ij_7} & \cdots & \alpha_0 a_{ij_0} & \alpha_1 a_{ij_1} \\ \alpha_7 a_{ij_7} & \alpha_6 a_{ij_6} & \cdots & -\alpha_1 a_{ij_1} & \alpha_0 a_{ij_0} \end{pmatrix} . \tag{17}$$

Here, α represents the list of parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)$ which will allow us to turn matrix elements OFF or ON by setting various α_i parameters to either 0 or 1, respectively. Then, $\omega(A_{N\times N})$ of Eq. (6) becomes the following matrix:

$$\omega(A_{N\times N};\alpha) = \begin{pmatrix} \omega(a_{11}) & \omega(a_{12};\alpha) & \cdots & \omega(a_{1N};\alpha) \\ \omega^T(a_{12};\alpha) & \omega(a_{22}) & \cdots & \omega(a_{2N};\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^T(a_{1N};\alpha) & \omega^T(a_{2N};\alpha) & \cdots & \omega(a_{NN}) \end{pmatrix} , \tag{18}$$

and we can begin to study its eigenvalue statistics. Given the binary choice of 0 or 1 for each of the α_i parameters, it is obvious that there are 256 different choices for α, and we shall study all of them numerically. Of course, if $\alpha = (1, 1, 1, 1, 1, 1, 1, 1)$, we return to our first study and if $\alpha = (0, 0, 0, 0, 0, 0, 0, 0)$, the eigenvalues of $\omega(A_{N\times N};\alpha)$ are trivially given by the values of $\omega(a_{ii})$ (where $i = 1, 2, \ldots, N$).

Let us begin with a summary of what we have observed and then move on to some specific examples. Shown in Table I are the results that were obtained for various binary values of α_k. Note that $\alpha_\#$ is defined to be the number of α_k-parameters (where $k = 1, 2, \ldots, 7$) that have been set to 1.

We observe that the eigenvalue statistics is dictated by $\alpha_{\#}$ — once this has been set, the value of α_0 does not change the resulting statistics. We further observe that the semicircle law holds for all choices of α and that the value of β' which gets used in Eqs. (8) and (9) is equal to the total number of α_k-parameters (where $k = 0, 1, \ldots, 7$) that have been set to 1; that is, $\beta' = \alpha_0 + \alpha_{\#}$. Also given in Table I are the degeneracies of the eigenvalues that we have observed numerically and the number of times that a given statistic was observed [which is equal to $\binom{7}{\alpha_{\#}}$].

TABLE I

Results for various ensembles of $\omega(A_{N\times N};\alpha)$ random matrices having binary α_k-parameters. $\alpha_{\#}$ is the number of α_k ($k = 1, 2, ..., 7$) that have been set to 1.

α_0	$\alpha_{\#}$	Eigenvalue statistics	Level repulsion ($S \to 0$)	Degeneracy of eigenvalues	Value of β' in Eq. (8)	Number of cases
0	0	—	—	8	—	1
1	0	GOE	linear	8	1	1
0	1	GUE	quadratic	8	1	7
1	1	GUE	quadratic	8	2	7
0	2	GSE	quartic	8	2	21
1	2	GSE	quartic	8	3	21
0	3	2 GSEs	—	4	3	35
1	3	2 GSEs	—	4	4	35
0	4	GSE	quartic	4	4	35
1	4	GSE	quartic	4	5	35
0	5	GUE	quadratic	2	5	21
1	5	GUE	quadratic	2	6	21
0	6	GOE	linear	1	6	7
1	6	GOE	linear	1	7	7
0	7	GOE	linear	1	7	1
1	7	GOE	linear	1	8	1

We will discuss three arbitrary cases in detail: $\alpha = (0, 1, 1, 0, 1, 1, 1, 0)$, $\alpha = (1, 0, 0, 0, 0, 1, 0, 1)$, and $\alpha = (0, 0, 0, 1, 1, 0, 1, 0)$. We will not show the eigenvalue densities for any of these cases since we have found that they all follow the semicircle law and look much like the density shown in Fig. 1. However, we remind the reader that the value of β' must be chosen appropriately in order for the semicircle law to hold (see Table I). Also, we emphasize that we have examined all 256 cases noted in Table I with the same amount of detail as is given below (although, in many cases, with smaller values of N).

Shown in Figs. 4, 6, and 8 are the NNSDs for $\alpha = (0,1,1,0,1,1,1,0)$, $\alpha = (1,0,0,0,0,1,0,1)$, and $\alpha = (0,0,0,1,1,0,1,0)$, respectively. Shown in Figs. 5, 7, and 9 are the number variances for the same choices of α. From these studies, it is obvious that the eigenvalues for the case where $\alpha = (0,1,1,0,1,1,1,0)$ follow GUE statistics and the eigenvalues for the case where $\alpha = (1,0,0,0,0,1,0,1)$ follow GSE statistics.

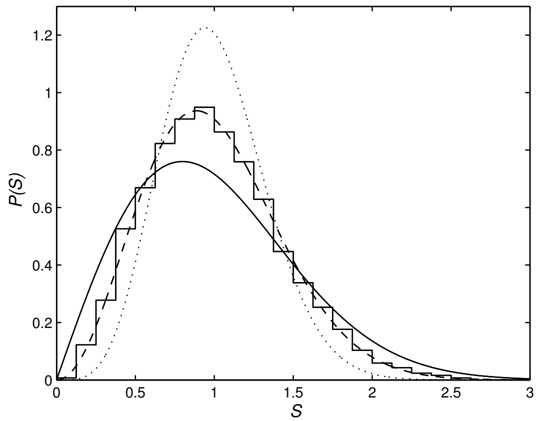

Fig. 4. Numerical study (histogram) of the NNSD for unfolded eigenvalues of a random matrix defined by $\omega(A_{N\times N};\alpha)$, with $N = 2000$ and $\alpha = (0,1,1,0,1,1,1,0)$. The solid, dashed, and dotted curves correspond to the Wigner surmise for the GOE, GUE, and GSE, respectively.

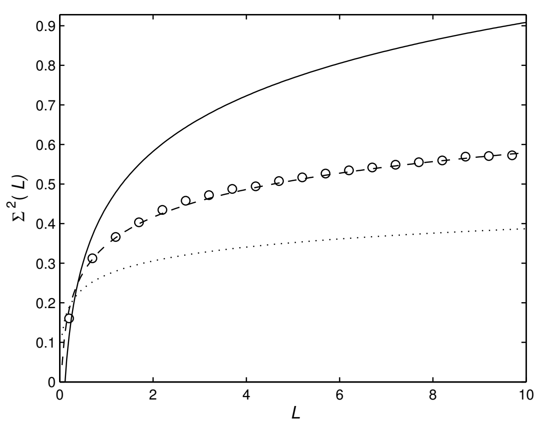

Fig. 5. Numerical study (open circles) of the number variance for unfolded eigenvalues of a random matrix defined by $\omega(A_{N\times N};\alpha)$, with $N = 2000$ and $\alpha = (0,1,1,0,1,1,1,0)$. The solid, dashed, and dotted curves correspond to the large-L analytical forms of $\Sigma^2(L)$ for the GOE, GUE, and GSE, respectively.

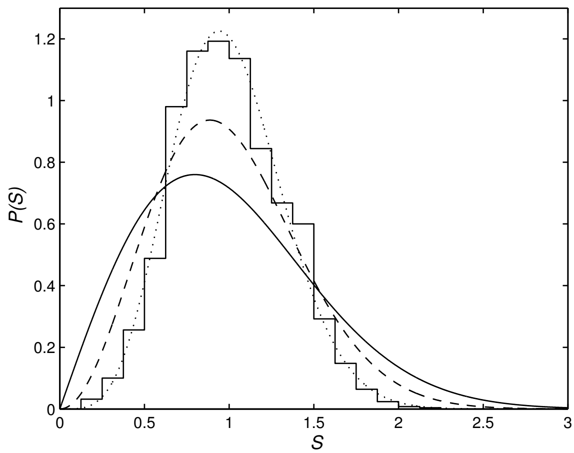

Fig. 6. The same as in Fig. 4 except with $\alpha = (1, 0, 0, 0, 0, 1, 0, 1)$.

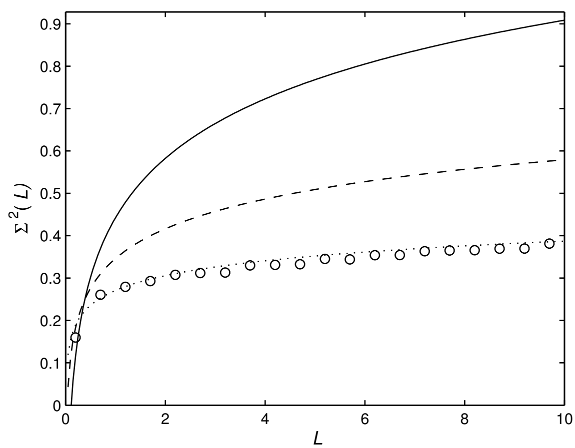

Fig. 7. The same as in Fig. 5 except with $\alpha = (1, 0, 0, 0, 0, 1, 0, 1)$.

However, the eigenvalues of the case where $\alpha = (0, 0, 0, 1, 1, 0, 1, 0)$ clearly do not follow statistics of any of the classical RMT ensembles and we must, therefore, describe them otherwise. We initially guess that the statistics are described by some superposition of two independent eigenvalue spectra each having either GOE, GUE, or GSE statistics — our intuition is guided by the appearance of the NNSD (*i.e.* the histogram in Fig. 8) for which level repulsion is now absent and $P(S)$ is close to $1/2$ as $S \to 0$. More precisely, if we let f_i represent the fraction of levels belonging to the i^{th} component spectrum, then the NNSD for a superposition of independent spectra is [1]

$$P(S) = E(S)\left\{ \sum_i f_i^2 \frac{P_i(f_i S)}{E_i(f_i S)} + \left[\sum_i f_i \frac{1-\Psi_i(f_i S)}{E_i(f_i S)}\right]^2 - \sum_i \left[f_i \frac{1-\Psi_i(f_i S)}{E_i(f_i S)}\right]^2 \right\}, \tag{19}$$

where

$$\Psi_i(S) = \int_0^S P_i(x)\,\mathrm{d}x\,, \tag{20}$$

$$E_i(S) = \int_S^\infty [1-\Psi_i(x)]\,\mathrm{d}x\,, \tag{21}$$

and

$$E(S) = \prod_i E_i(f_i S)\,. \tag{22}$$

Note that $\sum_i f_i = 1$. Based on our numerical experiments, we conjecture that the eigenvalue statistics for the case where $\alpha = (0,0,0,1,1,0,1,0)$ (or more generally, any time that $\alpha_\# = 3$; see Table I) are described by the superposition of two GSEs with $f_1 = f_2 = 1/2$. The resulting $P(S)$ is shown as the solid curve in Fig. 8. To further test our conjecture, we study the number variance for a superposition of i independent spectra, which is given by

$$\Sigma^2(L) = \sum_i \Sigma^2_{\beta,i}(f_i L)\,. \tag{23}$$

In our present case, we take $\Sigma^2_{\beta,1}(f_1 L)$ and $\Sigma^2_{\beta,2}(f_2 L)$ to both be given by the large-L analytical form of $\Sigma^2_{\beta=4}(L)$. The result is shown as the solid curve in Fig. 9. A more refined calculation is shown in Fig. 10, where we use Eq. (12) for the individual number variances. Note how we now properly describe the oscillations in the number variance when $\alpha = (0,0,0,1,1,0,1,0)$.

To summarize, we have numerically studied the eigenvalue statistics of real adjoints of $N \times N$ Hermitian octonion random matrices. A list of parameters, denoted by α, was used to control the various matrix elements by turning them OFF or ON. It was shown that when $\alpha = (1,1,1,1,1,1,1,1)$, GOE statistics resulted and the NNSD exhibited linear level repulsion — this is in contrast to 2×2 random matrices where it is known that the eigenvalues exhibit octic level repulsion [21]. Next, we examined the eigenvalues for

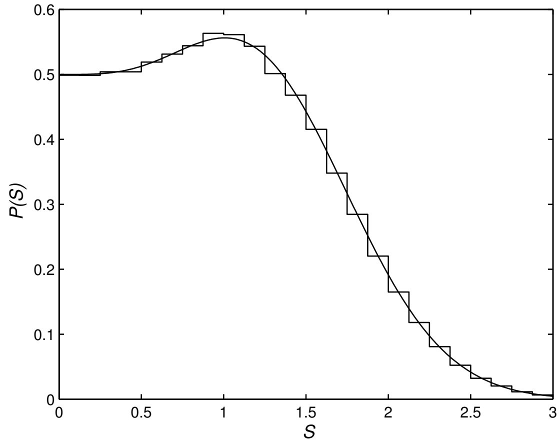

Fig. 8. The same as in Fig. 4 except with $\alpha = (0,0,0,1,1,0,1,0)$. The solid curve now represents $P(S)$ as given in Eq. (19) with $P_1(f_1S)$ and $P_2(f_2S)$ taken to be $P_{\mathrm{W}}(S;\beta=4)$ and $f_1 = f_2 = 1/2$. Note that the numerical results are now for an ensemble average of 40 trials, with $N = 2000$ for each trial.

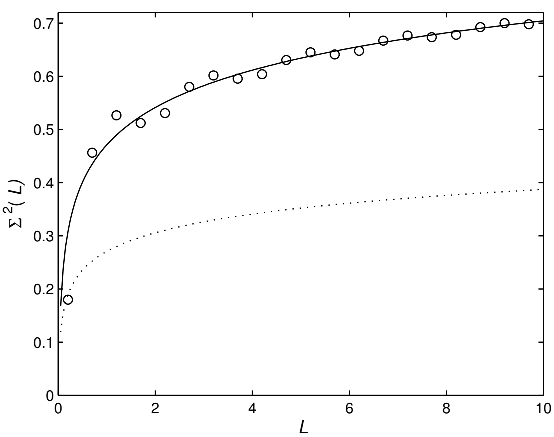

Fig. 9. The same as in Fig. 5 except with $\alpha = (0,0,0,1,1,0,1,0)$. The solid curve now represents the number variance given in Eq. (23) with $\Sigma^2_{\beta,1}(f_1L)$ and $\Sigma^2_{\beta,2}(f_2L)$ taken to be the large-L analytical form of $\Sigma^2_{\beta=4}(L)$ and $f_1 = f_2 = 1/2$. The dotted curve still corresponds to the large-L analytical form of $\Sigma^2_{\beta=4}(L)$. Note that the numerical results are now for an ensemble average of 40 trials, with $N = 2000$ for each trial. The error in the mean for each L is smaller than the size of the open circles.

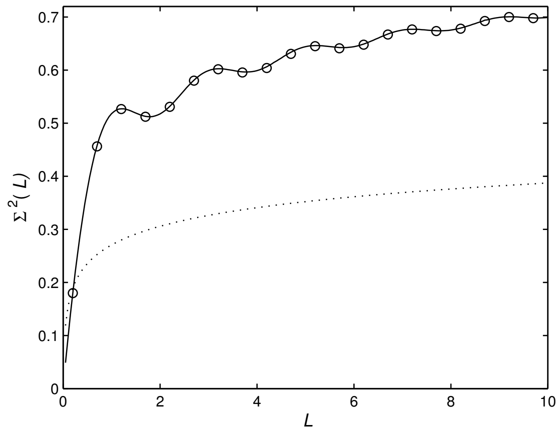

Fig. 10. The same as in Fig. 9 except that we are now using the $\Sigma^2_{\beta=4}(L)$ expression given in Eq. (12) for $\Sigma^2_{\beta,1}(f_1 L)$ and $\Sigma^2_{\beta,2}(f_2 L)$. The result, when $f_1 = f_2 = 1/2$, is shown as the solid curve.

all 256 binary choices of α and discovered that the resulting statistics were described by the GOE, GUE, or GSE, depending on the value of $\alpha_{\#}$ (see Table I). For cases where $\alpha_{\#} = 3$, we observed that none of the Gaussian ensembles of classical RMT could describe the resulting eigenvalue statistics and conjectured that they are, in fact, described by the superposition of two independent GSE spectra — we do not yet understand why this should be the case. The observations made in this paper are based purely on numerical studies; it would be highly satisfying if one could prove these results analytically.

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