THERMODYNAMIC EQUILIBRIUM IN RELATIVITY: FOUR-TEMPERATURE, KILLING VECTORS AND LIE DERIVATIVES

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Dedicated to Andrzej Bialas in honour of his 80th birthday

The main concepts of general relativistic thermodynamics and general relativistic statistical mechanics are reviewed. The main building block of the proper relativistic extension of the classical thermodynamics laws is the four-temperature vector β, which plays a major role in the quantum framework and defines a very convenient hydrodynamic frame. The general relativistic thermodynamic equilibrium condition demands β to be a Killing vector field. We show that a remarkable consequence is that all Lie derivatives of all physical observables along the four-temperature flow must then vanish.

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1. Introduction

Relativistic thermodynamics and relativistic statistical mechanics are nowadays widespreadly used in advanced research topics: high-energy astrophysics, cosmology, and relativistic nuclear collisions. The standard cosmological model views the primordial Universe as a curved manifold with matter content at (local) thermodynamic equilibrium. Similarly, the matter produced in high-energy nuclear collisions is assumed to reach and maintain local thermodynamic equilibrium for a large fraction of its lifetime.

In view of these modern and fascinating applications, it seems natural and timely to review the foundational concepts of thermodynamic equilibrium in a general relativistic framework, including — as much as possible — its quantum and relativistic quantum field features. I will then address the key physical quantity in describing thermodynamic equilibrium in relativity: the inverse temperature or four-temperature vector β. I will show how this can be taken as a primordial vector field defined on the sole basis of thermodynamic equilibrium and ideal thermometers, and its outstanding geometrical features in curved spacetimes.

2. Entropy in relativity

The extension of the classical laws of thermodynamics to special relativity raised the attention of Einstein and Planck themselves [1, 2]. Their viewpoint is still the generally accepted one, with a later alternative approach put forward in the 60s [3, 4], which will be discussed in Sect. 3.

The first question, when trying to extend classical thermodynamics to relativity, is how to deal with entropy. More specifically, should entropy be considered as a scalar or the time component of some four-vector, like energy? The non-controversial answer is that total entropy should be taken as a relativistic scalar, for various well-founded reasons. Here, a general relativistic argument based on the second law of thermodynamics, that is total entropy of the universe must increase in all physical processes. As in some finite portion of the spacetime entropy may decrease, as it is borne out by our daily experience, the only sensible choice for an extensive non-decreasing quantity is the result of an integration. Since in general relativity an integral can only be a scalar to be generally covariant, entropy must then be a scalar. Examples of integral scalars are well-known: the action, which is the integral over a finite four-dimensional region of spacetime of the Lagrangian density

$$A = \int_{\Omega} \mathrm{d}^4x \, \sqrt{-g}\mathcal{L} \, ;$$

the total electric charge, which is the integral over a 3D spacelike hypersurface Σ of a conserved current

$$Q = \int_{\Sigma} \mathrm{d}\Sigma \, n_\mu j^\mu \, ,$$

where n is the (timelike) normal unit vector to Σ and $\mathrm{d}\Sigma$ its measure. Similarly, total entropy should result from the integration over a 3D spacelike hypersurface of an entropy current s^μ,

$$S = \int_{\Sigma} \mathrm{d}\Sigma \, n_\mu s^\mu \, . \tag{1}$$

Even if this approach is apparently the most reasonable relativistic extension, it should be pointed out that the total entropy (1) is meaningful only if entropy current is conserved, *i.e.* if $\nabla_\mu s^\mu = 0$, which applies only at global thermodynamic equilibrium. In a non-equilibrium situation,

$$\nabla_\mu s^\mu \geq 0$$

and the total entropy will depend on the particular hypersurface Σ chosen. Otherwise stated, in non-equilibrium, the total entropy is an observer-dependent quantity as two inertial observers moving at different speed have two different simultaneity three-spaces. Only if $\nabla_\mu s^\mu = 0$, because of the Gauss' theorem, the total integral S in Eq. (1) is independent of Σ provided that the entropy flux at some timelike boundary vanishes. If Σ is a hypersurface at some constant time, however the time is defined, this also implies that total entropy will be time-independent: precisely our familiar classical definition of equilibrium.

3. Temperature and thermometers in relativity

The first physical quantity encountered in thermodynamics textbooks is temperature. It is then natural to wonder how relativity affects the classical temperature notion. There has been a long-standing debate about the way temperature changes with respect to Lorentz transformations (see *e.g.* [5] for recent summary). The debate stemmed from the possible ambiguity in the extension of the well-known thermodynamic relation (at constant volume)

$$T\mathrm{d}S = \mathrm{d}U\,. \tag{2}$$

If this is seen as a scalar relation, one would most likely conclude, like Einstein and Planck [1, 2], that $\mathrm{d}U/T$ must be generalized to be a scalar product of four-vectors $\beta = (1/T)u$ (see later on) and $\mathrm{d}P$, u being the four-velocity of the observer and P the four-momentum

$$\mathrm{d}S = \frac{1}{T} u \cdot \mathrm{d}P\,. \tag{3}$$

Conversely [3, 4], if relation (2) is seen as the time component of a four-vectorial relation, with $\mathrm{d}U = \mathrm{d}P^0$, then one would accordingly conclude that T is the time component of a four-vector $T^\mu = Tu^\mu$ and

$$Tu^\mu \mathrm{d}S = \mathrm{d}P^\mu\,. \tag{4}$$

These two different extensions of the classical thermodynamic relation involve two converse answers to a relevant physical question: what does a moving thermometer — with respect to the system which is in thermal contact with — measure? Or, tantamount, what does a thermometer at rest measure if it is put in thermal contact with a moving system with four-velocity u? It should be stressed that here by thermometer we mean an idealized gauge with zero mass, pointlike and capable of reaching equilibrium instantaneously (zero relaxation time) with the system which is in contact with. In the first option, the temperature measured by a thermometer at rest in a moving system is *smaller* by a factor γ, in the latter case is *larger* by a factor $1/\gamma$, where γ is the Lorentz contraction factor. To see how this comes about, we have to keep in mind that a thermometer which is kept at rest, by definition, can achieve equilibrium with respect to *energy* exchange with the system in thermal contact with it, and not with momentum. In other words, the energies — that is the time components of the four-momentum — of the thermometer and the system will be shared (interaction energy is neglected) so as to maximize entropy, thus,

$$\frac{\partial S}{\partial E}\Big|_{\rm T} = \frac{\partial S}{\partial E}\Big|_{\rm S}\,, \tag{5}$$

where S stands for system and T for thermometer. The left-hand side, in the rest frame of the thermometer, must be $1/T_{\rm T}$, *i.e.* the inverse temperature marked by its gauge, while the right-hand side is either γ/T in the Einstein–Planck option (3) or $1/\gamma T$ in the alternative option (4).

Without delving the controversy in depth, my viewpoint is that the Einstein and Planck's — hence the most widely accepted in the past [6] as well as today [7, 8] — relativistic extension of the temperature concept is the correct one. If entropy is a Lorentz scalar, it must be a function of the invariant mass, that is $S = S(\sqrt{E^2 - \boldsymbol{P}^2}\,)$. Hence,

$$\frac{\partial S}{\partial P^\mu} = \frac{\partial S}{\partial M}\frac{\partial}{\partial P^\mu}\sqrt{E^2 - \boldsymbol{P}^2} = \frac{\partial S}{\partial M}\frac{P_\mu}{M} = \frac{\partial S}{\partial M}u_\mu\,.$$

The derivative of the entropy with respect to the mass of the system, that is its rest energy, can be properly seen as the proper temperature, the one which would be measured by a thermometer at rest with the system, hence the above relation reads

$$\frac{\partial S}{\partial P^\mu} = \frac{1}{T}u_\mu \equiv \beta_\mu\,,$$

where we have introduced the inverse temperature four-vector, or simply, the four-temperature β, see Introduction. Hence, the entropy differential can be written as

$$\mathrm{d}S = \frac{\partial S}{\partial P^\mu}\mathrm{d}P^\mu = \beta_\mu \mathrm{d}P^\mu = \beta\cdot\mathrm{d}P$$

which is apparently the Einstein–Planck extension (3). Instead, the alternative option suffers from a serious difficulty: to make sense of a differential relation (4), the four-momentum vector of a relativistic thermodynamic system must be a function of a scalar, the entropy. This is clearly counterintuitive and against any classical definition and experimental evidence, as entropy has to do only with the internal state of a system and should be independent of its collective motion. Therefore, the alternative by Ott and followers should be refused.

4. Four-temperature $\boldsymbol{\beta}$ and the $\boldsymbol{\beta}$ frame

The four-temperature β is then the correct relativistic extension of the temperature notion. The four-temperature vector is ubiquitous in all relativistic thermodynamic formulae, such as the well-known Jüttner or Cooper–Frye distribution function

$$f(x,p) = \frac{1}{e^{\beta\cdot p} \pm 1}\,.$$

Yet, β is usually viewed as a secondary quantity obtained from previously defined temperature and an otherwise defined velocity u, with $\beta = (1/T)u$. In this section, we will overturn this view.

One can make the definition of four-temperature operational, like in classical thermodynamics for the temperature, by defining an ideal "relativistic thermometer" as an object able to instantaneously achieve equilibrium with respect to energy *and momentum* exchange. This implies that an ideal relativistic thermometer will istantaneously move at the same velocity as the system which is in contact with, besides marking its temperature, *i.e.* it will tell the β vector in each spacetime point

$$\left.\frac{\partial S}{\partial P^\mu}\right|_{\mathrm{T}} = \left.\frac{\partial S}{\partial P^\mu}\right|_{\mathrm{S}} \implies \beta^\mu_{\mathrm{T}} = \beta^\mu_{\mathrm{S}}\,.$$

Alternatively, one can retain the more traditional definition of thermometer, with an externally imposed four-velocity u_{T}. In the latter case, going to the thermometer rest frame, one has, from equation (5), the equality of the time components of the β vectors in that frame

$$\beta^0 = \beta^0_{\mathrm{T}}$$

or

$$\beta \cdot u_{\mathrm{T}} = \frac{1}{T_{\mathrm{T}}}\,.$$

Hence, a thermometer moving with four-velocity $u_{\rm T}$ in a system in local thermodynamical equilibrium, characterized by a four-vector field β, will mark a temperature

$$T_{\rm T} = \frac{1}{\beta(x) \cdot v} \,. \tag{6}$$

As the scalar product of two timelike unit vectors $u \cdot v \geq 1$ and

$$u \cdot v = 1 \qquad \text{iff} \qquad u = v \,,$$

one has, according to (6),

$$T_{\rm T} \leq T = \frac{1}{\sqrt{\beta^2}} \,, \qquad T_{\rm T} = T \qquad \text{iff} \qquad u = u_{\rm T}$$

that is, the temperature marked by an idealized thermometer is maximal if it moves with the same four-velocity of the (fluid) system. Thus,

$$T = 1/\sqrt{\beta^2}$$

is the comoving, or proper, temperature.

Thereby, we can establish a thought operational procedure to *define* a four-velocity, that is a frame, for a fluid based on the notion of local thermodynamical equilibrium at some spacetime point x:

— put (infinitely many) ideal thermometers in contact with the relativistic system at the spacetime point x, each with a different four-velocity $u_{\rm T}$;

— the ideal thermometer marking the *highest* temperature value T moves, by construction, with the four-velocity $u(x) = T\beta(x) = 1/\sqrt{\beta^2}\beta(x)$.

We can thus define a four-velocity of a fluid just by using an ideal thermometer. This makes the four-vector β a more fundamental quantity than the fluid velocity. We defined this frame as β frame [9] to distinguish it from the tradtional Landau and Eckart frames, from which it differs even in general global equilibrium states, as we explicitely showed in Ref. [10]. The β frame has many nice features and it is very convenient in general relativity, especially for quantum statistical mechanics, because at equilibrium it has a crucial feature: it is a *Killing vector field* as we will see in the next section.

5. Quantum relativistic statistical mechanics at equilibrium

In thermal quantum field theory, the usual task is to calculate mean values of physical quantities at thermodynamic equilibrium with an equilibrium density operator, whose familiar form is

$$\widehat{\rho} = (1/Z) \exp\left[-\widehat{H}/T_0 + \mu_0 \widehat{Q}/T_0\right] \,, \tag{7}$$

where T_0 is the temperature and μ_0 the chemical potential (the reason for the 0 superscript will become clear soon) coupled to a conserved charge $\widehat{Q}$, and Z the partition function. The above density operator can be obtained by maximizing the total entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with respect to $\widehat{\rho}$ with the constraints of fixed total *mean* energy and fixed total *mean* charge. If a further constraint of fixed *mean* momentum vector is included, the density operator becomes manifestly covariant

$$\widehat{\rho} = (1/Z) \exp\left[-\beta \cdot \widehat{P} + \mu_0 \widehat{Q}/T_0\right] , \tag{8}$$

where $\widehat{P}$ is the four-momentum operator and β is a four-vector Lagrange multiplier for energy and momentum. Form (8) is thus the covariant form of (7), which is a special case when $\beta = (1/T_0, \mathbf{0})$.

However, the density operator (8), is not the only form of global thermodynamic equilibrium, as one can add more constraints. For instance, one can include the angular momentum and obtain [11, 12]

$$\widehat{\rho} = (1/Z) \exp\left[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0 + \mu_0 \widehat{Q}/T_0\right] , \tag{9}$$

where $\widehat{J}_z$ is the angular momentum operator along some axis z, which represents a globally equilibrated spinning fluid with angular velocity ω.

The above (8) and (9) are, indeed, special cases of the most general thermodynamic equilibrium density operator, which can be obtained by maximizing the total entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with the constraints of given mean energy-momentum and charge *densities* at some specific time over some spacelike hypersurface Σ [9, 13, 14]. Therefore, the general equilibrium density operator can be written in a fully covariant form as [13, 15, 16]

$$\widehat{\rho} = (1/Z) \exp\left[-\int_{\Sigma} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta \widehat{j}^\mu\right)\right] , \tag{10}$$

where $\widehat{T}^{\mu\nu}$ is the stress-energy tensor operator, $\widehat{j}^\mu$ a conserved current and ζ is a scalar whose meaning is the ratio between comoving chemical potential and comoving temperature. The four-vector field β can be seen as a field of Lagrange multipliers and no longer needs to be constant and uniform at equilibrium.

Indeed, for the right-hand side of Eq. (10) to be a true, global equilibrium distribution, the integral must be independent of the particular Σ, which also means independent of time if Σ is chosen to be $t = \mathrm{const}$, as it was pointed out in Sect. 2. Provided that the flux at some timelike boundary vanishes, this condition requires the divergence of the vector field in the integrand to be zero. If the stress-energy tensor $\widehat{T}$ and the current $\widehat{j}$ are covariantly conserved, this requires ζ to be a constant scalar field and β a Killing vector field, that is fulfilling the equation

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \,. \tag{11}$$

This condition for thermodynamic equilibrium has been known for a long time (see *e.g.* [17] for a kinetic derivation and [18] for the above one). The density operator (10) is well-suited to describe thermodynamic equilibrium in a general curved spacetime possessing a timelike Killing vector field. It should be pointed out that extending the building blocks of quantum mechanics, that is operators and Hilbert spaces, to curved spacetimes, features several major difficulties, which can be partly circumvented by using the path integral formalism [19]. Thus, making full sense of expressions such as (10) in curved spacetimes may not be trivial and it has been the subject of long discussion and research which certainly goes beyond the scope of this work. Nevertheless, one can keep on using the operator formalism in an abstract algebraic sense, with the understood convention that traces are to be calculated by path integrals, so that conclusion (11) holds.

In Minkowski spacetime, the general solution of Eq. (11) is known

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu \,, \tag{12}$$

where b is a constant four-vector and ϖ a constant antisymmetric tensor, which, because of Eq. (12) can be written as an exterior derivative of the β field

$$\varpi_{\nu\mu} = -\tfrac{1}{2} \left(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu \right) \tag{13}$$

defined as *thermal vorticity*. Hence, by using Eq. (12), the integral in Eq. (10) can be rewritten as

$$\int_\Sigma \mathrm{d}\Sigma_\mu \, \widehat{T}^{\mu\nu} \beta_\nu = b_\mu \widehat{P}^\mu - \tfrac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} \tag{14}$$

and the density operator (10) as

$$\widehat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right] , \tag{15}$$

where the $\widehat{J}$s are the generators of the Lorentz transformations

$$\widehat{J}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda \, \left(x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) .$$

Therefore, besides the chemical potentials, the most general equilibrium density operator in Minkowski spacetime can be written as a linear combinations of the 10 generators of its maximal continuous symmetry group, the orthocronous Poincaré group with 10 constant coefficients. The density operator (15) can also be obtained by maximizing the entropy with the constraints of given *mean* values of all the generators of the Poincaré group, namely energy, momenta, angular momenta and boosts, with b and ϖ being the corresponding Lagrange multipliers. Note that, even if they do not all commute with each other, their mean values are actually constant and the constrained maximum problem can be solved and has precisely the solution (15) [20].

It can be readily seen that the familiar density operator (7) is obtained by setting $b = \frac{1}{T_0}(1,0,0,0)$ and $\varpi = 0$, what we define as *homogeneous thermodynamic equilibrium*. The rotating global equilibrium in Eq. (9) can be obtained as a special case of Eq. (15) by setting

$$b_\mu = (1/T_0, 0, 0, 0)\,, \qquad \varpi_{\mu\nu} = (\omega/T_0)\,(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu}) \tag{16}$$

i.e. by imposing that the antisymmetric tensor ϖ has just a "magnetic" part; thereby, ω gets the physical meaning of a costant angular velocity [11]. In fact, there is a third, not generally known, form which is conceptually independent of the above two, which can be obtained by imposing that ϖ has just an "electric" (or longitudinal) part, *i.e.*

$$b_\mu = (1/T_0, 0, 0, 0)\,, \qquad \varpi_{\mu\nu} = (a/T_0)\,(g_{0\mu}g_{3\nu} - g_{3\mu}g_{0\nu})\,. \tag{17}$$

The resulting density operator is

$$\widehat{\rho} = (1/Z)\exp\left[-\widehat{H}/T_0 + a\widehat{K}_z/T_0\right]\,, \tag{18}$$

$\widehat{K}_z$ being the generator of a Lorentz boost along the z axis. This represents a relativistic fluid with constant comoving acceleration along the z direction. Note that the operators $\widehat{H}$ and $\widehat{K}_z$ are both conserved and yet, unlike in the rotation case (9), they *do not* commute with each other. This makes the density operator (18) a very peculiar kind of thermodynamic equilibrium [21].

6. Killing vectors and Lie derivatives

In this section, I will prove a general property of any physical observable in general thermodynamic equilibrium:

The Lie derivative of any physical observable X along the four-temperature vector β vanishes at thermodynamic equilibrium.

This statement makes it clear what thermodynamic equilibrium physically implies for an observer moving along a Killing vector field in a general spacetime, as it will be discussed in Sect. 7.

A physical observable X in quantum statistical mechanics is always defined as the mean value of a corresponding quantum operator, which can be either local or resulting from an integration

$$X = \mathrm{tr}\left(\widehat{\rho}\widehat{X}(x)\right) .$$

With a density operator given by (10), the mean value will depend on the four-temperature field, the metric and ζ, in a functional sense

$$X = X[\beta, \zeta, g]$$

because so does the density operator $\widehat{\rho}$. This is the most general dependence that X can have upon the data, *i.e.* the background metric and the thermodynamic fields β and ζ (in fact, the only non-trivial dependence will be on β and g as ζ is constant at thermodynamic equilibrium). Expanding the functional dependence, a local mean value X will then depend, in general, on the derivatives of all orders of both β and g calculated in x. Indeed, all the derivatives at some point are what we need to know the supposedly analytic functions β and g in any other spacetime point. Furthermore, because of general covariance, we can choose an inertial set of coordinates in x so that the first derivatives of the metric vanish, and all derivatives in x of β and g at all orders in x can be expressed as combinations of covariant derivatives of any order of β and the Riemann tensor. In symbols

$$X(x) = X[\beta, \zeta, g] = X(\beta, \nabla\beta, \nabla\nabla\beta, \ldots, g, R, \nabla R, \nabla\nabla R, \ldots) . \tag{19}$$

Indeed, β being a Killing vector, it is known that its second covariant derivative can be expressed as

$$\nabla_\mu \nabla_\nu \beta_\lambda = R^\rho_{\ \mu\nu\lambda} \beta_\rho \tag{20}$$

so that, effectively, the dependence on the four-temperature field at equilibrium is just on the field and its covariant derivative. Therefore, Eq. (19) can be rewritten as

$$X(x) = X[\beta, \zeta, g] = X(\beta, \nabla\beta, g, R, \nabla R, \nabla\nabla R, \ldots) . \tag{21}$$

Altogether, X can be seen as an analytic function of infinitely many arguments and expanded in them. In general, the tensorial rank of X determines how the arguments can appear in its expansion: for instance, if X is a scalar, it will be expressed as all possible scalar combinations of the arguments with scalar coefficients depending on β^2, *e.g.*

$$c_1\left(\beta^2\right)R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}+c_2\left(\beta^2\right)R+c_3\left(\beta^2\right)R^{\mu\nu}R_{\mu\nu}+c_4\left(\beta^2\right)\nabla_\mu\beta_\nu\nabla^\mu\beta^\nu+\ldots,$$

where we have used the Ricci tensor and the curvature scalar.

A simple example of relation (21) is the well-known mean value of the stress energy at the homogeneous equilibrium in the Minkowski spacetime with constant $\beta=b$ with $\varpi=0$ (see Section 5)

$$T^{\mu\nu}(x)=\frac{h\left(\beta^2\right)}{\beta^2}\beta^\mu\beta^\nu+p\left(\beta^2\right)g^{\mu\nu},$$

where h is the enthalpy density and p the pressure, which are both functions of β^2, *i.e.* the proper temperature. In curved spacetimes or in general equilibria in flat spacetime defined by Eq. (12), there can be much more than the ideal form. Indeed, an expansion of the general relation (21) for the stress-energy tensor was envisaged in Refs. [22, 23] which was further studied and developed in several papers, *e.g.* Refs. [24–26] with path integral methods; the coefficients of the expansion have been calculated in some relevant cases [10, 27].

Therefore, in order to prove the statement at the beginning of this section, we just need to show that any argument of X in Eq. (19) has a vanishing Lie derivative along β. For β, this is trivial, for g, it is true by definition of Killing vector, *i.e.* Eq. (11) itself. To proceed and show that this holds for any other argument, we need first to prove the following:

Proposition. *For any vector field V, the Lie derivative along a Killing field β commutes with the covariant derivative, that is $\mathcal{L}_\beta(\nabla V)=\nabla\mathcal{L}_\beta(V)$.*

To show this, we expand the Lie derivative definition

$$\mathcal{L}_\beta(\nabla_\mu V_\nu)=\beta^\lambda\nabla_\lambda\nabla_\mu V_\nu+\nabla_\mu\beta^\lambda\nabla_\lambda V_\nu+\nabla_\nu\beta^\lambda\nabla_\mu V_\lambda\,. \tag{22}$$

Now, we use the commutator of two covariant derivatives

$$\nabla_\lambda\nabla_\mu V_\nu-\nabla_\mu\nabla_\lambda V_\nu=R^\rho_{\ \nu\mu\lambda}V_\rho \tag{23}$$

for the first term on the right-hand side of (22), and the Leibniz rule for the covariant derivative of the other two terms. Hence,

$$\begin{aligned}\mathcal{L}_\beta(\nabla_\mu V_\nu) &= \beta^\lambda\nabla_\mu\nabla_\lambda V_\nu+\beta^\lambda R^\rho_{\ \nu\mu\lambda}V_\rho+\nabla_\mu\left(\beta^\lambda\nabla_\lambda V_\nu\right)\\ &\quad -\beta^\lambda\nabla_\mu\nabla_\lambda V_\nu+\nabla_\mu\left(\nabla_\nu\beta^\lambda V_\lambda\right)-\nabla_\mu\nabla_\nu\beta^\lambda V_\lambda\\ &= \beta^\lambda R^\rho_{\ \nu\mu\lambda}V_\rho-\nabla_\mu\nabla_\nu\beta^\lambda V_\lambda+\nabla_\mu\mathcal{L}_\beta(V_\nu)\,,\end{aligned} \tag{24}$$

where we have again used the Lie derivative definition, for a vector field. Now, we can use Eq. (20), so that the first two terms on the right-hand side of Eq. (24) cancel

$$\begin{aligned}\beta^\lambda R^\rho{}_{\nu\mu\lambda}V_\rho - \nabla_\mu\nabla_\nu\beta^\lambda V_\lambda &= R_{\rho\nu\mu\lambda}V^\rho\beta^\lambda - R_{\rho\mu\nu\lambda}\beta^\rho V^\lambda \\ &= \left(R_{\lambda\nu\mu\rho} - R_{\rho\mu\nu\lambda}\right)V^\lambda\beta^\rho = \left(R_{\mu\rho\lambda\nu} - R_{\rho\mu\nu\lambda}\right)V^\lambda\beta^\rho \\ &= \left(R_{\rho\mu\nu\lambda} - R_{\rho\mu\nu\lambda}\right)V^\lambda\beta^\rho = 0\,,\end{aligned}$$

where we have used the symmetry properties of the Riemann tensor indices. Thus, Eq. (24) yields the sought relation

$$\mathcal{L}_\beta(\nabla_\mu V_\nu) = \nabla_\mu \mathcal{L}_\beta(V_\nu)$$

and this concludes the proof.

By using the Leibniz rule for the covariant derivative of a tensor product, it is straightforward to extend the above relation to the Lie derivative of any tensor field T, that is

$$\mathcal{L}_\beta(\nabla T) = \nabla \mathcal{L}_\beta(T)\,. \tag{25}$$

A straightforward consequence of the above relation is that $\mathcal{L}_\beta(\nabla\beta) = 0$ being $\mathcal{L}_\beta(\beta) = 0$.

The last step to prove the initial statement is to show that the Riemann tensor has vanishing Lie derivative along β, that is

$$\mathcal{L}_\beta(R) = 0 \tag{26}$$

which implies, in view of (25) that all Lie derivatives of $\nabla R, \nabla\nabla R, \ldots$ vanish. Let us take an arbitrary vector field V and write the Lie derivative of (23)

$$\mathcal{L}_\beta\left(\nabla_\lambda\nabla_\mu V_\nu - \nabla_\mu\nabla_\lambda V_\nu\right) = \mathcal{L}_\beta\left(R^\rho{}_{\nu\mu\lambda}V_\rho\right)\,.$$

By using Leibniz rule and (25), we get

$$\left(\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda\right)\mathcal{L}_\beta(V)_\nu = \mathcal{L}_\beta\left(R^\rho{}_{\nu\mu\lambda}\right)V_\rho + R^\rho{}_{\nu\mu\lambda}\mathcal{L}_\beta(V_\rho)\,.$$

By using again (23) for the left-hand side

$$R^\rho{}_{\nu\mu\lambda}\mathcal{L}_\beta(V_\rho) = \mathcal{L}_\beta\left(R^\rho{}_{\nu\mu\lambda}\right)V_\rho + R^\rho{}_{\nu\mu\lambda}\mathcal{L}_\beta(V_\rho)\,,$$

whence we conclude that

$$\mathcal{L}_\beta\left(R^\rho{}_{\nu\mu\lambda}\right)V_\rho = 0$$

for any vector field V. Thus, we obtain (26), which finally demonstrates the general statement at the beginning of the section.

7. Concluding remarks

The general stationarity equation implied by the vanishing of the Lie derivative for a scalar reads

$$\mathcal{L}_\beta(S) = \beta^\lambda \partial_\lambda S = \sqrt{\beta^2} u^\lambda \nabla_\lambda S \equiv \sqrt{\beta^2} DS = 0$$

implying that a scalar quantity does not change along the β flow. That is, a comoving observer with four-velocity $u = \beta/\sqrt{\beta^2}$ will measure the same temperature, energy density, pressure and any other scalar field.

Instead, for a vector field,

$$\mathcal{L}_\beta(V_\mu) = \beta^\lambda \nabla_\lambda V_\mu + (\nabla_\mu \beta_\lambda)\, V^\lambda = 0\,. \tag{27}$$

As β is a Killing vector, its covariant derivative is antisymmetric, thus, one can extend (13) to the general relativistic case, that is $\varpi_{\mu\lambda} = -\nabla_\mu \beta_\lambda$. If one sets

$$\Omega_{\mu\lambda} \equiv \frac{1}{\sqrt{\beta^2}} \varpi_{\mu\lambda}\,,$$

the Ω is an antisymmetric tensor such that, according to (27)

$$DV_\mu = \Omega_{\mu\lambda} V^\lambda\,. \tag{28}$$

These are the well-known (in general relativity) equations of motion of an orthonormal tetrad frame, the relativistic extension of the classical Poisson equations for the motion of a rigid frame. The consequence of (28), for a vector field V at thermodynamic equilibrium, is that its components are constant for a comoving observer only if he has an associated tetrad frame — which must include the normalized Killing vector itself as time direction — which is Lie-transported, that is with vanishing Lie derivative; the same holds for any tensor field.

It is also worth pointing out another interesting consequence of this formulation of general relativistic thermodynamics:

A free-falling ideal thermometer in a fluid at global thermodynamic equilibrium will mark a constant temperature $T_{\mathrm{T}} = 1/(\beta \cdot u)$ with β the four-temperature of the fluid and u the four-velocity of the thermometer.

This is a straightforward consequence of the well-known conservation theorem for a geodesic motion in spacetimes with Killing vectors [28] and equation (6).

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