ON LOCAL EQUILIBRIUM AND ERGODICITY*

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The main mathematical argument of the universal framework for local equilibrium proposed in *Analysis* **36**, 49 (2016) is condensed and formulated as a fundamental dichotomy between subsets of positive measure and subsets of zero measure in ergodic theory. The physical interpretation of the dichotomy in terms of local equilibria rests on the universality of time scale separation in an appropriate long-time limit.

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1. Introduction

A basic problem in the foundations of non-equilibrium statistical physics is the question why equilibrium quantities such as temperature, density, pressure or chemical potential often depend on time or position although equilibrium quantities are by definition translation invariant. Dependence of thermodynamic quantities on time or position in non-equilibrium statistical physics is usually assumed or postulated *ad hoc* in textbooks [1].

Mathematically, the problem of local equilibrium is a problem of multiple scales. A dilute gas between walls at different temperature can serve as an example. If the walls are centimetres apart, then heterogeneities (in density or temperature) develop on length scales $L_{\mathrm{h}}/\mathrm{cm}$ of centimetres. One may translate the heterogeneity length scale L_{h} into a heterogeneity time scale τ_{h} by division with a velocity. Room temperature molecular velocities (speed of sound) in hydrogen range around $10^3\,\mathrm{m/s}$, and hence $\tau_{\mathrm{h}} \approx 10^{-5}\,\mathrm{s}$ [1, p. 450]. Equilibration of local pressure and local temperature on small scales is due to incessant microscopic molecular collisions on scales much smaller than L_{h}. Molecular collisions in hydrogen occur during times of the order of $\tau_{\mathrm{c}} \approx 50\,\mathrm{fs}$ (time during which the trajectory is not straight) while the system relaxes into local equilibrium within times of the order of $\tau_{\mathrm{r}} \approx 50\,\mathrm{ps}$ (time of flight between two collisions) [1, p. 450].

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Defining L_{c} as the range of interactions, L_{r} as the mean free path, and L_{h} as the scale of heterogenities, the separation of length scales

$$L_{\mathrm{c}} \ll L_{\mathrm{r}} \ll L_{\mathrm{h}} \tag{1.1a}$$

corresponds to the separation of time scales

$$\tau_{\mathrm{c}} \ll \tau_{\mathrm{r}} \ll \tau_{\mathrm{h}} \tag{1.1b}$$

exemplified in the paragraph above. Equations of motion appropriate for these scales could be Hamilton's equations for the $(L_{\mathrm{c}}, \tau_{\mathrm{c}})$-scale, Boltzmann's equation on $(L_{\mathrm{r}}, \tau_{\mathrm{r}})$-scales, and the heat equation on $(L_{\mathrm{h}}, \tau_{\mathrm{h}})$-scales. In general, multiple scales induce a hierarchy of scales and scaling limits, and local equilibrium amounts to translation invariance and thermodynamic behaviour on a certain "local" scale of interest.

Gravitational systems (such as the solar system) exhibit local equilibria on multiple scales (sun, earth, planets, asteroids, moons) due to their large number of (some 10^{57} or so) particles. Local equilibria in a laboratory on planet earth are readily prepared for as few as 10^{25} particles. One is, therefore, led to investigate the dynamics of extremely small subsystems occupying only a tiny fraction of the phase space of the full dynamical system. Restricted dynamical subsystems of the solar system also appear naturally when the positions of some 10^{53} particles are reduced to their center of mass representing the position of a planet in celestial mechanics. It is then important to investigate induced (or restricted) dynamics of subsystems from a more abstract point of view. A suitable abstract framework is given by ergodic theory. My objective in this short note is to discuss and elucidate the essential part of Theorem 2 in [2] from this general and abstract viewpoint.

Section 2 recalls the general definition of dynamical systems and their associated transition and orbit maps, and Section 3 provides the setup of ergodic theory. Section 4 introduces the hitting function of a subset and defines induced transformations. Next, recurrence into sets of positive measures is discussed in Section 5. Basic results from ergodic theory concerning induced transformations on subsets of positive measure are given and used for taking the long-time limit in Section 6. The main result is formulated as Theorem 7.2 in Section 7. Finally, the discussion in Section 8 shows applications of the general theorems to anomalous transport and glassy relaxation in experiment.

2. Deterministic dynamical systems

The physical system is represented by a set M of states or configurations. For a classical Hamiltonian system, the set M could be the phase space or the manifold of constant energy. Physically, $M \cong \mathbb{R}^{6N}$ or $M \cong \mathbb{R}^{6N-1}$, where N is the number of molecules or particles and $\cong$ stands for "locally isomorphic". Mathematically, M is assumed to have suitable properties (*e.g.* locally compact Hausdorff) to permit the subsequent considerations.

The deterministic *time evolution* or *dynamical rule* Δ of the physical system is defined to be a continuous mapping [3]

$$\begin{aligned} \Delta : \mathbb{R} \times M &\to M \,, \\ (t, x) &\mapsto \Delta(t, x) =: x(t) \end{aligned} \tag{2.1a}$$

such that there exists a time instant $t_0 \in \mathbb{R}$ with

$$\Delta(t_0, x) = x \tag{2.1b}$$

for all $x \in M$, and such that the compatibility condition

$$\Delta\,[t_2 - t_1, \Delta(t_1 - t_0, x)] = \Delta(t_2 - t_0, x) \tag{2.1c}$$

holds for all $x \in M$ and time instants $t_0, t_1, t_2 \in \mathbb{R}$. Choosing $t_0 = 0$ and defining the length of the time span between t_2 and t_1 as the difference

$$\tau := t_2 - t_1 \in \mathbb{R} \,, \tag{2.2}$$

the compatibility condition can be written in terms of translating the initial instant as

$$(x(t_1))(\tau) = x(t_1 + \tau) \tag{2.3}$$

for all time instants $t_1 \in \mathbb{R}$ and time intervals $\tau \in \mathbb{R}$.

The dynamical rule determines two additional mappings by fixing either t or x. Firstly, for fixed $t \in \mathbb{R}$, the map

$$\begin{aligned} \Delta(t, \cdot) : M &\to M \,, \\ x &\mapsto x(t) \end{aligned} \tag{2.4}$$

is called *transition* map, because it maps the state at the initial instant t_0 to that at t. The transitions are homeomorphisms of M onto itself for each $t \in \mathbb{R}$ [3, Thm 1.2]. The transitions, abbreviated as $T^t := \Delta(t, \cdot)$, define a group homomorphism from the additive group $(\mathbb{R}, +)$ into the symmetric group $\mathrm{Sym}(M)$ of bijections from M to M. Secondly, for fixed $x \in M$, the dynamical rule defines the *orbit* map

$$\begin{aligned} \Delta(\cdot, x) : \mathbb{R} &\to M \,, \\ t &\mapsto x(t) \end{aligned} \tag{2.5}$$

as a curve or trajectory in M passing through the point x at the initial instant t_0.

3. Ergodicity

Consider now the standard setup of ergodic theory [4]. In ergodic theory, time is usually discretized by setting

$$T^0 = \mathbf{1} = \Delta(t_0, \cdot)\,, \qquad t_0 \in \mathbb{R}\,, \tag{3.1a}$$
$$T^1 = T := \Delta(t_1 - t_0, \cdot)\,, \qquad t_0, t_1 \in \mathbb{R}\,, \tag{3.1b}$$
$$T^k = TT^{k-1}\,, \qquad k \in \mathbb{Z}\,, \tag{3.1c}$$
$$x_k = x(t_k) = T^k x_0 = x(t_0 + k(t_1 - t_0))\,, \qquad t_k \in \mathbb{R}\,, \quad k \in \mathbb{Z}\,, \tag{3.1d}$$

and time evolution amounts to iteration of a single transition map T. Here, $\mathbf{1}$ is the identity map on M. Next, a σ-algebra $\mathfrak{M}$ of subsets of M is specified, *e.g.* the one generated by the topology on M. Finally, a complete and finite measure $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_+ := [0, \infty]$ is assumed to be given on $\mathfrak{M}$, *i.e.* all subsets with measure zero are assumed to belong to $\mathfrak{M}$ and $\mu(M) < \infty$ [4].

An *automorphism* of the measure space $(M, \mathfrak{M}, \mu)$ is a bijective map $T : M \to M$ that is *measurable* and *measure preserving*, *i.e.* for all $A \in \mathfrak{M}$

$$TA, T^{-1}A \in \mathfrak{M}\,, \tag{3.2a}$$
$$\mu(A) = \mu(TA) = \mu\left(T^{-1}A\right) \tag{3.2b}$$

holds true [4]. Here,

$$T^{-1}A := \{x \in M : Tx \in A\} \tag{3.2c}$$

is the inverse image (pre-image) of the set A under T. The measure μ is called *invariant* under the measure preserving transformation T. The set of all automorphisms on $(M, \mathfrak{M}, \mu)$ is denoted $\mathrm{Aut}(M, \mathfrak{M}, \mu)$ or $\mathrm{Aut}(M)$ for short. The composition of two automorphisms is again an automorphism and $\mathrm{Aut}(M)$ is a group. Its neutral element is the identity map $\mathbf{1}$ on M. The standard setup of ergodic theory is the quadruple $(M, \mathfrak{M}, \mu, T)$ representing a deterministic *discrete dynamical system.*

An element $x \in M$ is called *invariant* under T, if $Tx = x$. A measurable set $A \in \mathfrak{M}$ is called *invariant*, if $T^{-1}A = A$. For a subset $A \subset M$ of positive measure ($\mu(A) > 0$) define the *trace σ-algebra* $\mathfrak{M}_A$ of $\mathfrak{M}$ in A

$$\mathfrak{M}_A := \{B \cap A : B \in \mathfrak{M}\} \tag{3.3a}$$

and the (probability) measure μ_A

$$\mu_A(B) := \frac{\mu(B \cap A)}{\mu(A)}\,, \qquad B \in \mathfrak{M}(A) \tag{3.3b}$$

induced by μ on A. Let

$$A^c := M \setminus A \tag{3.4}$$

denote the complement of a subset $A \subset M$. If A is an invariant subset, then the automorphism T can be split by restriction of T to A and its complement into two automorphisms $T|_A : A \to A$ and $T|_{A^c} : A^c \to A^c$ such that the dynamical systems

$$(A, \mathfrak{M}_A, \mu_A, T|_A) \qquad \text{and} \qquad (A^c, \mathfrak{M}_{A^c}, \mu_{A^c}, T|_{A^c})$$

do not overlap or interact in any way. The decomposition $M = A \cup A^c$ for invariant A into independent ergodic components motivates the definition of *ergodicity* as a form of indecomposability: A dynamical system $(M, \mathfrak{M}, \mu, T)$, its automorphism T or its invariant measure μ is called *ergodic*, if every invariant set $A \in \mathfrak{M}$ satisfies $\mu(A) = 0$ or $\mu(A^c) = 0$. The set of all invariant elements (or any subset of it) is clearly an invariant set. Thus, in an ergodic system, invariant elements have measure one or zero.

4. Induced transformations

For an arbitrary subset $A \subset M$, not necessarily measurable, the *hitting function* $h_A : M \to \overline{\mathbb{N}}$ *of* A with $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ assigns to each $x \in M$ the smallest positive integer k such that $T^k x \in A$. Formally,

$$h_A(x) := \min\left\{k \geq 1 : T^k x \in A\right\}, \qquad x \in M \tag{4.1}$$

with $h_A(x) = \infty$ indicating that x never hits A or never returns into A after leaving it. The restriction of h_A to A is called *return time function*, the restriction of h_{A^c} to A is called *exit time function.*

Given a subset $B \subset M$ and $k \in \overline{\mathbb{N}}$, the hitting function h_A of A defines a partitioning of B into equivalence classes

$$(B \rightsquigarrow A)_k := \{x \in B : h_A(x) = k\} \tag{4.2}$$

of points having the same first passage time k for the passage from B to A. The sets are called *first passage sets from* B *to* A. Assuming $A \neq (A \rightsquigarrow A)_\infty$, the mapping T induces a transformation

$$\begin{aligned} T_{\tilde{A}} : \tilde{A} &\to \tilde{A}, \\ x &\mapsto T_{\tilde{A}} x := T^{h_A(x)} x, \end{aligned} \tag{4.3a}$$

where

$$\tilde{A} := A \setminus (A \rightsquigarrow A)_\infty \tag{4.3b}$$

is the set of points $x \in A$ with finite return time $1 \leq h_A(x) < \infty$. It is called the *induced transformation* and maps recurrent points to their points of first reentry. It is a standard construction in ergodic theory [5]. Note that $T_{\tilde{A}}$ is defined for an arbitray subset A, while $T|_A$ was defined only for invariant subsets.

5. Recurrence into sets of positive measure

Consider a discrete dynamical system $(M, \mathfrak{M}, \mu, T)$ and let $A \subset M$ be a subset of positive measure $\mu(A) > 0$. Then almost all points in A return to A under iteration of T and hence $\mu((A \rightsquigarrow A)_\infty) = 0$ by virtue of Poincare's recurrence theorem. As a consequence, the quadruple $(\tilde{A}, \mathfrak{M}_{\tilde{A}}, \mu_{\tilde{A}}, T_{\tilde{A}})$ with $T_{\tilde{A}}$ from (4.3a) is well-defined and $T_{\tilde{A}}$ is again measure preserving. If $(M, \mathfrak{M}, \mu, T)$ is ergodic, then also $(\tilde{A}, \mathfrak{M}_{\tilde{A}}, \mu_{\tilde{A}}, T_{\tilde{A}})$ is ergodic [4]. Now note

Theorem 5.1. *The first return time function $R_{\tilde{A}} : \tilde{A} \to \mathbb{N}$ on $\tilde{A}$, defined as the restriction*

$$R_{\tilde{A}} := h_{\tilde{A}}\big|_{\tilde{A}} \tag{5.1}$$

of the hitting function $h_{\tilde{A}}$ to $x \in \tilde{A}$, is integrable with respect to the induced measure $\mu_{\tilde{A}}$. The first return time function is an integer-valued random variable $R_{\tilde{A}} : (\tilde{A}, \mathfrak{M}_{\tilde{A}}) \to (\mathbb{N}, \mathcal{P}(\mathbb{N}))$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. Its distribution $P_{R_{\tilde{A}}} : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}_+$ with

$$P_{R_{\tilde{A}}} = \mu_{\tilde{A}} \circ R_{\tilde{A}}^{-1}, \tag{5.2a}$$

$$P_{R_{\tilde{A}}}(B) = \sum_{k \in B} p(k), \qquad B \subset \mathbb{N} \tag{5.2b}$$

is the image of $\mu_{\tilde{A}}$ under the map $R_{\tilde{A}}$, and $p : \mathbb{N} \to [0,1]$ with

$$p(k) = \mu_{\tilde{A}}\left(\left\{x \in \tilde{A} : R_{\tilde{A}} \in \{k\}\right\}\right) = \mu_{\tilde{A}}\left(\left(\tilde{A} \rightsquigarrow \tilde{A}\right)_k\right), \qquad k \in \mathbb{N} \tag{5.3}$$

is its distribution function. The expectation value of $R_{\tilde{A}}$

$$\langle R_{\tilde{A}} \rangle = \sum_{k=1}^{\infty} k p(k) = \int_{\tilde{A}} R_{\tilde{A}}(x) \mathrm{d}\mu_{\tilde{A}}(x) = \frac{1}{\mu\left(\tilde{A}\right)} \tag{5.4}$$

is finite, because $\mu(\tilde{A}) > 0$. If T is egodic, the time averages of $R_{\tilde{A}}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} R_{\tilde{A}}\left(T^k x\right) = \langle R_{\tilde{A}} \rangle \tag{5.5}$$

exist for almost all $x \in M$, and agree with the expectation value of $R_{\tilde{A}}$.

Proof. The theorem collects well-known statements. See [4] for proofs. □

6. The long-time limit

The idea is now to regard $(\tilde{A}, \mathfrak{M}_{\tilde{A}}, \mu_{\tilde{A}}, T_{\tilde{A}})$ as a discrete *dynamical subsystem* of the original system. This requires to interpret the iterates $T_{\tilde{A}}^k$ of the induced automorphism $T_{\tilde{A}} : \tilde{A} \to \tilde{A}$ in the same way as the iterates T^k of the original automorphism $T : M \to M$ as a time evolution. An obstacle to this analogy is the lack of synchronicity. The induced automorphism $T_{\tilde{A}}$ does not correspond to a well-defined time step. More precicely, for $A = M$, one has $\tilde{M} = M$ and $T_M = T$ by definition. Therefore, $R_M = 1$ is non-random and constant. For $A \subsetneq M$, however, $R_{\tilde{A}}$ is a random variable that can take any value in $\mathbb{N}$.

Absence of synchronicity means that a "time step" cannot be assigned to $T_{\tilde{A}}$ because such a "step" would never be complete at any finite epoch. Its completion would require an infinite number of iterations of T. This puzzle points to the need for changing scales and rescaling time.

To investigate the limit $k \to \infty$ of T^k, the discrete evolution is embedded into continuous time $\mathbb{R}$. The embedding of $\mathbb{Z}$ into $\mathbb{R}$ is given by the arithmetic progression

$$\mathbb{A} := t_0 + \tau\mathbb{Z} = \{t_0 + k\tau\}_{k\in\mathbb{Z}} \tag{6.1}$$

with the initial instant $t_0 \in \mathbb{R}$ and an arbitrary clock step or time scale $\tau \in \mathbb{R}$. A single transition T, expressed as a translation by τ in (2.3), can be written as a convolution

$$x(t_0 + \tau) = \int x(t_0 - r)\mathrm{d}\delta_{-\tau}(r) = \left(x * \delta_{-\tau}\right)(t_0) \tag{6.2}$$

for all $t_0, \tau \in \mathbb{R}$ with a Dirac measure supported on the negative half-axis. The convolution $*$ of a measure μ and a function f is defined as $(f*\mu)(x) := \int f(x-y)\mathrm{d}\mu(y)$. The *Dirac measure* $\delta_\tau : \mathcal{B} \to \{0, 1\}$ on the real line $\mathbb{R}$ concentrated at $\tau \in \mathbb{R}$ is defined for the measurable space $(\mathbb{R}, \mathcal{B})$ by[1]

$$\delta_\tau(B) = \begin{cases} 0 & \text{if } t \notin B \\ 1 & \text{if } t \in B \end{cases} \tag{6.3}$$

for all $B \in \mathcal{B}$. The k^{th} iterate of $T : M \to M$ is then

$$x(t_k) = T^k x_0 = x(t_0 + k\tau) = \left(x * \delta_{-k\tau}\right)(t_0) = \left(x * \delta_{-t_0-k\tau}\right)(0) \tag{6.4}$$

by virtue of Eq. (3.1d) and Eq. (6.2).

In continuous time, the return time function is a real-valued random variable $R_{\tilde{A}} : (\tilde{A}, \mathfrak{M}(\tilde{A})) \to (\mathbb{A}^+, \mathcal{P}(\mathbb{A}^+))$ on the arithmetic progression

$$\mathbb{A}^+ := t_0 + \tau\mathbb{N}\,, \tag{6.5}$$

[1] $\mathcal{B}$ denotes the canonical σ-algebra generated by open intervals.

where $\mathcal{P}(\mathbb{A}^+)$ is the power set of the set $\mathbb{A}^+ \subset \mathbb{R}$. Its *distribution* is the image measure $P_{R_{\tilde{A}}} : \mathcal{P}(\mathbb{A}^+) \to \overline{\mathbb{R}}_+$

$$P_{R_{\tilde{A}}} = \mu_{\tilde{A}} \circ R_{\tilde{A}}^{-1}\,, \tag{6.6a}$$

$$P_{R_{\tilde{A}}}(B) = \sum_{(t_0+k\tau)\in B} p(k) = \sum_{k=1}^{\infty} p(k)\delta_{t_0+k\tau}(B) \tag{6.6b}$$

for all $B \subset \mathbb{A}^+$. The step function $F_{R_{\tilde{A}}} : \mathbb{R}_+ \to [0,1]$

$$F_{R_{\tilde{A}}}(r) = P_{R_{\tilde{A}}}\left\{R_{\tilde{A}} \le r\right\} \tag{6.7}$$

is the *distribution function* of the random variable $R_{\tilde{A}}$.

The iterates $T_{\tilde{A}}^N$ are analogous to the iterates T^k. The analogy between T and $T_{\tilde{A}}$ emerges clearly when the return time is synchronous. Then $p(j) = 1$ for $j = 1/\mu(\tilde{A})$ and $p(i) = 0$ for $i \neq j$. The analogue of Eq. (6.2) for a single step is the transformation

$$\left(x * \widehat{P}_{R_{\tilde{A}}}\right)(t_0) = \int x(t_0-r)\mathrm{d}\widehat{P}_{R_{\tilde{A}}}(r) = \sum_{k=1}^{\infty} p(k) \int x(t_0-r)\mathrm{d}\delta_{-k\tau}(r)\,, \tag{6.8}$$

where $\widehat{P}_{R_{\tilde{A}}} := P_{-R_{\tilde{A}}}$ denotes the reflected return time distribution. The analogue of Eq. (6.4) for iterated convolutions T^k emerges from the random return time

$$S_N = R_{\tilde{A},1} + \cdots + R_{\tilde{A},N} \tag{6.9}$$

needed for N independent iterations $T_{\tilde{A}}^N$. The random sums S_N are concentrated on the arithmetic progressions

$$\mathbb{A}_N^+ := Nt_0 + \tau\mathbb{N} \tag{6.10}$$

with $\mathbb{A}_1^+ = \mathbb{A}^+$, $t_0 \in \mathbb{R}$ and $\tau \ge 0$.

The random variables $S_N : (\tilde{A}, \mathfrak{M}(\tilde{A})) \to (\mathbb{A}_N^+, \mathcal{P}(\mathbb{A}_N^+))$ form a random walk on the half axis with independent and identically distributed increments. Their distributions are the image measures

$$P_{S_N} = \mu_{\tilde{A}} \circ S_N^{-1}\,, \tag{6.11a}$$

$$P_{S_N}(B) = \sum_{k=1}^{\infty} p_N(k)\delta_{Nt_0+k\tau}(B)\,, \qquad B \subset \mathbb{A}_N^+ \tag{6.11b}$$

with $N \in \mathbb{N}$ and

$$p_N(k) = (p_{N-1} * p)(k) = \sum_{m=0}^{k} p_{N-1}(m)p(k-m) \tag{6.12}$$

for $N \geq 2$ and $p_1(k) = p(k)$. Note the analogy to Eq. (3.1c). As a consequence,

$$p_{N+M}(k) = (p_N * p_M)(k) \tag{6.13}$$

holds for all $N, M \geq 1$. The limit $N \to \infty$ is governed by

Theorem 6.1 (Law of large numbers [6, p. 235]). *In order that there exist centering constants $C_N \in \mathbb{R}$, such that for all $\epsilon > 0$*

$$\lim_{N\to\infty} P_{S_N}\left\{\left|\frac{S_N}{N} - C_N\right| > \epsilon\right\} = 0 \tag{6.14}$$

holds, it is necessary and sufficient that

$$\lim_{r\to\infty} r\left[1 - F_{R_{\tilde{A}}}(r)\right] = 0\,. \tag{6.15}$$

In this case, the constants are given as

$$C_N = \int_0^N r\,\mathrm{d}F_{R_{\tilde{A}}}(r) = \sum_{k=1}^{N} k\,p(k)\,. \tag{6.16}$$

Equivalently, the rescaled random sums $X_N = N^{-1}S_N - C_N$ converge in distribution [6, p. 247]

$$\operatorname*{w-lim}_{N\to\infty} P_{X_N} = \delta_{\langle R_{\tilde{A}}\rangle} \tag{6.17}$$

to a degenerate non-random limit (Dirac measure), where $\langle R_{\tilde{A}}\rangle$ is given in Eq. (5.4).

Equation (6.17) establishes the analogy between iterates $T_{\tilde{A}}^k$ of the induced automorphism $T_{\tilde{A}} : \tilde{A} \to \tilde{A}$ and iterates T^k of the original automorphism $T : M \to M$. If T is a transition with time step τ, then $T_{\tilde{A}}$ is a transition with time step $\tau\,\langle R_{\tilde{A}}\rangle = \tau/\mu(\tilde{A})$. It is then of interest to discuss the limit $\mu(A) \to 0$.

7. Subsets of measure zero

Consider a discrete dynamical system $(M, \mathfrak{M}, \mu, T)$ and let $A \subset M$ be a subset of vanishing measure $\mu(A) = 0$. Then $\tilde{A} \subset A$ has also vanishing measure $\mu(\tilde{A}) = 0$. The analogy between T and $T_{\tilde{A}}$ seems to break down because Eq. (3.3b) defining the induced measure $\mu_{\tilde{A}}$ becomes invalid and the expected recurrence time $\langle R_{\tilde{A}} \rangle$ in Eq. (5.4) diverges.

On the other hand, the induced transformation $T_{\tilde{A}}$ defined in Eq. (4.3a) and the measurable space $(\tilde{A}, \mathfrak{M}_{\tilde{A}})$ remain perfectly well-defined also for subsets of measure zero. Because the separation of time scales between T and $T_{\tilde{A}}$ is infinite, the study of induced automorphisms on subsets of measure zero can be viewed as the study of dynamics on scales that are longer than infinitely long. To extend the analogy between the original automorphism and the induced transformation to subsets of measure zero, another invariant measure is needed on A. If one exists, its properties are given by

Theorem 7.1. *Let $(M, \mathfrak{M}, \mu, T)$ and $(M, \mathfrak{M}, \nu, T)$ be two dynamical systems with the same automorphism T, but different invariant measures μ, ν. If both measures μ, ν are ergodic with respect to T, then either $\mu = \nu$ or $\mu \perp \nu$ are singular with respect to each other.*

Proof. This is statement (2) from Theorem 2 in [4, p. 15]. □

Let $A \subset M$ be such that $\mu(A) = 0$ and $\nu(A^c) = 0$. Then $\nu(A) > 0$ by virtue of Theorem 7.1. If also $\nu(\tilde{A}) > 0$, then the return time distributions

$$P^{\nu}_{S_N} = \nu_{\tilde{A}} \circ S_N^{-1}, \tag{7.1a}$$

$$P^{\nu}_{S_N}(B) = \sum_{k=1}^{\infty} p^{\nu}_N(k) \delta_{Nt_0 + k\tau}(B), \qquad B \subset \mathbb{A}^+_N \tag{7.1b}$$

are defined as images of $\nu_{\tilde{A}}$ instead of $\mu_{\tilde{A}}$ under S_N for each N. The probabilities are denoted $p^{\nu}_N(k)$. Two cases can arise, namely

$$\langle R_{\tilde{A}} \rangle_{\nu} = \sum_{k=1}^{\infty} k p^{\nu}_N(k) \begin{cases} < \infty \\ = \infty \end{cases} \tag{7.2}$$

depending on whether the return time distribution has finite or infinite expectation.

Theorem 7.2.

(a) For $\langle R_{\tilde{A}} \rangle_{\nu} < \infty$ finite, Theorem 6.1 applies with $p^{\nu}_N(k)$ instead of $p^{\mu}_N(k) = p_N(k)$. The limit $N \to \infty$ is again degenerate.

(b) If $\langle R_{\tilde{A}} \rangle_\nu = \infty$ *diverges, there exist rescaling constants* $D_N > 0$ *and centering constants* $C_N \in \mathbb{R}$ *such that the sequence of rescaled and centered random variables* $X_N = (S_N/D_N) - C_N$ *converges in distribution*

$$\operatorname*{w\text{-}lim}_{N\to\infty} P^\nu_{X_N} = P_\alpha \tag{7.3}$$

to a measure P_α *absolutely continuous with respect to the Lebesgue measure on the real line. The constant* $\alpha = \alpha(\nu, A)$ *obeys* $0 < \alpha < 1$ *and depends on the subset* A *and the invariant measure* $\nu_{\tilde{A}}$*.*

Proof. The theorem follows from application of [6, VI.1, XVII.5] to the case of $S_N \geq 0$. □

Case (a) in the theorem emerges from case (b) in the limit $\alpha \to 1$. The distribution function F_α of the measure P_α has the density

$$\frac{\mathrm{d}F_\alpha(x)}{\mathrm{d}x} = h_\alpha(x) = \begin{cases} 0 & \text{for} \quad x \leq 0\,, \\ \dfrac{1}{x}\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j x^{-\alpha j}}{j!\,\Gamma(-\alpha j)} & \text{for} \quad x > 0 \end{cases} \tag{7.4}$$

with respect to the Lebesgue measure. The parameter α and the function h_α are the same as in [2, Eq. (5.10)] and this establishes the connection to [2].

8. Discussion

The main theorem elucidates and condenses the local limit theorem from [2]. Induced automorphisms remain well-defined even on null sets. The importance of this observation stems from its universality, its scale independence, and its simplicity. A fundamental dichotomy exists in ergodic theory distinguishing induced automorphisms on null sets from sets of positve measure. Theorem 7.2 uses only standard theorems from ergodic theory and probability theory, and applies universally to all dynamical systems in physics [3]. Some applications of the main theorem to physics were outlined in [2, Section 6]. They include a solution of the reversed irreversibility problem and dielectric relaxation functions for glasses. Another application to dielectric response of glass forming materials was recently given for the Havriliak–Negami relaxation and its relatives in [7]. The universality of the theoretical framework is mirrored in the universality of anomalous relaxation, which is experimentally known to be largely material-independent.

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