A NOTE ON A PARADOXICAL PROPERTY OF THE SCHWARZSCHILD SOLUTION

By M. A. Abramowicz and J. P. Lasota

Institute of Astronomy, Polish Academy of Sciences, Warsaw*

(Received December 13, 1973)

This note contains an analysis of the motion of some special observers moving in the Schwarzschild gravitational field.

It appears unexpectedly that the old, good Schwarzschild solution still contains some surprising and new properties. The phenomenon we want to present in this note seems to be paradoxical because it violates our intuition based on Newtonian mechanics. However, it has a satisfactory explanation on the ground of the theory of relativity.

Let us consider the family $F(r_0)$ of observers (in general noninertial) moving on circular trajectories around the centre of the Schwarzschild field source at a given distance r_0 and with constant angular velocities Ω. More precisely, their trajectories in space are great circles on the surface of transitivity of the group $SO(3)$ at a given distance $r = r_0$. We assume that the observers of this family admit all physical values of Ω. (The observers with $r_0^2\Omega^2 \geqslant 1-2M/r_0$ are unphysical because they move faster than light.) Those, for which $\Omega = 0$ are called static observers.

Now, let us ask the following question. Is it possible for all observers of the family $F(r_0)$ to have equal accelerations?

Newtonian intuition prompts the answer: No! However, the correct answer is: Yes, it is possible at $r_0 = 3M$. This answer follows directly from the fact that at $r_0 = 3M$

$$a_k = u^i \nabla_i u_k = (0, 1/3M, 0, 0); \; i, k = 0, 1, 2, 3, \tag{1}$$

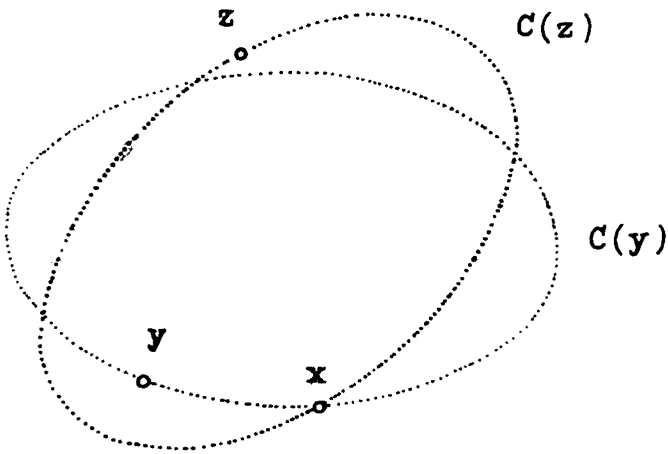
independently of Ω. Here u^k and ∇_i are, respectively, the four-velocity of an observer and the covariant derivative.

If we imagine that at the distance $r_0 = 3M$ there is a circular photon orbit, the answer is not so strange as one would think. Let $C(y)$ denote the space trajectory of an observer y and let $x \in C(y)$ mean that a static observer x is situated on this trajectory. The observer x can use radar to measure the acceleration of the observer y. Since $C(y)$ is the trajectory

* Address: Zakład Astronomii PAN, al. Ujazdowskie 4, 00-478 Warszawa, Poland.

of the radar signals, we see that neither the directions nor the redshift of the radar echo received by x can depend on time. Therefore, from the point of view of the observer x, the observer y moves with constant velocity along a straight line and so the relative acceleration vanishes. Since for two arbitrary observers y and z there exists an observer x such that $x \in C(y)$ and $x \in C(z)$, the relative acceleration between y and z vanishes (Fig. 1).

The world tube of the sphere $r = 3M$ forms a pictorial model of the three-dimensional spacetime of those of $F(3M)$ observers who never look at non-tangent directions. In this spacetime all physical observers ($\Omega^2 < 1/27$) move on timelike geodesics.



Let us examine more formally the motion of the observers of the family $F(r_0)$. The four-velocity of all these observers, who move along a given circle C, can be expressed by the formula

$$u^i = A(\eta^i + \Omega\xi^i), \tag{2}$$

where η^i is the timelike Killing vestor and ξ^i is the spacelike Killing vector tangent to C. The quantity A can be computed from the condition $(uu) = -1$. We shall denote the projection of u^i on the invariant spacelike hypersurface orthogonal to the trajectories of static observers by v^i. One can define the angular momentum density of an observer by

$$j = (v\xi). \tag{3}$$

Using this definition we can express the change δa^i of acceleration of an observer caused by the change $\delta\Omega$ of his angulur velocity:

$$\delta a_i = (\nabla_i j)\delta\Omega. \tag{4}$$

We see that on a given circle C accelerations of all observers are equal if, and only if, $\nabla_i j = 0$ for each Ω on this circle. In the Schwarzschild solution we have $\dfrac{j}{1-\Omega j} = -\Omega(\xi\xi)/(\eta\eta) = r^2 \sin^2\Theta/(1-2M/r)$, so $\nabla_i j = 0$ on the circle $r = 3M$, $\Theta = \pi/2$.

Formulae (3) and (4) which have exactly the same form as their Newtonian counterparts as well as formula (2) are valid also in the more general case of stationary (but non static), axially-symmetric fields. In this case

$$\frac{j}{1-\Omega j} = -\frac{(\eta\zeta)+\Omega(\xi\xi)}{(\eta\eta)+\Omega(\eta\zeta)}. \tag{5}$$

If $(\eta\xi) \neq 0$, it is impossible to have $\nabla_{i,}j = 0$ independently of Ω. One can also deduce this fact knowing that in stationary fields there are no circular trajectories with photons moving in both directions.

We would like to thank Professor A. Trautman for his kind interest in this work.