METHOD OF GENERATING STATIONARY EINSTEIN-MAXWELL FIELDS*

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We describe a method of generating stationary asymptotically flat solutions of the Einstein-Maxwell equations starting from a stationary vacuum metric. As a simple example, we derive the Kerr-Newman solution.

Recently a number of new stationary solutions was found [1–3] and new methods of generating stationary Einstein–Maxwell fields were discovered [4–6]. In this note I would like to describe another method of generating asymptotically flat solutions of the Einstein–Maxwell equations starting from stationary vacuum metrics.

The general stationary metric can be written in the form

$$ds^2 = f(dt+w_j dx^j)^2 - f^{-1} h_{ij} dx^i dx^j, \tag{1}$$

where $i, j = 1, 2, 3$ and the function f, w_i and h_{ij} do not depend on t. This notation closely follows that of Kinnersley [6].

The electromagnetic field is very conveniently described by the complex electromagnetic tensor $\mathcal{F}_{\mu\nu}$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + i {}^*F_{\mu\nu}, \tag{2}$$

where $F_{\mu\nu}$ is the Maxwell tensor and ${}^*F_{\mu\nu}$ is its dual. The source free Maxwell equations could be written as

$$\mathcal{F}_{[\mu\nu;\varrho]} = 0, \tag{3}$$

which assures the existence of the electromagnetic potential a_μ such that

$$\mathcal{F}_{\mu\nu} = a_{\nu;\mu} - a_{\mu;\nu}. \tag{4}$$

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The coupled Einstein–Maxwell field equations may be written as equations in a 3-space H with metric tensor h_{ij}. Let ∇ denote the covariant derivative in H. We define a twist vector

$$\bar{\tau} = f^2 \nabla \times \bar{w} + i(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*), \tag{5}$$

where Ψ is a complex function describing uniquely the electromagnetic field and * denotes complex conjugation.

Using part of the Einstein equations,

$$G_{jo} = 8\pi T_{jo}, \tag{6}$$

one can show that

$$\nabla \times \bar{\tau} = 0, \tag{7}$$

implying the existence of a real scalar potential χ such that

$$\bar{\tau} = \nabla \chi. \tag{8}$$

Let us now define a complex scalar potential for gravitation

$$\varepsilon = f - \Psi \Psi^* + i\chi. \tag{9}$$

Given h_{ij}, ε completely determines the metric and hence the gravitational field.

The Maxwell equations (3) and the remaining Einstein equations may now be written in terms of ε and Ψ. They assume the form

$$f \nabla^2 \varepsilon = (\nabla \varepsilon + 2\Psi^* \nabla \Psi) \nabla \varepsilon, \tag{10}$$

$$f \nabla^2 \Psi = (\nabla \varepsilon + 2\Psi^* \nabla \Psi) \nabla \Psi. \tag{11}$$

The curvature tensor of H is also determined by ε and Ψ through the relation,

$$f^2 R^{(3)}_{kj} = \tfrac{1}{2}\, \varepsilon_{,(j}\varepsilon^*_{,k)} + \Psi \varepsilon_{,(j}\Psi^*_{,k)} + \Psi^* \varepsilon^*_{,(j}\Psi_{,k)} - (\varepsilon + \varepsilon^*)\Psi_{,(j}\Psi^*_{,k)}. \tag{12}$$

The field equations in empty space where Ψ vanishes can be compactly written in the form

$$(\xi^* \xi - 1)\nabla^2 \xi = 2\xi^* \nabla \xi \cdot \nabla \xi, \tag{13}$$

where ξ is a complex Ernst potential defined by the relation,

$$\frac{\xi - 1}{\xi + 1} = f + i\chi. \tag{14}$$

Equation (13) possesses a number of invariant properties. Taking the complex conjugate, we see that if ξ is a solution of (13), so is ξ^*. Ehlers [7] some time ago noticed that one can replace ξ by $e^{i\alpha}\xi$ without altering the form of the equation. It is also invariant with respect to the following fractional transformation,

$$\xi \to \frac{(1+\beta)\xi + \beta^*}{1 + \beta^* + \beta\xi}, \tag{15}$$

where β is an arbitrary complex constant. When $\beta = -1$ (15) reduces to the inversion transformation $\xi \to \xi^{-1}$.

We shall now show that the Ernst potential ξ for the stationary vacuum spacetime could be treated as a complex electromagnetic potential in some stationary electrovac gravitational field. Let us assume that $\Psi = \sqrt{\kappa}\xi$ where ξ is any solution of (13), κ is a positive constant and

$$f = \kappa(\xi\xi^* - 1), \qquad \chi = \alpha, \tag{16}$$

α being a real constant. In this case $\varepsilon = -\kappa + i\alpha = \text{const}$. It is now apparent that Equation (10) is trivially satisfied and Equation (11) reduces to Equation (13). Therefore (16) describes a solution of coupled Einstein–Maxwell equations. In order to assure the asymptotic flatness of the gravitational field, f should tend to 1 at spacial infinity, implying that $\xi \to \sqrt{1 + 1/\kappa}$ asymptotically. Using the transformation (15), we can always satisfy this condition.

The remaining metric coefficients one obtains from Equation (12), which now simplifies to

$$f^2 R_{ik}^{(3)} = 2\kappa \Psi_{,(j} \Psi^*_{,k)}, \tag{17}$$

and Equation (5), which now reduces to

$$f^2 \nabla \times \bar{w} = i(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi). \tag{18}$$

Solutions of those equations provide us with w_i and h_{ij}.

As an example, let us consider the Kerr metric, which is described by the complex function $\xi = px - igy$, where x and y are oblate spheroidal coordinates and $p^2 + g^2 = 1$. Using the transformation (15) with $\beta = \kappa \pm \sqrt{\kappa(\kappa+1)}$ we obtain

$$\xi = \frac{(1+\beta)(px - igy) + \beta}{1 + \beta + \beta(px - igy)}, \tag{19}$$

which satisfies the required boundary condition at $x \to \infty$.

The metric we shall take in the form,

$$ds^2 = f(dt + w\,d\varphi)^2 - f^{-1}\left[e^{2\gamma}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right], \tag{20}$$

where

$$f = \frac{p^2x^2 + g^2y^2 - 1}{[px + 1 + \beta^{-1}]^2 + g^2y^2}. \tag{21}$$

Equation (17) leads to

$$e^{2\gamma} = p^2x^2 + g^2y^2 - 1, \tag{22}$$

and from (18) we obtain

$$w = -\frac{g(1-y^2)\,[2(1+\beta^{-1})px + 1 + (1+\beta^{-1})^2]}{p(p^2x^2 + g^2y^2 - 1)}. \tag{23}$$

Introducing the spherical coordinates r and θ, which are related to x and y by

$$x = \frac{1+\beta^{-1}}{mp}(r-m), \qquad y = \cos\theta, \tag{24}$$

and identifying g^2 with $(1+\beta^{-1})^2 a^2/m^2$, where a is the Kerr parameter, we obtain the Kerr–Newman solution with $e^2 = m^2(1+2\beta)/(1+\beta)^2$.

This procedure, when applied simultaneously with Kinnersley's method, leads to a new class of exact, stationary, asymptotically flat Einstein–Maxwell solutions. It also throws some light on the structure of the space of stationary Einstein–Maxwell solutions and indicates that there is a new relation between vacuum stationary solutions and Einstein–Maxwell solutions.

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