THE SEARCH FOR THE COMMON SYMMETRY OF PAIRING+ +QUADRUPOLE FORCES IN NUCLEAR THEORY

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Pairing forces connected with the R_3 group and quadrupole forces with the SU_3 group were taken together to generate the common symmetry group. It has been proved that the resulting group is the symplectic group in $(N+1)(N+2)$ dimension, where N is the major shell number. The special case of $Sp(6)$ for $N=1$ is discussed in detail.

It is not the aim of the paper to give account for the great usefulness of the pairing+ +quadrupole scheme in nuclear theory. It is even difficult to mention all papers dealing with the problem. Shortly speaking, the common one-particle shell-model potential is unable to describe all particularities in the real interactions among nucleons in the nucleus. Besides the one-particle potential we are dealing with the residual two-particle interactions which we divide, in an artificial way, into two parts: a short range part and a long range one. The approximation of short range forces is almost exactly the pairing forces which couple the two-particle state to the overall $J = 0$. The long range forces are approximated by the quadrupole forces which are the scalar product of two irreducible tensor operators proportional to the $\mathbf{Y}^{(2)}$.

In principle we know how to deal with pairing or quadrupole alone. The application of group theory is of great advantage in the treatment of these problems. The symmetry group connected with the quadrupole forces is the well known SU_3 group. The orthogonal groups R_3, R_3, and R_3 are suitable to treat the particular problems of pairing forces. A word has to be said about what we mean here by symmetry. Strictly speaking, a symmetry group of particular interaction is the group of transformations which leaves the interaction invariant. But the modern application of group theory introduces also such transformations, under which the Hamiltonian transforms in an established way. Usually one begins to study the problem by dividing the Hamiltonian into several simpler parts which are taken as generators of the group to be found. The next step is to construct the bases for the irreducible representations.

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The work deals with the first problem *i.e.* with the search for the group of transformations which are generated by the operators of the pairing+quadrupole forces.

The first two sections repeat the known problem of pairing and quadrupole forces taken separately but in a way suitable, for further application. The next section, deals with the commutation relations of the operators taken from pairing+quadrupole interactions. The search for the complete set of infinitesimal operators is the first part of the section, and the identification of the group is the second one. The last part is devoted to the special case of the general problem.

1. The pairing forces

The group theoretical approach to the pairing forces was firstly introduced by several authors independly [1–8]. It was proved in the papers we mentioned that the operators of pairing forces in $L-S$ coupling generate the R_3 group for one kind of nucleons and the R_8 group for both proton and neutron. In $j-j$ coupling there are, accordingly, the groups R_3 and R_8.

We restrict the problem by the following assumptions:

1. We start with zero order shell-model structure of levels in $L-S$ coupling, *i.e.* with one particle l levels.

2. We restrict the problem to, so-called, single closed shell nuclei, that is, we do not assume the mixing of protons and neutrons.

3. The residual interactions between the nucleons on the major shell (the harmonic oscillator shell) are taken into account but not the interactions between the nucleons on the different major shells. The last restriction is introduced not by pairing but by quadrupole forces, as will be seen later.

Under these assumptions the pairing interaction is conventionally written (see for example [5]) as

$$\mathscr{H}_p = -G\sum_{lml'm'}(-1)^{l+l'-m-m'}a^+_{lm\uparrow}a^+_{l-m\downarrow}a_{l'-m'\downarrow}a_{l'm'\uparrow} \equiv -G\mathscr{P}_+\mathscr{P}_- \tag{1}$$

where

$$\mathscr{P}_+ = \sum_{lm}(-1)^{l-m}a^+_{lm\uparrow}a^+_{l-m\downarrow}$$

$$\mathscr{P}_- = \sum_{lm}(-1)^{l-m}a_{l-m\downarrow}a_{lm\uparrow} \tag{2}$$

$\uparrow$, $\downarrow$ denote the third component of the ordinary spin and G is the strenght of the pairing forces taken as positive constant. The sums are taken over all l belonging to the major shell of the harmonic oscillator.

The quasispin method begins with the recognition of the commutation relations

$$[\mathscr{P}_+, \mathscr{P}_-] = 2\mathscr{P}_0 \tag{3}$$

where

$$\mathscr{P}_0 = \frac{1}{2}\sum_{lm}\{a^+_{lm\uparrow}a_{lm\uparrow}+a^+_{lm\downarrow}a_{lm\downarrow}-1\} \tag{4}$$

and

$$[\mathcal{P}_0, \mathcal{P}_+] = \mathcal{P}_+ \quad [\mathcal{P}_0, \mathcal{P}_-] = -\mathcal{P}_-. \tag{5}$$

We can treat the operators $\mathcal{P}_+, \mathcal{P}_-, \mathcal{P}_0$, according to their commutation relations (3), (5), as generators of infinitesimal transformations of the rotations in the abstract, so-called quasispin space, in three dimensions. The whole formalism of the ordinary spin may be immediately applied to the quasispin.

In particular, the square of the total quasispin will be connected with the $\mathcal{P}_+\mathcal{P}_-$ operators by the usual expression

$$\mathcal{P}^2 = \mathcal{P}_+\mathcal{P}_- + \mathcal{P}_0^2 - \mathcal{P}_0. \tag{6}$$

With the help of the relation (6) we can diagonalize the Hamiltonian (1) in the base $|\mathcal{P}, \mathcal{P}_0\rangle$ (4).

For more than one level, we have to add to the Hamiltonian (1) the single particle energy and the Hamiltonian takes on the form

$$\mathcal{H} = \sum_{lm} \varepsilon_l (a^+_{lm\uparrow} a_{lm\uparrow} + a^+_{lm\downarrow} a_{lm\downarrow}) - G \sum_{lml'm'} (-1)^{l+l'-m-m'} a^+_{lm\uparrow} a^+_{l-m\downarrow} a_{l'-m'\downarrow} a_{l'm'\uparrow}. \tag{7}$$

The single particle Hamiltonian (7) is not diagonal in the base $|\mathcal{P}, \mathcal{P}_0\rangle$ and we are faced by a problem essentially equivalent to the problem of adding several spins (see [9, 10]), which can also be exactly solved, using the method of angular momentum theory.

2. *The quadrupole forces and the SU_3 group*

The SU_3 group emerged in nuclear structure theory from the one particle potential of the oscillator type [12]

$$H = \frac{1}{2} (\boldsymbol{p}^2 + \boldsymbol{r}^2) \tag{8}$$

we adopt here, for simplicity, the system of units in which $m = \omega = \hbar = 1$.

The group symmetry of this Hamiltonian is the well-known SU_3 group. The infinitesimal operators of the SU_3 group are taken in following form [12]

$$-\sqrt{2} \sum_{q_1 q_2} (1q_1\, 1q_2 | 1q)\, \alpha^+_{q_1} \alpha_{q_2} = L_q \tag{9}$$

$$\sqrt{6} \sum_{q_1 q_2} (1q_1\, 1q_2 | 2q)\, \alpha^+_{q_1} \alpha_{q_2} = Q_q \tag{10}$$

where

$$\boldsymbol{\alpha} = \frac{i}{\sqrt{2}} (\boldsymbol{p} - i\boldsymbol{r}) \qquad q_1, q_2 = -1, 0, 1 \tag{11}$$

α^+ and α operators have the well-defined property of creating and anihilating the oscillator quanta. L_q are the angular momentum operators in "spherical" coordinates and

$$Q_q = \sqrt{\frac{4\pi}{5}} \{r^2 Y_{2q}(\vartheta_r, \varphi_r) + p^2 Y_{2q}(\vartheta_p, \varphi_p)\}. \tag{12}$$

The operators Q_q are like the quadrupole operators both in coordinate and momentum space. The commutation relations between them are

$$\begin{aligned}[L_q, L_{q'}] &= -\sqrt{2}\,(1q1q'|1q+q')\,L_{q+q'}\\ [Q_q, L_{q'}] &= -\sqrt{6}(2q1q'|2q+q')\,Q_{q+q'}\\ [Q_q, Q_{q'}] &= 3\sqrt{10}(2q2q'|1q+q')\,L_{q+q'}.\end{aligned} \tag{13}$$

Following Elliott [12, 13, 14] we denote the I.R. (irreducible representation) of the SU_3 group by two numbers (λ, μ). The one-particle functions of the harmonic oscillator well (8), degenerated in energy, form the basis for the special I.R.s of the SU_3 group, namely the representation $(\lambda, 0)$ where $\lambda = 0, 1, 2\ldots$ denote also the number of the major shell of harmonic oscillator. The dimension of the (λ, μ) basis

$$d = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+1) \tag{14}$$

is shortened for $(\lambda, 0)$ representations to the form

$$d = \frac{1}{2}(\lambda+1)(\lambda+2)$$

that is exactly the number of degenerated one-particle states on the $\lambda = N$ level (without spin degeneracy). In general, one needs three further quantum numbers to distinguish the function in the given I.R. However, for the $(\lambda, 0)$ representations, two number are enough to denote the functions. These are L and L_0 — the quantum numbers of orbital angular momentum. The Kronecker products of two I.R.s of SU_3 are given by Elliott [12] for the more useful representations.

Now we turn our attention to the quadrupole interaction, which we write, by definition, as

$$\boldsymbol{Q}^2 = \sum_{i>j} \boldsymbol{Q}_{ij}, \qquad \boldsymbol{Q}^2_{ij} = \frac{16\pi}{5}\sum_q (-1)^q r_i^2 r_j^2 Y_{2q}(i)\, Y_{2-q}(j). \tag{15}$$

We can consider the $\boldsymbol{Q}^2_{ij}$ as a quadrupole term of the general two-particle interaction $V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$ with the separable radial part of the form

$$v(r_i, r_j) = r_i^2 r_j^2. \tag{16}$$

We select, from (15), five one particle quadrupole operators of the form

$$Q_q = \sqrt{\frac{16\pi}{5}}\, r^2 Y_{2q} \tag{17}$$

which we write in the second quantization

$$\mathcal{Q}_q = \sum_{\nu\nu'} \langle \nu | Q_q | \nu' \rangle a_\nu^+ a_{\nu'} \tag{18}$$

where ν stands for $(Nlmm_s)$ and the summation is extended over the complete set of harmonic oscillator functions. The interaction (15) is now

$$\mathcal{Q}^2 = \sum_q (-1)^q \mathcal{Q}_q \mathcal{Q}_{-q}. \tag{19}$$

TABLE I

	$\mathscr{P}_+$	$\mathscr{P}_-$	$\mathscr{P}_0$	$\mathscr{L}_1$	$\mathscr{L}_0$	$\mathscr{L}_{-1}$	$\mathscr{Q}_2$
$\mathscr{P}_+$	0	$2\mathscr{P}_0$	$-\mathscr{P}_+$	0	0	0	$-\mathscr{B}_2^+$
$\mathscr{P}_-$	$-2\mathscr{P}_0$	0	$\mathscr{P}_-$	0	0	0	$-\mathscr{B}_2$
$\mathscr{P}_0$	$\mathscr{P}_+$	$-\mathscr{P}_-$	0	0	0	0	0
$\mathscr{L}_1$	0	0	0	0	$-\mathscr{L}_1$	$-\mathscr{L}_0$	0
$\mathscr{L}_0$	0	0	0	$\mathscr{L}_1$	0	$-\mathscr{L}_{-1}$	$2\mathscr{Q}_2$
$\mathscr{L}_{-1}$	0	0	0	$\mathscr{L}_0$	$\mathscr{L}_{-1}$	0	$\sqrt{2}\mathscr{Q}_1$
$\mathscr{Q}_2$	$\mathscr{B}_2^+$	$\mathscr{B}_2$	0	0	$-2\mathscr{Q}_2$	$-\sqrt{2}\mathscr{Q}_1$	0
$\mathscr{Q}_1$	$\mathscr{B}_1^+$	$\mathscr{B}_1$	0	$\sqrt{2}\mathscr{Q}_2$	$-\mathscr{Q}_1$	$-\sqrt{3}\mathscr{Q}_0$	0
$\mathscr{Q}_0$	$\mathscr{B}_0^+$	$\mathscr{B}_0$	0	$\sqrt{3}\mathscr{Q}_1$	0	$-\sqrt{3}\mathscr{Q}_{-1}$	0
$\mathscr{Q}_{-1}$	$\mathscr{B}_{-1}^+$	$\mathscr{B}_{-1}$	0	$\sqrt{3}\mathscr{Q}_0$	$\mathscr{Q}_{-1}$	$-\sqrt{2}\mathscr{Q}_{-2}$	$-\sqrt{18}\mathscr{L}_1$
$\mathscr{Q}_{-2}$	$\mathscr{B}_{-2}^+$	$\mathscr{B}_{-2}$	0	$\sqrt{2}\mathscr{Q}_{-1}$	$2\mathscr{Q}_{-2}$	0	$-6\mathscr{L}_0$
$\mathscr{B}_2^+$	0	$2\mathscr{Q}_2$	$-\mathscr{B}_2^+$	0	$-2\mathscr{B}_2^+$	$-\sqrt{2}\mathscr{B}_1^+$	0
$\mathscr{B}_1^+$	0	$2\mathscr{Q}_1$	$-\mathscr{B}_1^+$	$\sqrt{2}\mathscr{B}_2^+$	$-\mathscr{B}^+$	$-\sqrt{3}\mathscr{B}_0^+$	0
$\mathscr{B}_0^+$	0	$2\mathscr{Q}_0$	$-\mathscr{B}_0^+$	$\sqrt{3}\mathscr{B}_1^+$	0	$-\sqrt{3}\mathscr{B}_{-1}^+$	$2\mathscr{B}_2^+$
$\mathscr{B}_{-1}^+$	0	$2\mathscr{Q}_{-1}$	$-\mathscr{B}_{-1}^+$	$\sqrt{3}\mathscr{B}_0^+$	$\mathscr{B}_{-1}^+$	$-\sqrt{2}\mathscr{B}_{-2}^+$	$\sqrt{6}\mathscr{B}_1^+$
$\mathscr{B}_{-2}^+$	0	$2\mathscr{Q}_{-2}$	$-\mathscr{B}_{-2}^+$	$\sqrt{2}\mathscr{B}_{-1}^+$	$2\mathscr{B}_{-2}^+$	0	$2\mathscr{B}_0^+-8\mathscr{P}_+$
$\mathscr{B}_2$	$2\mathscr{Q}_2$	0	$\mathscr{B}_2$	0	$-2\mathscr{B}_2$	$-\sqrt{2}\mathscr{B}_1$	0
$\mathscr{B}_1$	$2\mathscr{Q}_1$	0	$\mathscr{B}_1$	$\sqrt{2}\mathscr{B}_2$	$-\mathscr{B}_1$	$-\sqrt{3}\mathscr{B}_0$	0
$\mathscr{B}_0$	$2\mathscr{Q}_0$	0	$\mathscr{B}_0$	$\sqrt{3}\mathscr{B}_1$	0	$-\sqrt{3}\mathscr{B}_{-1}$	$-2\mathscr{B}_2$
$\mathscr{B}_{-1}$	$2\mathscr{Q}_{-1}$	0	$\mathscr{B}_{-1}$	$\sqrt{3}\mathscr{B}_0$	$\mathscr{B}_{-1}$	$-\sqrt{2}\mathscr{B}_{-2}$	$-\sqrt{6}\mathscr{B}_1$
$\mathscr{B}_{-2}$	$2\mathscr{Q}_{-2}$	0	$\mathscr{B}_{-2}$	$\sqrt{2}\mathscr{B}_{-1}$	$2\mathscr{B}_{-2}$	0	$-2\mathscr{B}_0-8\mathscr{P}_-$

The table of commutators for the $Sp(6)$ group

$\mathscr{Q}_1$	$\mathscr{Q}_0$	$\mathscr{Q}_{-1}$	$\mathscr{Q}_{-2}$	$\mathscr{B}_2^+$	$\mathscr{B}_1^+$
$-\mathscr{B}_1^+$	$-\mathscr{B}_0^+$	$-\mathscr{B}_{-1}^+$	$-\mathscr{B}_{-2}^+$	0	0
$-\mathscr{B}_1$	$-\mathscr{B}_0$	$-\mathscr{B}_{-1}$	$-\mathscr{B}_{-2}$	$-2\mathscr{Q}_2$	$-2\mathscr{Q}_1$
0	0	0	0	$\mathscr{B}_2^+$	$\mathscr{B}_1^+$
$-\sqrt{2}\mathscr{Q}_2$	$-\sqrt{3}\,\mathscr{Q}_1$	$-\sqrt{3}\,\mathscr{Q}_0$	$-\sqrt{2}\mathscr{Q}_{-1}$	0	$-\sqrt{2}\mathscr{B}_2^+$
$\mathscr{Q}_1$	0	$-\mathscr{Q}_{-1}$	$-2\mathscr{Q}_{-2}$	$2\mathscr{B}_2^+$	$\mathscr{B}_1^+$
$\sqrt{3}\mathscr{Q}_0$	$\sqrt{3}\,\mathscr{Q}_{-1}$	$\sqrt{2}\,\mathscr{Q}_{-2}$	0	$\sqrt{2}\mathscr{B}_1^+$	$\sqrt{3}\mathscr{B}_0^+$
0	0	$\sqrt{18}\mathscr{L}_1$	$6\mathscr{L}_0$	0	0
0	$-\sqrt{27}\mathscr{L}_1$	$-3\mathscr{L}_0$	$\sqrt{18}\mathscr{L}_{-1}$	0	$\sqrt{6}\mathscr{B}_2^+$
$\sqrt{27}\mathscr{L}_1$	0	$-\sqrt{27}\mathscr{L}_{-1}$	0	$-2\mathscr{B}_2^+$	$\mathscr{B}_1^+$
$3\mathscr{L}_0$	$\sqrt{27}\mathscr{L}_{-1}$	0	0	$-\sqrt{6}\mathscr{B}_1^+$	$-\mathscr{B}_0^+-8\mathscr{P}_+$
$-\sqrt{18}\mathscr{L}_{-1}$	0	0	0	$8\mathscr{P}_+-2\mathscr{B}_0^+$	$-\sqrt{6}\mathscr{B}_{-1}^+$
0	$2\mathscr{B}_2^+$	$\sqrt{6}\mathscr{B}_1^+$	$2\mathscr{B}_0^+-8\mathscr{P}_+$	0	0
$-\sqrt{6}\,\mathscr{B}_2^+$	$-\mathscr{B}_1^+$	$\mathscr{B}_0^++8\mathscr{P}_+$	$\sqrt{6}\,\mathscr{B}_{-1}^+$	0	0
$-\mathscr{B}_1^+$	$-8\mathscr{P}_+-2\mathscr{B}_0^+$	$-\mathscr{B}_{-1}^+$	$2\mathscr{B}_{-2}^+$	0	0
$\mathscr{B}_0^++8\mathscr{P}_+$	$-\mathscr{B}_{-1}^+$	$-\sqrt{6}\,\mathscr{B}_{-2}^+$	0	0	0
$\sqrt{6}\,\mathscr{B}_{-1}^+$	$2\mathscr{B}_{-2}^+$	0	0	0	0
0	$-2\mathscr{B}_2$	$-\sqrt{6}\mathscr{B}_1$	$-2\mathscr{B}_0-$ $-8\mathscr{P}_-$	0	0
$\sqrt{6}\,\mathscr{B}_2$	$\mathscr{B}_1$	$8\mathscr{P}_--\mathscr{B}_0$	$-\sqrt{6}\mathscr{B}_{-1}$	0	$\sqrt{24}\mathscr{Q}_2$
$\mathscr{B}_1$	$2\mathscr{B}_0-8\mathscr{P}_-$	$+\mathscr{B}_{-1}$	$-2\mathscr{B}_{-2}$	$-4\mathscr{Q}_2$	$2\mathscr{Q}_1-\sqrt{108}\mathscr{L}_1$
$8\mathscr{P}_--\mathscr{B}_0$	$\mathscr{B}_{-1}$	$+\sqrt{6}\mathscr{B}_{-2}$	0	$\sqrt{72}\mathscr{L}_1-\sqrt{24}\mathscr{Q}_1$	$-16\mathscr{P}_0-6\mathscr{L}_0-$ $-2\mathscr{Q}_0$
$-\sqrt{6}\,\mathscr{B}_{-1}$	$-2\mathscr{B}_{-2}$	0	0	$16\mathscr{P}_0-4\mathscr{Q}_0+$ $+12\mathscr{L}_0$	$\sqrt{72}\mathscr{L}_{-1}-$ $-\sqrt{24}\mathscr{Q}_{-1}$

$\mathscr{B}_0^+$	$\mathscr{B}_{-1}^+$	$\mathscr{B}_{-2}^+$	$\mathscr{B}_2$
0	0	0	$-2\mathscr{Q}_2$
$-2\mathscr{Q}_0$	$-2\mathscr{Q}_{-1}$	$-2\mathscr{Q}_{-2}$	0
$\mathscr{B}_0^+$	$\mathscr{B}_{-1}^+$	$\mathscr{B}_{-2}^+$	$-\mathscr{B}_2$
$-\sqrt{3}\mathscr{B}_1^+$	$-\sqrt{3}\mathscr{B}_0^+$	$-\sqrt{2}\mathscr{B}_{-1}^+$	0
0	$-\mathscr{B}_{-1}^+$	$-2\mathscr{B}_{-2}^+$	$2\mathscr{B}_2$
$\sqrt{3}\mathscr{B}_{-1}^+$	$\sqrt{2}\mathscr{B}_{-2}^+$	0	$\sqrt{2}\mathscr{B}_1$
$-2\mathscr{B}_2^+$	$-\sqrt{6}\mathscr{B}_1^+$	$8\mathscr{P}_+-2\mathscr{B}_0^+$	0
$\mathscr{B}_1^+$	$-\mathscr{B}_0^+-8\mathscr{P}_+$	$-\sqrt{6}\mathscr{B}_{-1}^+$	0
$2\mathscr{B}_0^++8\mathscr{P}_+$	$\mathscr{B}_{-1}^+$	$-2\mathscr{B}_{-2}^+$	$2\mathscr{B}_2$
$\mathscr{B}_{-1}^+$	$\sqrt{6}\mathscr{B}_{-2}^+$	0	$\sqrt{6}\mathscr{B}_1$
$-2\mathscr{B}_{-2}^+$	0	0	$2\mathscr{B}_0+8\mathscr{P}_-$
0	0	0	0
0	0	0	0
0	0	0	$4\mathscr{Q}_2$
0	0	0	$\sqrt{72}\mathscr{L}_1+\sqrt{24}\mathscr{Q}_1$
0	0	0	$4\mathscr{Q}_0+12\mathscr{L}_0-$ $-16\mathscr{P}_0$
$-4\mathscr{Q}_2$	$-\sqrt{72}\mathscr{L}_1-\sqrt{24}\mathscr{Q}_1$	$16\mathscr{P}_0-4\mathscr{Q}_0-$ $-12\mathscr{L}_0$	0
$\sqrt{108}\mathscr{L}_1+2\mathscr{Q}_1$	$6\mathscr{L}_0-16\mathscr{P}_0-2\mathscr{Q}_0$	$\sqrt{72}\mathscr{L}_{-1}+$ $+\sqrt{24}\mathscr{Q}_{-1}$	0
$16\mathscr{P}_0+4\mathscr{Q}_0$	$\sqrt{108}\mathscr{L}_{-1}+2\mathscr{Q}_{-1}$	$-4\mathscr{Q}_{-2}$	0
$2\mathscr{Q}_{-1}-\sqrt{108}\mathscr{L}_{-1}$	$\sqrt{24}\mathscr{Q}_{-2}$	0	0
$-4\mathscr{Q}_{-2}$	0	0	0

$\mathcal{B}_1$	$\mathcal{B}_0$	$\mathcal{B}_{-1}$	$\mathcal{B}_{-2}$
$-2\mathcal{Q}_1$	$-2\mathcal{Q}_0$	$-2\mathcal{Q}_{-1}$	$-2\mathcal{Q}_{-2}$
0	0	0	0
$-\mathcal{B}_1$	$-\mathcal{B}_0$	$-\mathcal{B}_{-1}$	$-\mathcal{B}_{-2}$
$-\sqrt{2}\mathcal{B}_2$	$-\sqrt{3}\mathcal{B}_1$	$-\sqrt{3}\mathcal{B}_0$	$-\sqrt{2}\mathcal{B}_{-1}$
$\mathcal{B}_1$	0	$-\mathcal{B}_{-1}$	$-2\mathcal{B}_{-2}$
$\sqrt{3}\mathcal{B}_0$	$\sqrt{3}\mathcal{B}_{-1}$	$\sqrt{2}\mathcal{B}_{-2}$	0
0	$2\mathcal{B}_2$	$\sqrt{6}\mathcal{B}_1$	$2\mathcal{B}_0+8\mathcal{P}_-$
$-\sqrt{6}\mathcal{B}_2$	$-\mathcal{B}_1$	$\mathcal{B}_0-8\mathcal{P}_-$	$\sqrt{6}\mathcal{B}_{-1}$
$-\mathcal{B}_1$	$8\mathcal{P}_--2\mathcal{B}_0$	$-\mathcal{B}_{-1}$	$2\mathcal{B}_{-2}$
$\mathcal{B}_0-8\mathcal{P}_-$	$-\mathcal{B}_{-1}$	$-\sqrt{6}\mathcal{B}_{-2}$	0
$\sqrt{6}\mathcal{B}_{-1}$	$2\mathcal{B}_{-2}$	0	0
0	$4\mathcal{Q}_2$	$\sqrt{24}\mathcal{Q}_1-\sqrt{72}\mathcal{L}_1$	$4\mathcal{Q}_0-16\mathcal{P}_0-$ $-12\mathcal{L}_0$
$-\sqrt{24}\mathcal{Q}_2$	$\sqrt{108}\mathcal{L}_1-2\mathcal{Q}_1$	$16\mathcal{P}_0+6\mathcal{L}_0+2\mathcal{Q}_0$	$\sqrt{24}\mathcal{Q}_{-1}-\sqrt{72}\mathcal{L}_{-1}$
$-\sqrt{108}\mathcal{L}_1-2\mathcal{Q}_1$	$-16\mathcal{P}_0-4\mathcal{Q}_0$	$\sqrt{108}\mathcal{L}_{-1}-2\mathcal{Q}_{-1}$	$4\mathcal{Q}_{-2}$
$16\mathcal{P}_0+2\mathcal{Q}_0-$ $-6\mathcal{L}_0$	$-\sqrt{108}\mathcal{L}_{-1}-$ $-2\mathcal{Q}_{-1}$	$-\sqrt{24}\mathcal{Q}_{-2}$	0
$-\sqrt{72}\mathcal{L}_1-$ $-\sqrt{24}\mathcal{Q}_{-1}$	$4\mathcal{Q}_{-2}$	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

The next step the decisive one, was made first by Elliott. We restrict the interaction (19) to the particles on the major shell of harmonic oscillator only, which means that we restrict the summation over only (lmm_s) within a given shell N. The calculation of the matrix elements in (13) gives

$$\mathcal{Q}_q = \sum_l \left\{ -(2N+3) \left[\frac{l(l+1)\,(2l+1)}{(2l-1)\,(2l+3)} \right]^{\frac{1}{2}} U_q^{(2)}(l, l) + \right.$$

$$\left. + \left[\frac{6(l+1)\,(l+2)\,(N-l)\,(N+l+3)}{2l+3} \right]^{\frac{1}{2}} [U_q^{(2)}(l, l+2) + U_q^{(2)}(l+2, l)] \right\}$$

$$\equiv \sum_l \{C_1(l)\, U_q^{(2)}(l, l) + C_2(l)[U_q^{(2)}(l, l+2) + U_q^{(2)}(l+2, l)]\} \tag{20}$$

where, in general

$$U_q^{(t)}(l, l') = \frac{1}{\sqrt{2l+1}} \sum_{mm'm_s} (l'm'tq|lm)\, a^{+}_{lmm_s} a_{l'm'm_s'}. \tag{21}$$

Commutators between the defined Q_q operators give three new operators, *viz.*

$$\mathcal{L}_q = \sum_q \{l(l+1)\,(2l+1)\}^{\frac{1}{2}} U_q^{(1)}(l, l) \tag{22}$$

which are identified as angular momentum operators. In such a way we are led once more to the SU_3 group generated now by the eight operators (20) and (22). The commutation relations between these operators are the same as in (13). The commutators are obtained with the help of the relation

$$[U_q^{(t)}(l, l'),\, U_p^{(s)}(k, k')] = \sum_{r,v} (2r+1)^{\frac{1}{2}} (tq\, sp|rv) \times$$

$$\times \{(-1)^{t+s-r} \delta_{l'k} W(ts\, lk'; rl')\, U_v^{(r)}(l, k') - \delta_{lk'} W(tsl'k; rk')\, U_v^{(r)}(k, l'). \tag{23}$$

The SU_3 group of the harmonic oscillator Hamiltonian and SU_3 group of quadrupole interaction are the same group [15, 12]. The proof of this is based on the fact that the operators taken from (12)

$$\frac{1}{2} \{r^2 Y_{2q}(\vartheta_r, \varphi_r) + p^2 Y_{2q}(\vartheta_p, \varphi_p)\} \tag{24}$$

have the same matrix elements between the states of the same major oscillator shell, as the operators

$$r^2 Y_{2q}(\vartheta_r, \varphi_r). \tag{25}$$

It is not, however, true if we take these operators between the states of different shells: the operator (24) vanishes while the (25) does not. It means that the SU_3 symmetry is connected with the quadrupole interaction only on the boundary condition restricting the interaction between the particles on the same major shell of harmonic oscillator. The Hamiltonian with the harmonic oscillator well on the one hand and the quadrupole interaction on the other are, however, different in respect to the transformations of the same SU_3 group. The first is scalar while the second is the tensor operator of the second rank. Nevertheless, the well-known simple character of quadrupole interaction under the SU_3 group was widely exploited in nuclear structure theory. This is the reason for searching for the common symmetry group for pairing+quadrupole forces.

3. The pairing+quadrupole forces — the group $Sp\{(N+1)\,(N+2)\}$

The Hamiltonian consists now of three parts (1), (8) and (19)

$$\mathscr{H} = \mathscr{H}_0+\alpha\mathscr{H}_p+\beta\mathscr{Q}^2 \tag{26}$$

where α and β are the relative strength parameters. $\mathscr{H}_0$ is the one particle Hamiltonian of the harmonic oscillator which in the second quantization takes on the shape

$$\mathscr{H}_0 = \left(N+\frac{3}{2}\right)\sum_{lmm_s} \overset{+}{a}_{lmm_s} a_{lmm_s} \tag{27}$$

that is essentially equivalent to the $\mathscr{P}^0$ quasisipn operator (4). The orthogonal group R_3 is connected with three pairing operators and the SU_3 group — with five quadrupole operators. If these two sets of operators commute, the symmetry group would be the simple product of $R_3 \times SU_3$. But it is not the case and we have to commute the operators until reaching the closed set of them that can be attached to the symmetry group. The group to be found has to have the R_3 and SU_3 as subgroups.

The operators which have to be commuted are taken from (2) and (4) ($\mathscr{P}$ operators) and from (20) and (22) ($\mathscr{Q}$ and $\mathscr{L}$ operators).

The first step is to calculate twenty-four commutators between three pairing operators and eight SU_3 operators. Nine commutators which can be symbolically written as

$$[\mathscr{L}, \mathscr{P}] = 0 \tag{28}$$

are equel to zeros because the pairing operators are scalars in the angular momentum space. Five further commutators are, by calculation, also equal to zeros; they are

$$[\mathscr{Q}_q, \mathscr{P}_0] = 0 \qquad q = 2, 1, 0, -1, -2. \tag{29}$$

We are left with the rest

$$[\mathscr{Q}_q, \mathscr{P}_+] \equiv B_q^+ \qquad [\mathscr{Q}_q, \mathscr{P}_-] \equiv \mathscr{B}_q \tag{30}$$

which, after straightforward calculation, gives

$$\mathscr{B}_q^+ = \sum_l \{C_1(l) A_q^{+(2)}(l, l) + 2C_2(l) A_q^{+(2)}(l, l+2)\} \tag{31}$$

$$\mathscr{B}_q = \sum_l \{C_1(l) A_q^{(2)}(l, l) + 2C_2(l) A_q^{(2)}(l, l+2)\}$$

where

$$A_q^{+(t)}(l, l') = \frac{1}{\sqrt{2l+1}} \sum_{mm'} (-1)^{l'-m'}(l'm'tq|lm)\,(\overset{+}{a}_{lm\uparrow}\overset{+}{a}_{l'-m'\downarrow} - \overset{+}{a}_{lm\downarrow}\overset{+}{a}_{l'-m'\uparrow}) \tag{32}$$

$A_q^{(t)}$ is the hermitian conjugate of $A_q^{+(t)}$, and C_i are the same as in the relation (20).

In such a way the first step of commutation has introduced ten new operators $\mathscr{B}_l^+$, $\mathscr{B}_q$ with $q = 2, 1, 0, -1, -2$ which, however, are built in the same way from $A_q^{+(2)}$ and $A_q^{(2)}$ operators as the $\mathscr{Q}_q$ were from the $U_q^{(2)}$. Note that

$$A_q^{(t)}(l, l+2) = (-1)^t A_q^{(t)}(l+2, l). \tag{33}$$

The second step is to commute the ten new operators with themselves and with eleven $\mathfrak{P}$, $\mathscr{L}$ and $\mathscr{Q}$ operators to obtain 155 commutators. Thirty of them give zeros:

$$[\mathscr{B}^+, \mathfrak{P}_+] = [\mathscr{B}, \mathfrak{P}_-] = [\mathscr{B}^+, \mathscr{B}^+] = [\mathscr{B}, \mathscr{B}] = 0. \tag{34}$$

Fifty of them give the operators already introduced:

$$[\mathscr{B}_q^+, \mathscr{L}_{q'}] = -\sqrt{6}(2q1q'|2q+q')\,\mathscr{B}_{q+q'}^+ \tag{35}$$

$$[\mathscr{B}_q, \mathscr{L}_{q'}] = -\sqrt{6}(2q1q'|2q+q')\,\mathscr{B}_{q+q'}$$

$$[\mathscr{B}_q^+, \mathfrak{P}_-] = [\mathscr{B}_q, \mathfrak{P}_+] = 2\mathscr{Q}_q \tag{36}$$

$$[\mathscr{B}_q^+, \mathfrak{P}_0] = -\mathscr{B}_q^+\ [\mathscr{B}_q, \mathfrak{P}_0] = \mathscr{B}_q.$$

Unfortunately, seventy-five remaining commutators of the type

$$[\mathscr{Q}, \mathscr{B}],\ [\mathscr{Q}, \mathscr{B}^+] \text{ and } [\mathscr{B}^+, \mathscr{B}]$$

give new operators. For the sake of exactness we write down the results of the second step of calculations.

$$[\mathscr{B}_q^+, \mathscr{B}_{q'}] = -4\sum_{r,l} \sqrt{2r+1}\,(2q2q'|r\nu)\times$$

$$\times\{[C_1^2(l)\,W(22ll; rl) + C_2^2(l-2)\,W(22ll; rl-2) + C_2^2(l)\,W(22ll; rl+2)]\,U_\nu^{(r)}(l, l) +$$

$$+ C_2(l)[C_1(l+2)\,W(22l+2l; rl+2) + C_1(l)\,W(22ll+2; rl)][U_\nu^{(r)}(l+2, l) + U_\nu^{(r)}(l, l+2)] +$$

$$+ C_2(l)C_2(l+2)\,W(22l+4l; rl+2)[U_\nu^{(r)}(l+4, l) + U_\nu^{(r)}(l, l+4)]\} +$$

$$+ \frac{4}{5}(-1)^q \delta_{q-q'} \sum_l \{C_1^2(l) + 2C_2^2(l)\}, \tag{37}$$

$$[Q_q, \mathscr{B}_{q'}^+] = 2\sum_{rl} \sqrt{2r+1}\,(2q2q'|r\nu)\times$$

$$\times\{[C_1^2(l)\,W(22ll; rl) + C_2^2(l-2)\,W(22ll; rl-2) + C_2^2(l)\,W(22ll; rl+2)]A_\nu^{+(r)}(l, l) +$$

$$+[1+(-1)^r]C_2(l)\,[C_1(l+2)\,W(22l+2l; rl+2) + C_1(l)\,W(22ll+2; rl)]\,A_\nu^{+(r)}(l, l+2) +$$

$$+[1+(-1)^r]C_2(l)\,C_2(l+2)\,W(22l+4l; rl+2)\,A_\nu^{+(r)}(l, l+4); \tag{38}$$

the summation over r is extended for 0, 1, 2, 3, 4.

The commutators

$$[\mathcal{Q}_q, \mathcal{B}_{q'}] \tag{39}$$

have exactly the shape of (38) with the change $A^+ \to A$. The main part of calculations leading to the results given above was obtaining the commutators between the U and A. Using the standard method we get

$$[U_q^{(t)}(l, l'), A_p^{+(s)}(k, k')] = \sum_r \sqrt{2r+1}\,(tqsp|r\nu)\,(-1)^{t+r}\times \tag{40}$$
$$\times\{(-1)^s\delta_{l'k}W(tslk'; rl')\,A_\nu^{+(r)}(l, k') + \delta_{l'k'}W(tslk; rl')A_\nu^{+(r)}(l, k)\},$$

$$[U_q^{(t)}(l, l'), A_p^{+(s)}(k, k')] = \sum_r \sqrt{2r+1}\,(tqsp|r\nu)\times \tag{41}$$
$$\times\{\delta_{lk'}W(tsl'k; rl)A_\nu^{(r)}(k, l') + (-1)^{t+s}\delta_{lk}W(tsl'k'; rl)A_\nu^{(r)}(l', k')\},$$

$$[A_q^{+(t)}(l, l'), A_p^{(s)}(k, k')] = -\sum_r \sqrt{2r+1}\,(tqsp|r\nu)\times$$
$$\times\{(-1)^t\delta_{l'k'}W(tslk; rl')\,U_\nu^{(r)}(l, k) + (-1)^s\delta_{lk}W(tsl'k'; rl)\,U_\nu^{(r)}(l', k') + \tag{42}$$
$$+\delta_{lk'}W(tsl'k; rl)\;U_\nu^{(r)}(l', k') + (-1)^{t+s}\delta_{l'k}W(tslk'; rl')\,U_\nu^{(r)}(l, k')\} +$$
$$+2(2t+1)^{-1}\delta_{ts}\delta_{q-p}(-1)^q[(-1)^t\delta_{lk}\delta_{l'k'} + \delta_{lk'}\delta_{l'k}].$$

The third and further steps of commutations, although possible with the help of general formulas (23) and (40)–(42), are rather complicated and would lead to the linear combinations of A^+, A, U operators with different coefficients and with all possible ranks. The number of all such linear independent combinations is, of course, equal to the number of the operators A^+, A, U which can be constructed for one major shell. We change, at the moment, the base of infinitesimal operators and turn to the calculation of all possible A^+, A, U operators, which by the (23) and (40)–(42) form a complete set of operators. By the definitions (21), (32) and with the help of (33) we get the following numbers

$$\frac{1}{4}\{(N+1)\,(N+2)\}^2$$

$$\frac{1}{8}(N+2)\,(N^2+3N+4)$$

$$\frac{1}{8}(N+2)\,(N^2+3N+4)$$

for the U, A^+ and A operators, respectively, on the N major shell of harmonic oscillator. The total number of the operators in the closed set is

$$\frac{1}{2}(N+1)\,(N+2)\,\{(N+1)\,(N+2)+1\} \tag{43}$$

which suggests that the symmetry group connected with them is either the symplectic group in $(N+1)\,(N+2)$ dimension or the orthogonal group in $(N+1)\,(N+2)+1$ dimensions.

To choose the right group we have, once more, to change the basis of infinitesimal operators from A^+, A, U into

$$\begin{aligned}\mathscr{R}_{ij} &\equiv a_{i\uparrow}^{+}a_{j\uparrow}+a_{j\downarrow}a_{i\downarrow}^{+}\\ \mathscr{S}_{ij} &\equiv a_{i\uparrow}^{+}a_{j\downarrow}^{+}-a_{i\downarrow}^{+}a_{j\uparrow}^{+}\\ \mathscr{T}_{ij} &\equiv a_{i\downarrow}a_{j\uparrow}-a_{i\uparrow}a_{j\downarrow}\end{aligned} \tag{44}$$

where indexes i, j denote (lm) and (l, m') from the major shell N. The more complicated operators A^+, A, U are built exactly from the (44) operators and the number of these two sets of operators are equal. This means that the (44) operators can be equivalently taken as generators of the same group. The set (44) is of course closed because of the following commutation relations

$$\begin{aligned}[\mathscr{R}_{ij}, \mathscr{R}_{kl}] &= \delta_{jk}\mathscr{R}_{il}-\delta_{li}\mathscr{R}_{kj}\\ [\mathscr{S}_{ij}, \mathscr{S}_{kl}] &= [\mathscr{T}_{ij}, \mathscr{T}_{kl}] = 0\\ [\mathscr{R}_{ij}, \mathscr{S}_{kl}] &= \delta_{jk}\mathscr{S}_{il}+\delta_{jl}\mathscr{S}_{ik}\\ [\mathscr{R}_{ij}, \mathscr{T}_{kl}] &= -\delta_{ik}\mathscr{T}_{jl}-\delta_{il}\mathscr{T}_{jk}\\ [\mathscr{S}_{ij}, \mathscr{T}_{kl}] &= \delta_{jk}\mathscr{R}_{ij}+\delta_{ik}\mathscr{R}_{jl}+\delta_{jl}\mathscr{R}_{ik}+\delta_{il}\mathscr{R}_{jk}.\end{aligned} \tag{45}$$

Moreover, the relations (45) are known [19] to be the commutation relations for the infinitesimal operators of the symplectic group. This is the very end of searching for the symmetry group emerging from the pairing plus quadrupole forces. The group we have found is

$$Sp(n) \quad \text{where} \quad n = (N+1)\,(N+2). \tag{46}$$

The result of (46) is not unimportant. If we take the simplest two particle operators from the N shell

$$a^+a;\ a^+a^+;\ aa \tag{47}$$

the resulting group for them is the orthogonal group in $2(N+1)\,(N+2)$ dimensions [7]. The usefulness of the result (46) can be measured by simplification given by the group $Sp\{(N+1)\,(N+2)\}$ as compared with the group $R\,\{2(N+1)\,(N+2)\}$.

Let us take as a simple illustration the $N = 1$ shell, with the only spatial state $l = 1$. There are six spin-spatial one particle states and sixty-six operators a^+a, a^+a^+, aa respectively. They form the closed set of generators of the group R (12). The result (46) shows that one can take a simpler group, namely Sp (6) group to deal with pairing and quadrupole forces. In this simpler case we can take, instead of (44) operators, the physical operators $\mathscr{P}$, $\mathscr{L}$, $\mathscr{Q}$ and $\mathscr{B}$, whose number is exactly twenty-one, as for the Sp (6) group, and which form the complete set of infinitesimal operators (see Table). With the help of the Table I we can construct, using the standard technique of group theory, the Casimir operator for the Sp (6) group. It gives

$$\mathscr{C} = \frac{1}{96}\{8\mathscr{P}^2+3\mathscr{L}^2+\mathscr{Q}^2-\sum_q(-1)^q\mathscr{B}_q^+\mathscr{B}_{-q}-40\mathscr{P}_0\} \tag{48}$$

with the meaning of the letters given before.

The $\mathscr{B}$ operators take on the form of (31) but without the terms containing $A(l, l+2)$. These simplifications lead to the conclusion that the $\mathscr{B}$-part of (48) is the kind of interaction which couples the pair to the total $L = 2$; it is the so-called pairing-quadrupole interaction. If we include it into the total Hamiltonian and adopt the coefficients given in (48), we can get

$$\mathscr{H} \equiv k\{8\mathscr{P}_+\mathscr{P}_- + \mathscr{Q}^2 + \mathscr{B}^2\} = k\{96\mathscr{C} - 8\mathscr{P}_0(\mathscr{P}_0 - 6) - 3\mathscr{L}^2\} \tag{49}$$

where

$$\mathscr{B}^2 \equiv \sum_q (-1)^q (\mathscr{B}_q^+ \mathscr{B}_{-q}) \tag{50}$$

is the pairing-quadrupole interaction and k is the overall proportionality factor.

The Sp (6) group is the 3rd rank group with three weight-operators which can be chosen as $\mathscr{P}_0$, $\mathscr{L}_0$ and $\mathscr{Q}_0$. We can calculate an eigenvalue of the Casimir operator in these terms. The result is

$$\langle \mathscr{C} \rangle = \frac{1}{96}\{8\omega_1(\omega_1+6) + 3\omega_2(\omega_2+4) + \omega_3^2\} \tag{51}$$

where

$$\omega_1 = \max(\mathscr{P}_0) \qquad \omega_2 = \max(\mathscr{L}_0) \qquad \omega_3 = \max(\mathscr{Q}_0)$$

under the usual restriction.

The relations (49) and (51) enable us to calculate the energy of pairing+quadrupole+ $+\mathscr{B}^2$ forces among the nucleons on the $l = 1$ shell in term of the highest weight of the $Sp(6)$ operators and of the total angular momentum eigenvalue.

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