TRANSVERSITY AMPLITUDES AND THEIR APPLICATION TO THE STUDY OF COLLISIONS OF PARTICLES WITH SPIN

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The principal properties of the transversity amplitudes (*i.e.* amplitudes with spins projected on the normal to the reaction plane) are presented. The properties which are important for applications, as the simplicity of crossing relations and the behaviour under spatial reflections are discussed in detail. The applications to the study of behaviour of scattering amplitudes at thresholds and pseudothresholds and to the description of spin correlations in decay distributions are also presented. Finally many other applications (including those proposed by other authors) are shortly discussed.

1. *Introduction*

The systematic description of the scattering of spinning particles was initiated by the classical paper by Wigner [1], where the irreducible unitary representations of the Poincaré group were investigated. The study of Wigner was so thorough that the majority of relevant papers concerning the scattering of particles with spin, which have appeared since then, may be treated as examples to his considerations. On the other hand, this work was so far ahead of the interests and needs of the contemporaries, that it met with a response only after about twenty years. During this time it was thrown into the shade by the results of Dirac and his spinor formalism (it was however Dirac who suggested Wigner to study the Poincaré group). Both the quantum mechanics and the field theory have been formulated in the language of spinors. This status was petrified by editing several classical textbooks of these parts of physics. It appeared afterwards that using spinors is a rather complicated way of describing simple things. The restriction of horizons to the Dirac equation led sometimes to erroneous conclusions, *e.g.* that the spin is a relativistic effect (for a discussion *cf.* Lévy-Leblond [2]).

It is only in late fifties and early sixties when an increase in interest in unitary representations of the Poincaré group could be observed. Certainly the most important was the work of Jacob and Wick [3]. These authors proposed the formalism of helicity ampli-

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tudes (*i.e.* with spins projected on the momentum direction). It is interesting that their paper has rather a compilatory character. The notion of helicity was known since a long time (and applied mainly for massless particles) and many of relevant properties of helicity amplitudes can be found in earlier literature. The merit of Jacob and Wick was in presenting the properties of helicity amplitudes in a compact form, ready for practical applications. Thus since that time the word "helicity" appears more and more often in the scientific journals. An important role in the popularization of this formalism (especially in Europe) played the paper by Gottfried and Jackson [4] discussing the resonance decay distributions and the paper by Wick [5] where the studies of three-particle helicity states were started. The helicity amplitudes were found appropriate for investigations of multiparticle states (*cf.* Werle [6], Berman and Jacob [7]). Let us note that the following properties were decisive for the usefulness of helicity amplitudes: relatively simple relations implied by parity conservation, simple partial wave expansion, treating all spins on the same footing (no problems with higher spins), and finally applicability of the formalism both to the massive and massless particles.

Several years ago we could observe a revival of spinorial amplitudes. They were found convenient for solving some theoretical problems, although they are not suitable for phenomenological analyses. The new flush of popularity was provoked by the work of Stapp [8], who proposed to express the S-matrix theory in terms of an improved version of spinors (called the M-functions). The M-functions have a property simplifying the construction of a theory: they have no kinematical singularities [9–11]. They are not very convenient, however, in practical applications. For instance the parity conservation is rather difficult to account for (*cf.* [12]).

Another proposal, also related directly to Wigner's work, is to use the amplitudes with spins projected on an axis fixed in space. They are sometimes called the "canonical amplitudes" [13–15]. Their partial wave expansion gives the L–S coupling which is the relativistic generalization of the Russell-Saunders coupling.

In this paper we discuss the properties and some applications of the transversity amplitudes, with the spin projections on the normal to the reaction plane. These amplitudes were proposed in Refs [16] and [17], although some of their properties were known before. The reason to introduce such amplitudes was the simplification of the helicity crossing relations. The most important property of the transversity amplitudes is the diagonality of their crossing matrix. Some simplifications occur also because these amplitudes have definite parity. For instance the parity conservation means that some of them simply vanish. The principal applications are: study of kinematical singularities at thresholds and pseudothresholds, description of resonance decays, especially joint decay distributions, and finally the formulation of the quark model with additivity.

Sections 2 and 3 contain the discussion of the properties of the transversity amplitudes. The fourth section deals with the properties of density matrices and statistical tensors in the transversity basis. The applications to the study of the kinematical singularities of scattering amplitudes and the solution of the problem of singularities in cross-sections and density matrices are presented in Sections 5 and 6. Finally Section 7 contains a survey of other applications with respective references.

2. *The transversity amplitudes*

In the elementary particle physics the amplitudes of processes

$$a+b \to c+d, \tag{2.1}$$

$$\bar{c}+a \to d+\bar{b}, \tag{2.2}$$

$$\bar{d}+a \to c+\bar{b} \tag{2.3}$$

are connected by crossing relations. For the spinless particles it means simply that the amplitudes of each of these processes are represented by the same function of the total energy squared s and of the momentum transfer squared t. We say that processes (2.1)–(2.3) are the three channels of the same reaction.

The crossing relations are more complicated when particles have spins. A convenient method of describing a reaction with such particles is the use of helicity amplitudes [3]. The spin of each of the particles is then projected on its momentum direction. The measurable quantities such as the cross-sections, statistical tensors or density matrices are simply expressed in terms of the helicity amplitudes. The crossing relations contain the matrices of spin rotations through the angles called the crossing angles. The momentum direction of a particle is indeed different when observed from the centre of mass of the t channel than when observed from the centre of mass of the s channel. The crossing may also be performed by a complex Lorentz transformation and the crossing angles are the spin reorientation angles (called the Wigner angles, *cf.* [1], [5]).

The crossing relations were first derived by Trueman and Wick [20] and by Muzinich [21]. Recently it was found, however, that their formulae have some errors in signs [22] as these authors have not taken into account that the normal to the scattering plane changes the direction when crossed. The paper by Cohen-Tannoudji, Morel and Navelet [22] contains also the first rigorous derivation of the crossing relations and their overall signs. In the following we use their formulae which read

$$H^{(s)}_{cd,ab} = (-1)^{\sigma} e^{i\pi(d-a)} \sum_{a'b'c'd'} d^{s_a}(\chi_a)_{a'a} d^{s_b}(\chi_b)_{b'b} d^{s_c}(\chi_c)_{c'c} d^{s_d}(\chi_d)_{d'd} H^{(t)}_{d'b',c'a'} \tag{2.4}$$

where $H^{(s)}_{cd,ab}$ denotes the s-channel helicity amplitudes and the indices a, b, c, d denote the spin projections of corresponding particles. Quantity $\sigma = 1$ if and only if the particles a and d are fermions, otherwise $\sigma = 0$. Furthermore, $H^{(t)}$ is the t-channel helicity amplitude analytically continued to the s-channel physical region. The crossing angles are defined by the following formulae

$$\cos\chi_a = -\frac{(s+m_a^2-m_b^2)(t+m_a^2-m_c^2)+2m_a^2\Delta}{S_{ab}T_{ac}},$$

$$\sin\chi_a = -\frac{2m_a\sqrt{\Phi}}{S_{ab}T_{ac}} \tag{2.5}$$

$$\cos\chi_b = \frac{(s+m_b^2-m_a^2)(t+m_b^2-m_d^2)-2m_b^2\Delta}{S_{ab}T_{bd}},$$

$$\sin\chi_b = -\frac{2m_b\sqrt{\Phi}}{S_{ab}T_{bd}}, \tag{2.6}$$

$$\cos\chi_c = \frac{(s+m_c^2-m_d^2)(t+m_c^2-m_a^2)-2m_c^2\Delta}{S_{cd}T_{ac}},$$

$$\sin\chi_c = \frac{2m_c\sqrt{\Phi}}{S_{cd}T_{ac}}, \tag{2.7}$$

$$\cos\chi_d = -\frac{(s+m_d^2-m_c^2)(t+m_d^2-m_b^2)+2m_d^2\Delta}{S_{cd}T_{bd}},$$

$$\sin\chi_d = \frac{2m_d\sqrt{\Phi}}{S_{cd}T_{bd}} \tag{2.8}$$

where S_{xy} and T_{xy} are non-negative quantities given by the formulae

$$S_{xy}^2 = [s-(m_x-m_y)^2]\,[s-(m_x+m_y)^2], \tag{2.9}$$

$$T_{xy}^2 = [t-(m_x-m_y)^2]\,[t-(m_x+m_y)^2]. \tag{2.10}$$

Furthermore,

$$\Phi = stu-(m_b^2-m_d^2)\,(m_a^2-m_c^2)s-(m_a^2-m_b^2)\,(m_c^2-m_d^2)t- \\ -(m_a^2-m_b^2-m_c^2+m_d^2)\,(m_a^2m_d^2-m_b^2m_c^2) \tag{2.11}$$

and

$$\Delta = m_b^2+m_c^2-m_a^2-m_d^2. \tag{2.12}$$

The matrices $d^s(\chi)$ are representation matrices of rotations around the y-axis.

These helicity crossing relations are rather complicated both because of the rotation matrices and the long formulae for the crossing angles. Thus the question was asked whether it is possible to simplify these relations [23]. The problem was solved in Ref. [16].

The helicity crossing matrix is a simple product of four matrices $d^s(\chi)$ representing the rotations round the y-axis which in the helicity basis is perpendicular to the reaction plane. Each of these d-matrices can be diagonalized using a transformation carrying the z-axis into the y-axis. Physically it corresponds to choosing the normal to the production plane as the spin quantization axis. In Ref. [16] the word "transversity" has been proposed to denote the spin projection on the perpendicular to the reaction plane. An example of the above-mentioned transformation is the rotation through the Euler angles $\left(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right)$. It is represented by the unitary matrix

$$u(s)_{ab} = D^s\left(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right)_{ab}. \tag{2.13}$$

For spin 0 this is a unit matrix, for spin $\frac{1}{2}$

$$u\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \tag{2.14}$$

and using the Clebsch-Gordan coefficients we can write down the u-matrix for any spin. For instance,

$$u(1) = \frac{1}{2}\begin{pmatrix} 1 & i\sqrt{2} & -1 \\ i\sqrt{2} & 0 & i\sqrt{2} \\ -1 & i\sqrt{2} & 1 \end{pmatrix}. \tag{2.15}$$

The algebraic properties of these matrices were discussed in Refs [16] and [17]. The principal property is represented by the equality

$$[u(s)d^s(\psi)u^+(s)]_{ab} = e^{i\psi a}\delta_{ab} \tag{2.16}$$

which says that matrices $u(s)$ diagonalize the rotation matrices $d^s(\psi)$.

3. *Properties of the transversity amplitudes*

Starting from the helicity amplitudes is the most convenient way of defining the transversity amplitudes. It is enough then to rotate the spin projection axis of each particle using matrices (2.12).

$$T_{cd,ab} = \sum_{a'b'c'd'} u^*(s_c)_{cc'}u^*(s_d)_{dd'}H_{c'd',a'b'}u(s_a)_{a'a}u(s_b)_{b'b}. \tag{3.1}$$

Here $T_{cd,ab}$ is the transversity and $H_{cd,ab}$ is the helicity amplitude. The summation runs over all primed indices.

It is also possible to determine the transversity amplitudes quite independently, by constructing systematically the one-particle states. It is well known that the spin projection of a moving particle depends on the choice of the transformation from the rest system to the given state of motion. Such a transformation is called the "boost" and it does not change the spin projection by definition. Thus

$$|\boldsymbol{p} \neq 0, \sigma\rangle = B(p) \quad |\boldsymbol{p} = 0, \sigma\rangle \tag{3.2}$$

where σ is the spin projection, the same on both sides of this equality. This definition of spin projection was already given by Wigner [1]. The physical meaning of σ depends on the boost $B(p)$. For each Lorentz transformation $B(p)$ we can write

$$B(p) = B_z(p)R(\boldsymbol{p}) \tag{3.3}$$

where $B_z(p)$ is a rotation-free transformation along the z-axis from the rest system to a moving system and $R(p)$ is a rotation. The boost $B_z(p)$ is fixed for a given momentum p and the physical interpretation of the spin projection depends thus on the choice of rotation $R(\boldsymbol{p})$. In particular for $R(\boldsymbol{p}) = 1$ the quantity σ is the spin projection on an axis fixed in space and the particle states with such a projection are sometimes called "canonical" (*cf.* [13], [14]).

Helicity is obtained by taking for $R(\boldsymbol{p})$ the rotation through Euler angles $(\varphi, \vartheta, 0)$ or $(\varphi, \vartheta, -\varphi)$. Here ϑ, φ are the polar angles of the particle momentum $\boldsymbol{p}$. We have thus two choices for helicity and the corresponding states differ only by a phase factor. The first convention was used for instance in Refs [5] and [20] and is apparently more popular. The second convention can be met in the paper by Jacob and Wick [3].

Now transversity can be obtained using the helicity rotation $R(\boldsymbol{p})$ followed by the rotation (2.12)

$$R(\boldsymbol{p}) = R\left(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right) R(\varphi, \vartheta, 0). \tag{3.4}$$

In a two-body process the helicity frames are usually chosen in such a way that their z-axes are directed along particle momenta and the y-axes are perpendicular to the reaction plane in the centre-of-mass system. The transversity systems have then the z-axes along the normal and the y-axes opposite to the particle momenta (*cf.* Fig. 1).

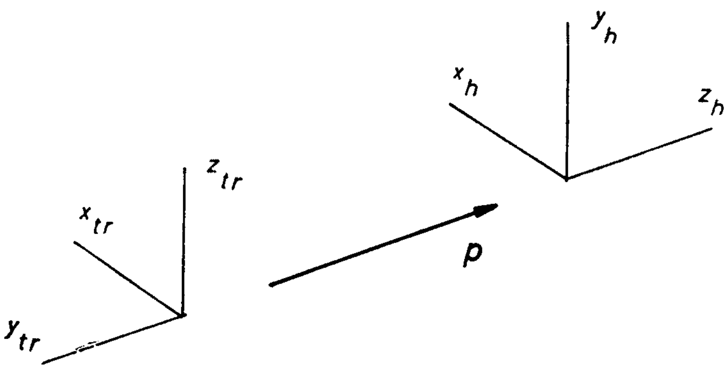

Fig. 1

If parity is conserved in the process, the helicity amplitudes fulfil the relation

$$H_{cd,ab}(s, t) = \eta(-1)^{s_a+s_b+s_c+s_d+a+b+c+d} H_{-c,-d,-a,-b} \tag{3.5}$$

where η is the product of particle intrinsic parities. This relation is valid in a phase convention differing from that of Jacob and Wick by a factor of $(-1)^{s-\lambda}$ for particles b and d. This convention was discussed in detail in Ref. [24] and is used more and more often (*cf. e.g.* Ref. [22]).

Relation (3.5) can be expressed by the transversity amplitudes, making use of Eq. (3.1). Then

$$T_{cd,ab}(s, t) = \eta(-1)^{a+b+c+d} T_{cd,ab}(s, t) \tag{3.6}$$

and this means that parity conservation requires that

$$T_{cd,ab} = 0 \text{ unless } \eta(-1)^{a+b-c-d} = 1. \tag{3.7}$$

If $\eta = +1$, we see that the transversity is conserved (modulo 2). In contrast to the helicity amplitudes (and all amplitudes with spin projection in the reaction plane) the space reflection invariance makes some transversity amplitudes vanish. This leads to simplifications in calculations.

Similarly one can find relations following from other symmetries. For instance, the time reversal invariance gives

$$T_{cd,ab} = T^{T}_{ab,cd}(-1)^{c-d-a+b} \tag{3.8}$$

the invariance under charge conjugation gives

$$T_{cd,ab} = T^{C}_{cd,ab} \tag{3.9}$$

and finally by exchanging the initial particles and simultaneously the final particles we get

$$T_{cd,ab} = T^{E}_{dc,ba}. \tag{3.10}$$

The most important property of the transversity amplitudes is, however, their behaviour under crossing. From Eqs (2.4) and (3.1) we obtain

$$T^{(s)}_{cd,ab} = (-1)^{\sigma}(-1)^{s_a-s_b+s_c-s_d}e^{i\pi(a-d)}\, e^{-ia\chi_a+ib\chi_b-ic\chi_c+id\chi_d}\, T^{(t)}_{-d,-b,-c,-a}. \tag{3.11}$$

The simplicity of this formula, when compared with Eq. (2.4), is remarkable. The crossing matrix is now diagonal. The transversity amplitudes do not mix under crossing, because the spin reorientation is now performed by rotations round the z-axes. The amplitudes get only a phase factor. Furthermore, parity conservation (3.6) leads to further simplification of relations (3.11). Some amplitudes vanish then in both the s-channel and the t-channel and some of equalities (3.11) become trivial. On the other hand, for helicity amplitudes the relations (3.5) lead even to the complication of crossing relations (2.4).

These two properties, the simplicity of the crossing and the behaviour under space reflection are decisive for applications.

The question how to do the partial wave expansion for transversity is interesting. It is possible to find an expansion similar to the Fourier transformation [25], but its properties are still not exactly known.

4. *Spin density matrices and statistical tensors*

The polarization of resonances produced in particle collisions can be described in terms of spin density matrices. For example if the particles c and d in reaction (2.1) are resonances, and the target is unpolarized the density matrix of resonance d is given by

$$\varrho_{dd'} = N \sum_{abd} T_{cd,ab} T^{*}_{cd',ab}, \tag{4.1}$$

where $T_{cd,ab}$ are scattering amplitudes and

$$N^{-1} = \sum_{abcd} |T_{cd,ab}|^2 \tag{4.2}$$

is chosen to have Tr $\varrho = 1$. In the most complicated case of scattering of a polarized beam on a polarized target we have the following expression for the so-called joint density matrix

$$\varrho^{cc'}_{dd'} = N \sum_{aa'bb'} T_{cd,ab}\varrho_{aa'}\varrho_{bb'}T^{*}_{c'd',a'b'} \tag{4.3}$$

where $\varrho_{bb'}$ describes the target polarization and $\varrho_{aa'}$ describes the beam polarization. The oint density matrix contains information not only about the polarization of single resonances but also about the correlations between them.

Another equivalent method of describing the polarization is the use of statistical tensors T^J_M connected with density matrices by

$$T^J_M = \sum_{mm'} (-1)^{s+m-J} \langle s, -m; s, m' | JM \rangle \varrho_{mm'} \tag{4.4}$$

where s is the particle spin and $\langle s, a; s, b | JM \rangle$ are Clebsch-Gordan coefficients. A detailed discussion of properties of the statistical tensors with emphasis on joint decays is presented in Ref. [26].

In the transversity basis the density matrices and statistical tensors are simpler and easier to handle. Five years ago Dalitz noted [7] that with the normal to the scattering plane as the spin quantization axis every second element of a density matrix (4.1) has to vanish. Thus

$$\varrho_{dd'} = 0 \qquad \text{for odd } d-d'. \tag{4.5}$$

This is a consequence of parity conservation in the production process, assuming the initial particles are unpolarized. In fact, it is sufficient for initial particles not to have a polarization in the reaction plane. Then in the helicity basis one gets

$$\varrho_{dd'} = (-1)^{d-d'} \varrho_{-d,-d'} \tag{4.6}$$

and for the statistical tensors in the transversity frame

$$T^J_M = 0 \qquad \text{for odd } M \tag{4.7}$$

while in the helicity frame the relation is

$$T^J_{-M} = (-1)^{J-M} T^J_M. \tag{4.8}$$

All these relations follow from definition (4.4) and Eqs (4.5) and (4.6).

A great simplification occurs for joint decays where in the transversity system we have

$$\varrho^{mm'}_{nn'} = 0 \qquad \text{for } m-m'+n-n' \text{ odd} \tag{4.9}$$

and

$$T^{J_1 J_2}_{M_1 M_2} = 0 \qquad \text{for odd } M_1 + M_2. \tag{4.10}$$

A proof of the property (4.10) is given in Ref. [26].

5. *Transversity amplitudes and kinematical singularities*

Much attention was recently payed to the study of kinematical singularities of scattering amplitudes (*cf.* [9], [10], [28–31]). It appeared that the helicity amplitudes may have some singularities which are different from the dynamical singularities of the S matrix. They may arise on the border of the physical region (what corresponds to the forward or backward scattering), on the thresholds and pseudothresholds and at the vanishing square of total energy s. Here thresholds are the lines $s = (m_a + m_b)^2$ or $s = (m_c + m_d)^2$ and pseudothresholds are $s = (m_a - m_b)^2$ or $s = (m_c - m_d)^2$. The easiest to consider are singularities on the border of the physical region. It has been known since long ago that they are factorizable, *i.e.* they can be removed by multiplying the helicity amplitudes by appropriate factors (*cf.* [32 — 33]). Functions $\overline{H}$ defined by

$$H_{cd,ab}(s,t) = \left(\sin\frac{\vartheta}{2}\right)^{|\lambda-\mu|}\left(\cos\frac{\vartheta}{2}\right)^{|\lambda+\mu|}\overline{H}_{cd,ab}(s,t) \tag{5.1}$$

are free from these singularities. Here ϑ is the scattering angle, $\lambda = a-b$, $\mu = c-d$. On the other hand, the threshold and pseudothreshold singularities are more troublesome. They are not factorizable and it is impossible to remove them by multiplying them by any factors, as people often tried (*cf.* [28—29]). It appeared that besides singularities, the helicity amplitudes satisfy on thresholds and pseudothresholds some constraint relations. Ignoring these constraints leads to the appearance of non-existing singularities in cross-sections (*cf.* also [34], [19]).

The problem of kinematical singularities of helicity amplitudes was eventually solved by Cohen-Tannoudji, Morel and Navelet [22]. They found that the threshold singularities are best described in terms of the transversity amplitudes (*cf.* also [18]). It is known that the crossed channel amplitudes have no singularities at the s-channel thresholds and pseudothresholds. Thus all such singularities have to be present in the crossing matrix. This matrix is particularly simple just in the transversity basis

$$T^{(s)}_{cd,ab} = \text{const}\, e^{-ia\chi_a+ib\chi_b-ic\chi_c+id\chi_d}\, T^{(t)}_{-d,-b,-c,-a}. \tag{5.2}$$

We have to remember that the crossing angles depend on the Mandelstam variables s and t. This dependence is exactly known (*cf.* Eqs (2.5)–(2.11)). Therefore to find the singularities of $T^{(s)}$ it is enough to calculate the singularities of the factors like $e^{-ia\chi_a}$. As they have factorizable singularities at thresholds and pseudothresholds, the same can be said about the singularities of the transversity amplitudes. Thus in the case when all four particles have different masses, the singular part of the amplitude is

$$T_{cd,ab} \sim \varphi_{ab}^{(a+b)\varepsilon}\, \varphi_{ab}^{(c+d)\varepsilon}\, \psi_{ab}^{(a-b)\varepsilon\varepsilon_{ab}}\, \psi_{cd}^{(c-d)\varepsilon\varepsilon_{cd}} \tag{5.3}$$

where

$$\varphi_{ik}^2 = s-(m_i+m_k)^2 \tag{5.4}$$

vanishes at the threshold and

$$\psi_{ik}^2 = s-(m_i-m_k)^2 \tag{5.5}$$

vanishes at the pseudothreshold. Further, ε is the sign of the expression

$$s(t-u)+(m_a^2-m_b^2)\,(m_c^2-m_d^2) \tag{5.6}$$

whereas ε_{ik} is the sign of

$$m_i-m_k. \tag{5.7}$$

The cases of non-different masses were discussed in detail in Ref. [18].

Some important conclusions follow from Eq. (5.3). Firstly, the factorizability of singularities at thresholds and pseudothresholds. Secondly, the absence of any constraints for transversity amplitudes. It may happen that a transversity amplitude vanishes for instance at a threshold (to this end it is enough to have an exponent in Eq. (5.3) positive and even). A zero of the transversity amplitude means a vanishing linear combination of helicity amplitudes. We see therefore that two apparently distinct effects — kinematical singularities and threshold constraints — are in fact closely connected and represent particular cases of the transversity threshold behaviour — poles and zeros respectively. It is now clear that ignoring threshold constraints must give wrong results.

It is necessary to realize that for the simple behaviour of transversity amplitudes at thresholds and pseudothresholds (factorizable singularities or zeros) we pay with more complicated behaviour on the border of the physical region (non-factorizable singularities, linear relations between amplitudes). Thus on the physical border the transversity amplitudes are complicated while the helicity amplitudes have at most factorizable singularities. At the thresholds and pseudothresholds the situation is just the opposite: transversity amplitudes are simple and factorizable while helicity amplitudes are non-factorizable and satisfy constraint relations.

The problems of finding the kinematical singularities of the helicity and transversity amplitudes are therefore complementary.

6. *Kinematical singularities in cross-sections*

The proper accounting of kinematical singularities is important in high-energy collisions models. For instance in the Regge pole model the cross-sections and other observables from the s-channel are expressed in terms of the t-channel amplitudes. These amplitudes may possess kinematical singularities at pseudothresholds, *i.e.* for $t = (m_a - m_c)^2$ or $(m_b - m_d)^2$. As these lines are close to the physical region of the s-channel, the question was asked whether these singularities influence the behaviour of cross-sections at small momentum transfer t. Lin [34] noticed first that cross-sections cannot have any kinematical singularities, as they can be expressed in terms of the singularity-free s-channel amplitudes. On the other hand, the crossing matrix is orthogonal in the physical region and the cross-sections can be given both in terms of the s-channel and t-channel amplitudes. We have

$$\frac{d\sigma}{dt} \sim \sum_{\text{spins}} |H^{(s)}|^2 = \sum_{\text{spins}} |H^{(t)}|^2. \tag{6.1}$$

Then Jackson and Hite [30] discovered that the kinematical singularities in cross-sections are exactly cancelled by kinematical threshold constraints. This brought in question some earlier papers (*cf.* [29], [35]) where people ignored threshold constraints and studied formulae for cross-sections with explicit kinematical singularities.

A further explanation of this problem was given in Ref. [19] and almost independently in Ref. [36]. First, the notion of the singularity in cross-section was precised by defining an analytic function equal to the transition probability in the physical region (for simplicity a flux factor is dropped)

$$\frac{d\sigma}{dt} = F(s, t) = \sum_{mi} M_{mi}^{(s)}(s_0 + i\varepsilon, t) M_{mi}^{(s)*}(s_0 + i\varepsilon, t^*). \tag{6.2}$$

The cross-section (6.1), as it stands, is not analytic in the Mandelstam variables s and t. Hence its singularities beyond the physical region were not defined.

The mechanism of cancellation of the singularities becomes obvious in the transversity basis. We can use then the fact that the threshold and pseudothreshold singularities are then determined by the phase factors in the crossing matrix (5.2)

$$M^{(t)}_{ca,db} = e^{i\chi}R(s, t) \tag{6.3}$$

where

$$\chi = -a\chi_a + b\chi_b - c\chi_c + d\chi_d \tag{6.4}$$

and the function $R(s, t)$ has no singularities in the t variable. Because the crossing angles are real analytic functions of t, *i.e.*

$$\chi^*(s, t^*) = \chi(s, t) \tag{6.5}$$

as can be seen from Eqs (2.5)–(2.11), we find that the expression

$$M^{(t)}_{ca,db}(s_0+i\varepsilon, t)\, M^{(t)*}_{ca,db}(s_0+i\varepsilon, t^*) \tag{6.6}$$

has no kinematical singularities. If M behaves like $e^{-i\chi}$, then M^* behaves like $e^{i\chi}$ and inversely. In any case the product (6.6) is regular, as the singular factors cancel out. As a pole in $e^{i\chi}$ is at the same time a zero in $e^{-i\chi}$, we see how the cancellation mechanism works. Everything is very simple in transversity — one of the factors in (6.6) has a pole and the other a zero of the same order. The situation is quite similar with the branch points.

An analogous discussion can be repeated for the density matrix elements and components of statistical tensors [19]. The result is that the helicity and transversity density matrix elements do not have any kinematical singularities on the t-channel thresholds or pseudothresholds, because they can be expressed directly by non-singular s-channel amplitudes. However the non-diagonal elements of the density matrices in the Jackson system may have some kinematical singularities, because they are related to the t-channel amplitudes (*cf.* Ref. [4]). It is best seen in the transversity basis, where we can use Eq. (6.3) to write

$$\varrho_{mn} = e^{i(m-n)\chi}R(s, t) \tag{6.7}$$

where χ is a crossing angle and $R(s, t)$ is nonsingular in t. Since for unpolarized initial particles $m-n$ is even in the transversity basis and the factor $e^{i\chi}$ can have at most a simple branch point, we see at once that the Jackson-system density matrix element can have a zero or a pole, but not a branch point. When we consider the situation now in the conventional Jackson frame (with the z-axis in the reaction plane), the elements which were non--diagonal in transversity become mixed with those diagonal. As a result, in this reference frame the density matrix elements can have not only kinematical singularities, but also have to satisfy constraint relations.

These kinematical singularities are very peculiar. The crossing angles have a branch line on the physical border. The behaviour of scattering amplitudes (or density matrix elements) beyond the physical region is different on the two Riemann sheets. If the amplitude has a pole on one Riemann sheet, it has simultaneously a zero on the other sheet. However inside the physical region the singular factor $e^{i\chi}$ has always its module equal to one. The simplest function with this type of behaviour is

$$f(z) = e^{i\chi} \tag{6.8}$$

where

$$\cos\chi = (z+1)^{-1}. \tag{6.9}$$

The point $z = 0$ simulates then the physical border and $z = -1$ corresponds to the t-channel threshold or pseudothreshold.

7. *Other applications of transversity amplitudes*

It is necessary to stress here that some of the properties of the spin projection on the perpendicular to the scattering plane were known since a long time. For instance Dalitz [27] and Ademollo, Gatto and Preparata [37–38] knew the simplifications (3.7) and (4.5) following from the parity conservation. The possibility of simplifying the crossing relations was mentioned by Surkov in Ref. [39].

Most of the properties and application were found however recently. Here is a short discussion of these papers. They are quoted in the reference list as [40–80].

In the study of consequences of the quark model for the spin correlations in two-body processes it appeared that the predictions can be summarized in a simple way in terms of the transversity statistical tensors. On the other hand, in the quark model there is an ambiguity, as the reference frame in which the additivity assumption holds is not specified. The most interesting predictions are therefore those independent of the orientation of the additivity frame. The transversity amplitudes are convenient to this end as the additivity frame differs from the transversity frame at most by a rotation around the z-axis. These problems are discussed in Refs [40–47].

The most numerous group of papers concerns the classical application of the transversity amplitudes to the study of kinematical singularities. Here we have Refs [12], [22], [30], [36], [48–65].

Simple crossing relations for transversity are discussed in Refs [22], [48], [57], [66–72]. Here of special interest is the paper [69] where the transversity amplitudes are used in crossing relations for multiparticle processes.

Other applications to the two-body kinematics can be found in Refs [26], [73–75]. The systematic construction of the transversity states starting from the one-particle states is performed in Refs [70], [76–80].

Among the papers quoted above, Refs [57], [64], [67], [79] are review talks from conferences or summer schools.

Finally Refs [81–83] are examples of experimental work presenting the decay correlations in the transversity basis. Comparisons with the quark model predictions can also be found there.

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