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# EXPLICIT REALIZATION OF $E(2)$ SYMMETRY COMPATIBLE WITH THE ANALYTIC $S$-MATRIX

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An explicit mapping procedure is used to determine an approximate off zero-momentum transfer-squared symmetry group for the inelastic binary connected part.

## *I. Introduction*

After the discovery of the importance of the special $SO(3,1)$ symmetry of the scattering amplitude at zero momentum transfer $Q = 0$, attempts have been made to study the situation when the momentum transfer is non-zero. One may distinguish here two approaches. The first one is due to Delbourgo *et al.* [1]. These authors consider the expansion of the scattering amplitude in terms of the matrix elements of the irreducible unitary representations of the physical homogeneous Lorentz Group as the essential mathematical consequence of the $Q = 0$ symmetry, and therefore, they concentrate on and succeed in obtaining expansions in terms of the physical Lorentz Group for $Q \neq 0$ which go smoothly into the Toller expansion as $Q \to 0$ [2]. The other approach presented in a paper by Bali *et al.* [3] is concerned with the question of what happens to the symmetry group for $Q$'s slightly off the zero point. Their stated purpose was to show that for $Q = 0$ (and for multi-particle reactions involving six or more particles), we have an approximate symmetry of the scattering amplitude with respect to a group which is isomorphic to the Lorentz Group and as $Q \to 0$ tends to the group of Lorentz transformations. (Strictly speaking, they realized their program for the $O(4)$ subgroup of the complex Lorentz Group $SL(2, C) \otimes SL(2, C)$). This property makes the $Q = 0$ symmetry compatible with analytic continuation and enables them, in principle, as sketched in [3], to obtain Lorentz poles from $S$-matrix equations.

It is known that at $Q^2 = 0$ ($Q \neq 0$), we have another special symmetry of the scattering amplitude, namely with respect to the $E(2)$ group. In this note we study, in the spirit of Bali *et al.* [3], the question of the formulation of the approximate zero-momentum transfer-squared symmetry and we follow closely their method. This case has two interesting

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aspects. First, the discontinuous change in the number of momentum components, in addition to the momentum transfer $Q$, necessary to specify the initial and final momentum states pointed out in [3] obviously does not occur in the binary inelastic case, hence we can treat this case. Second, as we shall see, we are able to obtain an isomorphism with the physical $E(2)$ symmetry group, *i.e.*, we are able to realize the original strong version of their formulation for this case.

## *II. Statement of approximate $Q^2 = 0$ symmetry in the absence of spin*

We adopt the notation of Reference [3] and consider the Toller description of a connected part in which the total number of particles $N$ is divided into two sets $A$ and $B$, with momenta $p_{1A}, p_{2A}, \ldots; p_{1B}, p_{2B}, \ldots$ such that

$$Q = \sum_{i=1}^{N_A} p_{iA} = \sum_{i=1}^{N_B} p_{iB} \tag{2.1}$$

where $N = N_A + N_B$.
Within the set $A$, particles with positive energy are out-going, and negative energy in-coming. The reverse is true for $B$ and there is at least one in-coming and one out-going particle in each set. The connected part is denoted by $M_Q(P_A^Q, P_B^Q)$. For each light-like four-vector $\tilde{Q}$ there is a physical $E(2)$ subgroup of the physical Lorentz Group which leaves it invariant and which we denote as $E_{\tilde{Q}}(2)$. For example the three generators of $E_{(\omega,0,0,\omega)}(2)$ may be expressed in terms of the generators of the Lorentz Group as

$$J_{12},\ \pi_1 = J_{10} - J_{13},\ \pi_2 = J_{20} - J_{23} \tag{2.2}$$

and the group element may parametrized in the "Eulerian" manner

$$E_{(\omega,0,0,\omega)}(\varphi, \xi, \psi) = e^{-i\varphi J_{12}} e^{-i\xi\pi_1} e^{-i\psi J_{12}}. \tag{2.3}$$

In general, for any light-like $\tilde{Q}$ we have by definition

$$E_{\tilde{Q}}(\alpha)\, \tilde{Q} = \tilde{Q}$$

and from Lorentz invariance

$$M_{\tilde{Q}}(P_A^{\tilde{Q}}, P_B^{\tilde{Q}}) = M_{\tilde{Q}}(E_{\tilde{Q}}(\alpha)\, P_A^{\tilde{Q}}, E_{\tilde{Q}}(\alpha)\, P_B^{\tilde{Q}}). \tag{2.4}$$

For $Q^2 \neq 0$, $E(2)$ is no longer even an approximate symmetry of the connected part since the action of its elements on $P_A^Q, P_B^Q$ leads off the mass shell. This is seen if we note that the connected part is described by the momenta $Q, p_{iA}, p_{iB}$ subject to constraints imposed by (2.1) and the mass shell conditions. In particular, $P_A^Q$ is the space of $p_{iA}$ such that

$$\begin{aligned} p_{1A}^2 &= m_{1A}^2 \\ p_{2A}^2 &= m_{2A}^2 \\ &\cdots \\ p_{N_A-1,A}^2 &= m_{N_A-1A}^2 \\ (Q - p_{1A} - p_{2A} - \cdots - p_{N_A-1})^2 &= m_{NA}^2 \end{aligned} \tag{2.5}$$

and similarily for $P_B^Q$.

The last mass shell condition is not invariant for $Q^2 \neq 0$ with respect to $E(2)$ acting on $P_A^Q$.

Our aim is to show that for small $Q^2 \neq 0$ there exists a group $E_Q^A$ which is isomorphic to $E_{\tilde{Q}}(2)$, with elements denoted by $E_A^Q(\alpha)$ under which Eqs (2.5) are invariant and with the property

$$\lim_{Q \to \tilde{Q}} E_Q^A(\alpha)\, P_A^Q = E_{\tilde{Q}}(\alpha)\, P_A^{\tilde{Q}} \tag{2.6}$$

where $\tilde{Q}$ is light-like. Similarily for $P_B^Q$, with a corresponding meaning for $E_Q^B(\alpha)$. Since $E(2)$ is a subgroup of the Lorentz Group, then from Lorentz invariance, we have

$$M_Q(P_A^Q, P_B^Q) = M_{E_{\tilde{Q}}(\alpha)Q}(E_{\tilde{Q}}(\alpha)\, P_A^Q, E_{\tilde{Q}}(\alpha)\, P_B^Q) \tag{2.7}$$

and from (2.6) and (2.4)

$$M_{\tilde{Q}}(P_A^{\tilde{Q}}, P_B^{\tilde{Q}}) = \lim_{Q \to \tilde{Q}} M_Q(E_Q^A(\alpha)\, P_A^Q, E_Q^B(\alpha)\, P_B^Q) \tag{2.8}$$

which express the approximate invariance.

## *III. Explicit construction of $E_Q^A$ for $N_A = 2$*

We shall show that, at least for two-particle reactions a group with the properties described in Section II exists. The sets $A$ and $B$ now consist of two particles, and, for instance, the space $P_A^Q$ is the space of one vector $p_1 = p = (p^0, \mathbf{p})$ subject to the constraints

$$p^2 = m_1^2$$
$$(Q-p)^2 = m_2^2 \neq m_1^2. \tag{3.1}$$

For $Q^2 \neq 0$, operations of the elements of the group $E(2)$ will lead outside this space. However, as explained in Section II, we are going to show that there exists a group isomorphic to $E(2)$ which spans the space $P_A^Q$ and does not lead outside of it. In order to do this, we construct a one-to-one mapping of the space $P_A^Q$ onto a space $P_A^{\tilde{Q}}$ which is carried into itself by the group $E_{\tilde{Q}}(2)$. This reference space is constructed in the following way. As we shall see at the end of this section, we can assume, without loss of generality, that

$$\tilde{Q} = (\omega, 0, 0, \omega). \tag{3.2}$$

Then, if we set

$$\tilde{p}(0) = \left(\frac{x}{2\omega} + \frac{m_1^2\omega}{2x}, 0, 0, -\frac{x}{2\omega} + \frac{m_1^2\omega}{2x}\right) \tag{3.3}$$

where $\quad x = \dfrac{m_1^2 - m_2^2}{2}$

we can express any element of $P_A^{\tilde{Q}}$ as

$$\tilde{p}(\alpha) = E_{\tilde{Q}}(\alpha)\, \tilde{p}(0). \tag{3.4}$$

From the parametrization (2.3) it is seen that only two of the three parameters $\alpha$ are active; in particular, the angle $\psi$ is inactive. The vectors $\tilde{p}(\alpha)$ satisfy the constraints

$$\tilde{p}^2(\alpha) = m_1^2$$
$$(\tilde{Q}-\tilde{p}(\alpha))^2 = m_2^2. \tag{3.5}$$

We explicitly define a mapping $f_Q$: $P_A^Q \rightarrow P_A^{\tilde{Q}}$ by means of the following

$$p^0(\alpha) = (1+\varepsilon\,(Q,\alpha))\,\tilde{p}^0(\alpha) \text{ energy components}$$
$$\boldsymbol{p}(\alpha) = (1+\varepsilon'(Q,\alpha))\boldsymbol{p}(\alpha) \qquad \text{space components} \tag{3.6}$$

where $\varepsilon$ and $\varepsilon'$ are to be determined from the constraints (3.5). Note that the map $f_Q$ is not a covariant one, and the frame of reference in which it is to be applied will be specified later. After inserting (3.6) into (3.5), we obtain

$$(1+\varepsilon)^2\tilde{p}^{0^2}(\alpha)-(1+\varepsilon')^2\tilde{\boldsymbol{p}}^2(\alpha) = m_1^2$$
$$(1+\varepsilon)\tilde{p}^0(\alpha)Q^0-(1+\varepsilon')\tilde{\boldsymbol{p}}(\alpha)\cdot\boldsymbol{Q} = x+\frac{Q^2}{2}. \tag{3.7}$$

For $Q^0 \neq 0$, eliminating $\varepsilon$, we get

$$(1+\varepsilon')^2\{(Q^0)^{-2}(\tilde{\boldsymbol{p}}\cdot\boldsymbol{Q})^2-\tilde{\boldsymbol{p}}^2\}+2(1+\varepsilon')(Q^0)^{-2}\left(x+\frac{Q^2}{2}\right)\times$$
$$\times\,\tilde{\boldsymbol{p}}'\cdot\boldsymbol{Q}'+(Q^0)^{-2}\left(x+\frac{Q^2}{2}\right)^2-m_1^2 = 0. \tag{3.8}$$

The determinant of this quadratic equation

$$\Delta = 4\left\{(Q^0)^{-2}\tilde{\boldsymbol{p}}^2\left(x+\frac{Q^2}{2}\right)^2+(Q^0)^{-2}(\tilde{\boldsymbol{p}}\cdot\boldsymbol{Q})^2m_1^2-\tilde{\boldsymbol{p}}^2m_1^2\right\} \tag{3.9}$$

must be positive in order to make our mapping real, and this leads to the requirement that

$$Q^{0^2} \leqslant \frac{(x+Q^2/2)^2}{m_1^2}. \tag{3.10}$$

It is obvious that there exist $Q$'s satisfying (3.10). (For spacelike $Q$'s a frame of reference where this condition is satisfied always exists, for time-like $Q$'s such a frame exists when $Q^2 \leqslant (m_1-m_2)^2$ or $Q^2 \geqslant (m_1+m_2)^2$.) But we are not interested in arbitrary $Q$'s rather since we desire to establish a smooth limiting procedure such that $Q \rightarrow \tilde{Q}$, we are only concerned with those $Q$'s which are in the neighbourhood of $\tilde{Q}$. This means that $\tilde{Q}$ must also obey (3.10) or from (3.2) we find that

$$|\tilde{Q}^0| = |\omega| \leqslant \left|\frac{x}{m_1}\right| \tag{3.11}$$

This can obviously be satisfied because from its definition, $|x|>0$. Now one can readily construct a sequence $\{Q_n\}$ such that each $Q_n$ satisfies (3.10) and the $\lim_{n\to\infty} Q_n = \tilde{Q}$ where $\tilde{Q}$ satisfies (3.11). We then fix the real map $f_Q$ by requiring that both $\varepsilon$ and $\varepsilon'$ vanish as $Q_n \to \tilde{Q}$.

However, we are not finished yet, because, it is essential that the map $f_Q$ be $1-1$ onto. From Eqs (3.6) it is evident that for given set of parameters $\alpha$ we get a unique vector $(p^0, \boldsymbol{p})$. To establish the converse, we write formula (3.6) in the form

$$\begin{aligned} \tilde{p}^0(\alpha) &= (1+\varepsilon)^{-1} p^0(\alpha) \\ \tilde{\boldsymbol{p}}(\alpha) &= (1+\varepsilon')^{-1} \boldsymbol{p}(\alpha) \end{aligned} \tag{3.12}$$

and insert this expression into (3.5). This leads to our equation for $(1+\varepsilon')^{-1}$ of the form (3.8) with $\tilde{p}(\alpha)$ replaced by $p(\alpha)$ and $Q$ replaced by $\tilde{Q}$. And the condition which guarantees a real solution is $|\tilde{Q}^0| = |\omega| \leqslant |x/m_1|$ which is precisely condition (3.11). Thus, the coefficients $\varepsilon, \varepsilon'$ are uniquely determined by the vectors $p(\alpha)$ and $\tilde{Q}$, and in particular, from (3.4) and (3.12), we have

$$E_{\tilde{Q}}^{-1}(\alpha) \begin{Vmatrix} (1+\varepsilon)^{-1} p^0 \\ (1+\varepsilon')^{-1} p^1 \\ (1+\varepsilon')^{-1} p^2 \\ (1+\varepsilon')^{-1} p^3 \end{Vmatrix} = \begin{Vmatrix} \dfrac{x}{2\omega} + \dfrac{m_1^2\omega}{2x} \\ 0 \\ 0 \\ -\dfrac{x}{2\omega} + \dfrac{m_1^2\omega}{2x} \end{Vmatrix}. \tag{3.13}$$

Further, the two active parameters $\alpha$ are determined by the condition that after applying the transformation $E_{\tilde{Q}}^{-1}(\alpha)$ to the vector $((1+\varepsilon)^{-1}p^0, (1+\varepsilon')^{-1}\boldsymbol{p})$, the first two spatial components of the new vector vanish. Thus the map $f_Q$ is one to one onto and the group of transformations $E_Q^A(\alpha)$ introduced in Section II are linear transformations in the space $P_A^Q$ induced by the transformations $\tilde{p}(\alpha') \to \tilde{p}(\alpha'')$ in the space $P_A^{\tilde{Q}}$ with $\alpha'' = \alpha'\alpha$. Since when $Q \to \tilde{Q}$, $\varepsilon$ and $\varepsilon'$ vanish and $p(\alpha) \to \tilde{p}(\alpha)$, thus, we see that $\lim_{Q\to\tilde{Q}} E_Q^A(\alpha)$ indeed satisfies (2.6).

Now the existence of a real $1-1$ onto map $f_Q$ given by (3.6), which explicitly shows the existence of an approximate symmetry group as defined in the sense of Section II, is valid for any sequence $\{Q_n\}$ with a limiting light-like vector $\tilde{Q} = (\omega, 0, 0, \omega)$ satisfying (3.10) and (3.11) respectively. We can now easily extend our mapping procedure to any arbitrary sequence $\{Q'_n\}$ with a limiting light-like vector $\tilde{Q}'$ with the following prescription. Given any such sequence, select any sequence $\{Q_n\}$ satisfying (3.10) with its limit satisfying (3.2) and (3.11). Then, $E_{Q'_n}^A(\alpha)$ is determined as shown schematically

$$\begin{array}{ccc} \{(P_A^{Q'_n}, E_{Q'_n}^A(\alpha))\} & \xrightarrow[1-1 \text{ onto}]{Ln} & \{(P_A^{Q_n}, E_{Q_n}^A(\alpha))\} \\ \Big\downarrow{\scriptstyle n\to\infty} & & {\scriptstyle 1-1 \atop \text{onto}}\Big\downarrow{\scriptstyle f_a} \\ \{(P_A^{\tilde{Q}'_n}, E_{\tilde{Q}'} = L^{-1}E_{\tilde{Q}}(\alpha)L)\} & \xrightarrow[1-1 \text{ onto}]{L} & \{(P_A^{\tilde{Q}}, E_{\tilde{Q}}(\alpha))\} \end{array}$$

where $L_n$ is the Lorentz transformation connecting $Q'_n$ and $Q_n$, and $E_{Q'}(\alpha)$ and $E_{\tilde{Q}}(\alpha)$ are in general equivalent representations of $E(2)$ which leave the light-like vectors $Q'$ and $\tilde{Q}$ invariant, respectively. Thus, we have achieved an explicit realization of the formulation of approximate symmetry stated in Section II.

### *IV. Inapplicability of the construction given in section II for $N_A > 2$*

If we repeat our construction for a space $P_A^Q$ consisting of more than two particles, then it turns out that the mapping is no longer real. For instance, for three particles, the space $P_A^Q$ consists of vectors $p_1$ and $p_2$ subject to the constraints

$$p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad (Q - p_1 - p_2)^2 = m_3^2.$$

With $\tilde{Q} = (\omega, 0, 0, \omega)$, the space $P_A^{\tilde{Q}}$ invariant under $E(2)$ may be parametrized as follows

$$\tilde{p}_1(0) = (a, 0, 0, \sqrt{a^2 - m_1^2})$$

$$\tilde{p}_2(0) = (b, 0, \sqrt{b^2 - c^2 - m_2^2}, c)$$

$$\tilde{p}_{1,2}(\alpha) = E(\alpha)\tilde{p}_{1,2}(0)$$

where $a^2 \geqslant m_1^2$, $b^2 \geqslant c^2 + m_2^2$ and $c = (m_3^2 - m_1^2 - m_2^2 - 2ab + 2\omega(a + b - \sqrt{a^2 - m_1^2}))/2(\omega - - \sqrt{a^2 - m_1^2})$ with $\tilde{p}_{1,2}$ satisfying $\tilde{p}_1^2 = m_1^2, \tilde{p}_2^2 = m_2^2, (\tilde{Q} - \tilde{p}_1 - \tilde{p}_2)^2 = m_3^2$. But, when we map $\tilde{p}_{1,2}$ onto $p_{1,2}$ by the method of Section III, *i.e.*, use Eqs (3.6) for the connection between $p_1(\alpha)$ and $\tilde{p}_1(\alpha)$ and let $p_2(\alpha) = \tilde{p}_2(\alpha)$, then it turns out that $\varepsilon$, $\varepsilon'$ become complex for some values of $\alpha$. So that at least by this method, the extension to more than two particles is impossible.

### *Conclusion*

Using an explicit mapping procedure, we have been able to determine an approximate zero-momentum transfer-squared symmetry group for the inelastic binary connected part which is isomorphic to the physical symmetry group $E(2)$. This is a realization for the case of $E(2)$ symmetry of the original strong version of the smooth analytic $S$-matrix approach to exact symmetry given by Bali, *et al.*

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