THE OPTICAL THEOREM FOR PARTIALLY POLARIZED PARTICLES

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The optical theorem for partially polarized particles is derived in its most general form. It is shown that it can be used to determine experimentally the imaginary part of all spin amplitudes nonvanishing in the forward direction.

It is well known that the number of the spin amplitudes non-vanishing in the forward direction is in general larger than the number of those amplitudes which correspond to the purely elastic scattering (with no spin flip) and consequently fulfil the optical theorem in its usual form. On the other hand the optical theorem is one of the most valuable means of determining the imaginary part of the amplitude giving its value at the forward direction. It is, then, unsatisfactory that the optical theorem cannot be used for a part of the amplitudes which may play an important role in the region of small scattering angles. The aim of this paper is to show that one can formulate the optical theorem in a shape which will enable to calculate the imaginary part of all spin amplitudes nonvanishing in the forward direction.

Let us consider elastic scattering of the particles with spins s_1 and s_2 ($s_1 \geqslant s_2$). It is clear that there are $n_0 = (2s_1+1)\,(2s_2+1)$ amplitudes which correspond to non-spin flip transitions and consequently fulfil the optical theorem. This number is reduced by parity conservation to

$$n_0' = \frac{1}{2}\,[(2s_1+1)(2s_2+1)+b_1b_2]. \tag{1}$$

(In the following $b_i(f_i) = 1$ if the i^{th} particle is a boson (fermion), and is equal to zero otherwise.)

On the other hand, the amplitudes non-vanishing in the forward direction are those which correspond to the transitions $(\lambda_1, \lambda_2 \to \lambda_1', \lambda_2')$ if $\lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$ (λ_i — helicity of

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the i^{th} particle). There are n_1 amplitudes of this kind, where

$$n_1 = (2s_2+1)^2(2s_1-2s_2+1) + \frac{2}{3}s_2(2s_2+1)(4s_2+1). \tag{2}$$

This number is reduced by parity conservation to

$$n_1' = \frac{1}{2}(n_1+b_1b_2), \tag{2'}$$

whereas T-invariance imposes q further conditions on those amplitudes, q being equal to

$$q = \frac{1}{3}s_2(2s_2+1)\left(3s_1-s_2+\frac{1}{2}\right) - (f_1f_2+b_1b_2)\frac{1}{2}\left(s_2^2+s_2f_2+\frac{1}{4}f_2\right). \tag{3}$$

In general $N = n_1'-q$ is much larger than n_0' and the difference increases when s_i become large, as $N \sim s^3$ and $n_0' \sim s^2$.

To find a form of the optical theorem suited to all the spin amplitudes let us assume that the particles in the initial state are partially polarized, so that the state is described by a certain density matrix ϱ_{12}. Using the unitarity condition it is easy to see that

$$\sum_{(\lambda)} \varrho_{\lambda_1\lambda_2}^{\lambda_1'\lambda_2'}[\langle k, 0, 0; \lambda_1'\lambda_2'|\hat{T}|k, 0, 0; \lambda_1\lambda_2\rangle - \langle k, 0, 0; \lambda_1\lambda_2|\hat{T}|k, 0, 0; \lambda_1'\lambda_2'\rangle^*]$$

$$= i\sum_{(\lambda)}\sum_{n} \varrho_{\lambda_1\lambda_2}^{\lambda_1'\lambda_2'}\langle k, 0, 0; \lambda_1'\lambda_2'|\hat{T}|n\rangle\langle n|\hat{T}^+|k, 0, 0; \lambda_1\lambda_2\rangle; \tag{4}$$

$(k, 0, 0)$ denotes a $\vec{k}$ vector pointing in the z direction.

However, T-invariance together with the normal choice of the phase factor (see *e.g.* [1]) gives us

$$\langle k, 0, 0; \lambda_1'\lambda_2'|\hat{T}|k, 0, 0; \lambda_1\lambda_2\rangle = \langle k, 0, 0; \lambda_1\lambda_2|\hat{T}|k, 0, 0; \lambda_1'\lambda_2'\rangle,$$

which enables us to write (after having introduced proper kinematical factors) the optical theorem in the following form:

$$\sum_{(\lambda)} \varrho_{\lambda_1\lambda_2}^{\lambda_1'\lambda_2'}\,\mathrm{Im}\, f_{\lambda_1\lambda_2,\lambda_1'\lambda_2'}(0) = \frac{k}{4\pi}\sigma_{\text{tot}}^{(\varrho_{12})}(k), \tag{5}$$

where $\sigma_{\text{tot}}^{(\varrho_{12})}$ represents the total cross-section for the scattering of particles 1 and 2, which are described in the initial state by the density matrix ϱ_{12}.

It can be seen that relation (5) is not trivial from the fact that if we try to diagonalize the density matrix ϱ_{12} by changing the representation, we may do it only for one particle by a suitable choice of the quantization axis; however, the same choice does not necessarily make the density matrix of the second particle diagonal as well. Looking at it from another point of view, the total density matrix ϱ_{12} can be diagonalized in any case, but the transformation which is in general needed, has not the geometrical sense of a simple rotation in physical space. It is clear, however, that Eq. (5) becomes trivial in two border cases, namely, when the particles are totally polarized or totally unpolarized. *E.g.* if the particles are totally unpolarized, the density matrix is diagonal and has the form:

$$\varrho_{12} = \frac{1}{(2s_1+1)(2s_2+1)} \delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}.$$

Inserting this form into Eq. (5) we get

$$\frac{1}{(2s_1+1)(2s_2+1)} \sum_{\lambda_1\lambda_2} \operatorname{Im} f_{\lambda_1\lambda_2,\lambda_1\lambda_2}(0) = \frac{k}{4\pi}\, \sigma_{\text{tot}}^{(\text{unpol})}(k), \tag{6}$$

which is merely a sum of the "usual" optical theorems for the amplitudes $(\lambda_1\lambda_2 \to \lambda_1\lambda_2)$.

Let us take an example. The formula (5) may have a practical meaning for the nucleon-nucleon scattering. This scattering is known to be described by 5 independent spin amplitudes 3 of which do not vanish in the forward direction. (These are $(++,\ ++)$, $(+-,\ +-)$, $(++,\ --)$.) However, only two of them are of the type $(\lambda_1\lambda_2 \to \lambda_1\lambda_2)$ and fulfil the optical theorem in its standard form. Assume now that the $N-N$ scattering amplitude in the forward direction in the C.M. frame is of the form

$$f(0) = f_1(0) + (\vec{\sigma}_1\vec{\sigma}_2) f_2(0) + \frac{1}{k^2} (\vec{\sigma}_1\vec{k})(\vec{\sigma}_2\vec{k}) f_3(0), \tag{7}$$

where $\vec{k}$, as before, is the C.M. momentum. Notice that

$$f_1(0) = \frac{1}{2}[f_{++,++}^{(0)} + f_{+-,+-}^{(0)}],$$

$$f_2(0) = \frac{1}{2} f_{++,--}^{(0)},$$

$$f_3(0) = \frac{1}{2}[-f_{++,++}^{(0)} + f_{+-,+-}^{(0)} - f_{++,--}^{(0)}].$$

Let us now assume that the density matrix for the initial particles may be written in the form

$$\varrho_{12} = \frac{1}{4}(1+\vec{\sigma}_1\vec{P}_1)(1+\vec{\sigma}_2\vec{P}_2), \tag{8}$$

where $\vec{P}_1$ and $\vec{P}_2$ are the polarization vectors of the two nucleons. Then after a simple calculation we get

$$\operatorname{Im} f_1(0) + (\vec{P}_1\vec{P}_2) \operatorname{Im} f_2(0) + \frac{1}{k^2}(\vec{P}_1\vec{k})(\vec{P}_2\vec{k}) \operatorname{Im} f_3(0) = \frac{k}{4\pi}\, \sigma_{\text{tot}}^{(\vec{P}_1,\vec{P}_2)}(k). \tag{9}$$

As can be seen from Eq. (9), it is possible to obtain the imaginary part of all the three amplitudes $f_1(0), f_2(0), f_3(0)$ in the forward direction by measuring the total cross-section for partially polarized nucleons. It must be noticed, however, that both the beam and target have to be polarized.

To see how complicated do things get in the case of higher spins, let us consider $1^- + \frac{1}{2}^+$ particles scattering. Denoting spin-vector matrices for the spin-1 particle by $\vec{S}$, we may write the following form of the amplitude for the forward scattering:

$$f(0) = f_1(0) + (\vec{\sigma}\vec{S})f_2(0) + \frac{1}{k^2}(\vec{\sigma}\vec{k})(\vec{S}\vec{k})f_3(0) + \frac{1}{k^2}(\vec{S}\vec{k})(\vec{S}\vec{k})f_4(0). \tag{10}$$

(There are, according to the formulae (2) and (3), four amplitudes nonvanishing in the forward direction.) The density matrix in the initial state can be written in the form

$$\varrho_{12} = \frac{1}{6}(1+\vec{P}\vec{\sigma})(1+\sqrt{3}\vec{\Pi}\vec{S}+\sqrt{6}A_{ij}S_iS_j), \tag{11}$$

where $\vec{P}$ is the polarization vector for spin $\frac{1}{2}$ particle, $\vec{\Pi}$ — the same vector for the spin 1 particle (it is defined as $\frac{1}{2}\sqrt{3}\langle\vec{S}\rangle$), and A_{ij} is the alignment tensor for that particle defined as

$$A_{ij} = \frac{1}{\sqrt{24}}\langle 3(S_iS_j+S_jS_i) - 2\vec{S}^2\delta_{ij}\rangle.$$

Simple trace calculation gives us the following formula:

$$\begin{aligned}\operatorname{Im} f_1(0) + \frac{2}{\sqrt{3}}\left[\vec{P}\vec{\Pi}\operatorname{Im} f_2(0) + \frac{1}{k^2}(\vec{P}\vec{k})(\vec{\Pi}\vec{k})\operatorname{Im} f_3(0)\right] + \frac{1}{\sqrt{6}}\frac{1}{k^2}A_{ij}k_ik_j\operatorname{Im} f_4(0)\\ = \frac{k}{4\pi}\sigma_{\mathrm{tot}}^{(\vec{P},\vec{\Pi},A_{ij})}(k).\end{aligned} \tag{12}$$

In this case to determine the imaginary parts of all forward scattering amplitudes one needs not only linearly polarized particles, but also some alignment of the vector particle, specifically, the nonvanishing (3, 3) component of the alignment tensor, if $\vec{k}$ points in the z direction. This illustrates difficulties one may encounter in practical application of the Eq. (5) for high spin targets (as, *e.g.* for some nuclei).

REFERENCES

[1] J. Werle, *Relativistic Theory of Reactions*, PWN, Warszawa 1966.