Vol. B1 (1970) ACTA PHYSICA POLONICA Fasc. 4

LOCALIZABLE AND NON-LOCALIZABLE FIELD THEORIES

By E. Kapuścik*

International Centre for Theoretical Physics, Trieste**

(Received November 5, 1969)

The notion of localizability of field theory is discussed. New criteria for localizability are found and the relationship between Lagrangian and localizability is sketched.

Introduction

The content of any quantum field theory can be formulated in several different ways. Among them the so-called Wightman formulation proved to be most convenient for precise definitions of various concepts used in field theory and general discussions of basic properties of field theory.

Recently, the class of field theories discussed has extended considerably, concentrating on non-renormalizable field theories. While the Wightman formulation can be adapted to any new field theory without any fundamental trouble, adaptation of the Lagrangian formulation is not a trivial task. Only for a restricted class of non-renormalizable interactions do computational methods exist which allow these cases to be treated in close analogy to the usual Lagrangian field theory.

The distinction of the non-renormalizable field theories follows from the usual power-counting theorem by which for these theories we must introduce an infinite number of counter-terms into the interaction Lagrangian. But it turns out that the difference goes much deeper. The notion of non-renormalizability is closely connected with the notions of localizability of fields and locality of the interaction. This makes some non-renormalizable field theories completely different from the usual ones.

In what follows we shall discuss the notion of localizability of field theories. In Sec. 1 we briefly review this concept and give the known criteria of localizability. Then in Sec. 2 we discuss some such new criteria and indicate their advantages and disadvantages. Finally, in Sec. 3 we briefly sketch the connection of the types of Lagrangians with the notion of localizability.

---

* Address: Instytut Fizyki Jądrowej, Kraków 23, Radzikowskiego 152, Polska.

** On leave of absence from Institute of Nuclear Physics, Cracow, Poland.

1. Strictly localizable field theories

The usual Wightman formulation of quantum field theory is based on the following requirements [1], [2]:

a) The physical states form a Hilbert space.

b) The fields are covariant with respect to the Poincaré group.

c) The energy is positive.

d) The theory is causal.

From the first two requirements it follows that a field cannot be an operator-valued function of the coordinates [3], [4]. The example of free field theory shows that is possible to treat the field as an operator-valued generalized function, but the question immediately arises how to choose the test function space [5]. Unfortunately, this question cannot be answered on the basis of the above-mentioned physical requirements and usually the fields were treated as operator-valued distributions for which the vacuum expectation values of products of field operators were tempered distributions in the space S′ (for definition of S′ see Ref.[5]). It has been believed that the requirement of temperedness appears very natural from the physical point of view, since it reflects the symmetry between the coordinate and momentum space and it is fulfilled in any order of perturbation theory in all renormalizable field theories. But it simultaneously excludes a large class of field theories, one of which is the field describing weak interactions.

In the past there have been several suggestions that the postulate of temperedness must be changed [6], [7], [8] but only Jaffe [4], [9] has shown that the requirements a)–d) can be incorporated in a theory with a much wider class of fields. These fields he called strictly localizable fields and by

Definition: we say that a field is strictly localizable in a certain compact region of space-time if it can be averaged (in order to yield a well-defined operator) with some test function which vanishes outside this region.

The reason for using the compact sets as a region of strict localizability can be explained as follows. The property of the set to be compact is equivalent to the properties of boundedness and closeness of the given set. The requirement of boundedness for strict localizability is obvious, while the closeness is connected with the fact that the notion of strict localizability requires that it must be possible to go out from the region of localizability by an infinitesimally small step and this is only possible for closed sets.

In the mathematical language, the above definition means that the field theory will be strictly localizable in a compact set k of space-time iff among the test functions we can find functions which have the compact set k as their support. If the set k can be an arbitrary compact set then we shall simply speak of localizable field theories.

So we see that the requirement of localizability can serve as a restriction on the choice of test function space. Simultaneously, we see that in the case of localizable field theories we can formulate the causality condition in the sense of local commutativity of fields.

The class of strictly localizable fields is very wide since, besides tempered fields, it includes all entire functions of free field, some not entire functions and some non-renormalizable field theories. In momentum space the localizable fields admit a non-tempered, *e.g.* exponential, character of spectral functions and so the off-mass-shell amplitudes in such theories can grow faster than any polynomial at large energies.

From a physical viewpoint, the smallest class of test functions must contain the fuctions with compact support in momentum space. However, the Fourier transforms of such functions, *i.e.* the test functions in coordinate space, do not contain functions with compact support and consequently such a theory may not be localizable. For this reason the class of test functions in momentum space must be larger than the set of functions with compact support.

Jaffe proposed to use as a test function the functions $f(x)$ whose Fourier transforms $\tilde{f}(p)$ are such that

$$\sup_{p\in R^4} g(A||p||^2)\,(1+||p||^2)^n|D^m\tilde{f}(p)| < \infty \tag{1}$$

where n, m and A are integers and

$$D^m = \frac{\partial^{|m|}}{\partial^{m_0}_{p_0}\partial^{m_1}_{p_1}\partial^{m_2}_{p_2}\partial^{m_3}_{p_3}}, \qquad |m| = m_0+m_1+m_2+m_3.$$

The function $g(t)$, called a Jaffe indicator function, is an entire function which will characterize the momentum space growth of the off-mass-shell amplitudes. The property of strict localizability can be translated into a property of the growth of the indicator function $g(t)$. Jaffe has proved the following

Theorem: The quantum field theory is strictly localizable if and only if

$$\int_0^\infty \frac{\ln g(t^2)}{1+t^2}\,dt < \infty. \tag{2}$$

For the localizable theories the test function space always contains sufficiently many functions with compact support. This fact is a content of the second fundamental theorem proved by Jaffe.

The negative feature of Jaffe-type spaces is that there is no one smallest space containing all the others. Hence there is no one test function space suitable for all strictly localizable fields. Each field will dictate which test function space is appropriate for that field and the relevant test functions will vary from problem to problem.

Another possible approach to strictly localizable fields is to consider the theory with test functions from the S_α-type spaces of Gel'fand and Shilov [5]. In this case the test functions are defined to satisfy the condition

$$\sup |p^k D^q\tilde{f}(p)| < C_q k_0^{\alpha_0 k_0} k_1^{\alpha_1 k_1} k_2^{\alpha_2 k_2} k_3^{\alpha_3 k_3} A_0^{k_0} A_1^{k_1} A_2^{k_2} A_3^{k_3} \tag{3}$$

where $k = \{k_0, k_1, k_2, k_3\}$, $p^k = p_0^{k_0}\, p_1^{k_1}\, p_2^{k_2}\, p_3^{k_3}$, with arbitrary integers $k_0, k_1 \ldots k_3$, C_q and A's constants depending on f, and D^q is similarly defined as before. The numbers $\alpha_0 \ldots \alpha_3$ are all positive.

The properties of field theory defined as a generalized operator-valued function with an S_α test function in momentum space have been extensively studied by Constantinescu

[10], [11]. The property of being a strictly localizable field theory is in this case formulated as a condition for the numbers α and we have the

Theorem: The field theory is strictly localizable if and only if

$$\alpha_i > 1 \quad \text{for} \quad i = 0, 1, 2, 3.$$

The S_α spaces have the property that

$$S_{\alpha_1} \subset S_{\alpha_2}$$

for

$$\alpha_1 < \alpha_2,$$

and thus the minimal space containing all strictly localizable test functions is the space $S_{1,1,1,1}$. However, in spite of the above theorem this space does not contain functions with compact support and thus cannot serve as a test function space for a strictly localizable field theory. Nevertheless, the investigation of this type of theory is interesting since it is a limiting case between strictly localizable theories and those where the notion of localizability and thus locality is not defined. Note that for this limiting case it is still possible to have a theory where the interaction has a local character [12].

2. *Properties of test functions for localizable and non-localizable theories*

In the proofs of theorems concerned with classes of functions, the notion of quasi-analytic classes of functions is frequently used. The importance of this class of functions is connected with the fact that among functions which form such a class there is no function with compact support. So, if we can show that our test functions form a quasi-analytic class, we surely know that the theory is not localizable. The converse is also true since, due to Mandelbrojt's theorem, every class of functions which is not quasi-analytic contains a non-trivial, positive function with compact support.

There are several criteria of the quasi-analyticity of a given class of functions. Among them the best known are the Carleman-Denjoy and Ostrowski theorems. They have been used by Jaffe [4] in deriving his bound (see Eq. (1)). Before discussing other such criteria, we discuss another much simpler criterion based on the Bernstein theorem.

First of all, we shall say that a function is of limited spectrum if its Fourier transform has a compact support. For such a function it is clear that it cannot vary too rapidly, since in its harmonic decomposition the high frequencies are absent. The Bernstein theorem serves as a measure of this phenomenon and says:

Theorem: If the Fourier transform of the function $f(t)$ has a compact support, then

$$|f'(t)| \leqslant 2\pi L \cdot B(f)$$

where $2L$ is the measure of support of the Fourier transform of $f(t)$ and $B(f)$ is the upper bound of the function $f(t)$, *i.e.*

$$|f(t)| \leqslant B(f).$$

The proof of this theorem is straightforward and can be found in Ref. [13].

This theorem admits an immediate generalization to higher derivatives in four-dimensional space in the following form:

Theorem: If the function $\tilde{f}(p_0 \cdots p_3)$ is of limited spectrum, then

$$\left| \frac{\partial^{|m|} \tilde{f}(p_0 \cdots p_3)}{\partial_{p_0}^{m_0} \partial_{p_1}^{m_1} \cdots \partial_{p_3}^{m_3}} \right| \leqslant (2\pi)^{|m|} L_0^{m_0} L_1^{m_1} \ldots L_3^{m_3} B(f) \tag{4}$$

where L_i measure the support of the Fourier transform of $\tilde{f}(p)$ in each variable and $|m| = \sum_{i=0}^{3} m_i$.

Using the above theorem we immediately come to the following

Theorem: If the quantum field theory is a localizable field theory then among the test functions in momentum space there are functions which satisfy the condition (4).

The converse of this criterion is also true, since if no test function satisfies the condition (4) then in coordinate space no function will have compact support.

Note that the above criterion simultaneously gives information about the magnitude of the region of localizability in the coordinate space without the need to calculate the Fourier transform. Also, there is no need to calculate the asymptotic behaviour and no need to know the indicator function.This will be extremely useful for complicated forms of test functions.

The second advantage of the above criterion is the fact that it does not use any topological property of the test function space. This fact is very important since no physical fact serves as an indication of how to choose the topology. Moreover, in the present state of knowledge, any argument based on a topology in order to rule out some field theory is evidence rather against the chosen topology than against the considered field theory. Since no physical requirement restricts the choice of topology we can freely change it in order to achieve the assumed continuity properties. Apart from the aesthetic reason, even if in this way we reach the discrete topology in the set of test functions there is no physical reason to reject this possibility!

Unfortunately, our criterion has the disadvantage that it is only a necessary condition for localizability, but not a sufficient one. It serves therefore rather as a criterion for deciding whether the theory is non-localizable. In order to formulate a sufficient criterion, we must know the inverses of Bernstein's theorem, but they are rather poorly investigated and we cannot therefore discuss them here.

The second property of localizable field theories we want to discuss is based on the fact that each function of limited spectrum can be represented in the form (for simplicity we write only the one-dimensional case)

$$f(t) = \sum_{n=-\infty}^{+\infty} f\left(\frac{n}{2L}\right) \frac{\sin 2\pi L(t - n/2L)}{2\pi L(t - n/2L)} \tag{5}$$

where L has the same meaning as above [13]. From this representation formula we see that in order to calculate the localizable field operator for an arbitrary localizable test function it is enough to know the value of the field operator for one function $\frac{\sin t}{t}$. Then, using the translations and changing the scale, we get the value of the field operator for each term in (5). Moreover, we do not need to know the test function in all points but only in denumerably many equidistant points. This property greatly simplifies the calculation problem and reflects the regularity properties of localizable field theories. But, once again, we cannot be completely happy since we cannot prove that this simplification is the property of only localizable and not of non-localizable field theories.

In order to get a more constructive result, we now return to the theory of quasi-analytic classes. First, we introduce some new definitions. We perform all our discussion for one variable since the passage to the realistic case is straightforward.

Definition: A positive function $g(x)$ is called a weight function if

$$\int_{-\infty}^{+\infty} \frac{\ln g(x)}{1+x^2}\, dx = \infty. \tag{6}$$

Definition: The weight function $g(x)$ increases normally if $x\,\dfrac{g'(x)}{g(x)} > 0$ and is strictly increasing to infinity together with $|x|$.

As a criterion for the function $g(x)$ to be a weight function we give here the following

Theorem [14]: The function $g(x)$ is a weight function if an only if

$$\sup_{P_n} \int_{-\infty}^{+\infty} \frac{\ln |P_n(x)|}{1+x^2}\, dx = \infty$$

where the sup is taken over the set of all algebraic polynomials $P_n(x)$ which satisfy the condition

$$|P_n(x)| \leqslant (1+|x|)g(x).$$

Next, in order to formulate the desired criterion, we need the following

Theorem: If $f(t)$ is a function which is measurable and bounded on $(-\infty, +\infty)$, then for any $\sigma > 0$ there exists among the entire functions $g_\sigma(t)$ of degree not greater than σ a function $g_\sigma(f;t)$ which deviates least from $f(t)$, *i.e.* it is such that

$$\operatorname*{vrai\,sup}_{-\infty<t<\infty} |f(t)-g_\sigma(f;t)| = \inf_{g_\sigma(t)} \operatorname*{vrai\,sup}_{-\infty<t<\infty} |f(t)-g_\sigma(t)|.$$

The above lower bound is called the best approximation of the function $f(t)$ by an entire function of degree not higher than σ and is denoted by $A_\sigma(f)$.

Now we are ready to formulate an important theorem due to Bernstein which can be used as a criterion for localizability of field theory.

Theorem [15]: For the class of all functions $f(x)$ satisfying

$$A_\sigma(f) < \frac{M(f)}{g(\sigma)}, \quad (\sigma > 0,\ M(f) \text{ a constant}),$$

where $g(x)$ is non-decreasing on $(0, \infty)$, to be quasi-analytic it is sufficient (and in the case when $g(x)$ increases normally also necessary) that $g(x)$ is a weight function.

The above criterion uses only test functions in coordinate space and the role of the function $g(x)$ is quite different from that of the Jaffe indicator function. For this reason it will not be so easy to translate the above criterion into the possible growth of Wightman functions.

On the basis of the above discussion, we finally get the

Theorem: For strict localizability of field theory it is necessary, and under the condition of the theorem above also sufficient, that the function $g(x)$ is not a weight function.

3. *Lagrangians and localizability*

Interest in more general than tempered distribution field theories has increased in connection with so-called phenomenological Lagrangians discussed in symmetries. But, unfortunately, the relationship between a given type of Lagrangian and the localizability of the corresponding field theory is not yet known. This is connected with the fact that such relationship depends on the method chosen for constructing the Green functions of the field theory from a given Lagrangian. There is no one unique way and, in fact, one cannot exist since the Lagrangians have only a symbolic meaning. Nevertheless, several interesting methods have been proposed [16]–[18].

In his latest papers, Efimov [19] used a sharp and drastic cut-off method which allows one to calculate (or rather give a meaning to) the Green functions in momentum space for a large class of non-polynomial interactions. The resulting two-point Green functions have very large orders of growth and Efimov did not show that this is the minimal order of growth needed in such theories. It is easy to see that actually this is not the case by comparing the behaviour of imaginary parts of his Green functions with the order of growth of the obtained real parts.

The shortest way to decide to which kind of theory the one given by the Lagrangian

$$L_{\text{int}} = \sum_{n=0}^{\infty} \frac{u_n}{n!} : \varphi^n(x):$$

belongs is to take the rough estimate of the two-point spectral function $\varrho(s)$ in the second order with respect to L_{int}. After simple calculations we get

$$\varrho(s) = \sum_{n=2}^{[s]} \frac{u_{n+2}^2}{n!} \Omega_n(s)$$

where $\Omega_n(s)$ is the n-particle phase space volume. From this formula specifying the coefficients u_n we can get the high-energy behaviour of the spectral functions and then look for an appropriate test function space. However, this method is based on the assumption of usual analytical properties of the amplitude in the second order with respect to L_{int} which are assured only in localizable local theories and for this reason such an approach is in fact a formal one.

The author would like to thank Professors Abdus Salam and P. Budini as well as the International Atomic Energy Agency for hospitality at the International Centre for Theoretical Physics, Trieste. Illuminating discussions with Professor Abdus Salam and Dr J. Strathdee are gratefully acknowledged.

REFERENCES

[1] R. F. Streater, A. S. Wightman, *PCT, Spin, Statistics and All That*, Benjamin, New York 1964.
[2] R. Jost, *The General Theory of Quantized Fields*, AMS, Providence, Rhode Island 1965.
[3] A. S. Wightman, *Ann. Inst. Henri Poincaré*, **1**, 403 (1964).
[4] A. M. Jaffe, *Phys. Rev.*, **158**, 1454 (1967).
[5] I. M. Gel'fand, G. E. Shilov, *Generalized Functions*, Vol. 2, Acad. Press, New York 1964.
[6] B. Schroer, *J. Math. Phys.*, **5**, 1361 (1964).
[7] W. Güttinger, *Nuovo Cimento*, **10**, 1 (1958).
[8] Nguen Van Hieu, *Ann. Phys.* (USA), **33**, 428 (1965).
[9] A . Jaffe, Harvard University preprint 1968.
[10] F. Constantinescu, Munich preprint 1968.
[11] F. Constantinescu, Munich preprint 1969.
[12] G. V. Efimov, Preprint ITF-Kiev 68–52, 1968.
[13] J. Arsac, *Fourier Transforms and the Theory of Distributions*, Prentice-Hall, Inc., Englewood Cliffs, NJ 1966.
[14] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Oxford 1963.
[15] S. Bernstein, *Leçons sur les Propriétés Extrémales des Polynomes*, Gauthier-Villars, Paris 1926.
[16] M. K. Volkov, *Commun. Math. Phys.*, **7**, 289 (1968).
[17] G. V. Efimov, *Soviet Phys. JETP* (USA), **17**, 1417 (1963).
[18] B. W. Lee, B. Zumino, CERN preprint 1053 (1969).
[19] G. V. Efimov, preprint ITF-Kiev 68–54, 1968.