KINEMATICAL SINGULARITIES OF THE THREE-BODY DECAY HELICITY AMPLITUDES

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The kinematical behaviour of the three-body decay helicity amplitudes at the boundary of the physical region and of the decay transversity amplitudes at the thresholds and pseudo-thresholds is discussed. The three-body decay amplitudes possess the threshold and pseudo-threshold singularities in s, t and u.

1. *Introduction*

The kinematical behaviour of the two-body scattering helicity amplitudes seems at the moment to be a closed problem. Kinematical singularities of such amplitudes were dicussed in a number of papers (*cf.* Refs [1–6]). The most complete discussion was given in the study by Cohen-Tannoudji, Morel and Navelet [6].

In our previous paper [7] we derived the crossing relations for the helicity amplitudes between the two-body scattering channels and the three-body decay channel. We assumed there that the analytic properties of spinor amplitudes allow such a crossing.

Using the results from this paper we discuss here the kinematical behaviour of the three-body decay helicity amplitudes.

As in Ref. [7] the three-body decay channel reaction is

$$d\colon\ 0 \to 1, 2, 3,$$

the scattering channels reactions being

$$s\colon\ 2, 3 \to 0, \bar{1}$$

$$t\colon\ 3, 1 \to 0, \bar{2}$$

$$u\colon\ 1, 2 \to 0, \bar{3},$$

$(m_0 > m_1+m_2+m_3)$.

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The obtained results are presented in Sec. 2. We collect there also the notation used in this paper. Some examples of the three-body decay helicity amplitudes are given in Sec. 3. In Sec. 4 we discuss the kinematical behaviour of the decay helicity amplitudes at the boundary of the physical region. Our method follows that of Ref. [6], that is we use the Joos expansion of spinor amplitudes in terms of covariant polynomials [8–10] and study the terms singular at the boundary of the physical region of thus obtained helicity amplitudes.

Sec. 5 is devoted to the discussion of the threshold and pseudothreshold singularities of decay transversity amplitudes [11]. For these amplitudes the kinematical singularities at thresholds and pseudothresholds are factorizable [6, 12]. The method makes use of the crossing relations for the helicity amplitudes between the scattering and decay channels obtained in Ref. [7].

The details of the calculations are given in the Appendices.

2. *Results and notation*

In this Section we present the kinematical behaviour of the decay helicity amplitudes at the boundary of the physical region and the threshold and pseudothreshold singularities of the corresponding transversity amplitudes. The notation used in these formulae is explained below.

The helicity decay amplitudes $M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0}$ behave at the boundary of the physical region as

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0} = \left(\sin\frac{\theta_{23}}{2}\right)^{|\lambda_0-\lambda_1+\lambda_2+\lambda_3|}\left(\sin\frac{\theta_{31}}{2}\right)^{|\lambda_0-\lambda_1+\lambda_2-\lambda_3|}\times$$
$$\times\left(\sin\frac{\theta_{12}}{2}\right)^{|\lambda_0-\lambda_1-\lambda_2+\lambda_3|}R(s,t), \tag{2.1}$$

where $R(s, t)$ is kinematically regular at the boundary of the physical region. Angles θ_{ij} are the angles between the three-momenta of i-th and j-th particles in the rest system of particle 0. Analytic expressions for sines and cosines of these angles are given below (formula (2.7)).

Formula (2.1) means that

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0} \sim \begin{cases} \sqrt{\Phi}^{|\lambda_0-\lambda_1+\lambda_2+\lambda_3|} \text{ when } \theta_{23}\to 0;\ \theta_{31},\theta_{12}\to\pi, \\ \sqrt{\Phi}^{|\lambda_0-\lambda_1+\lambda_2-\lambda_3|} \text{ when } \theta_{31}\to 0;\ \theta_{12},\theta_{23}\to\pi, \\ \sqrt{\Phi}^{|\lambda_0-\lambda_1-\lambda_2+\lambda_3|} \text{ when } \theta_{12}\to 0;\ \theta_{23},\theta_{31}\to\pi. \end{cases} \tag{2.2}$$

Here $\Phi(s, t) = 0$ is the boundary equation of the physical region (Ref. [6]). Inside this region $\Phi(s, t) > 0$. We choose here the positive determination of $\sqrt{\Phi}$.

The method used to obtain formula (2.1) will be sketched in Sec. 4.

To describe the kinematical behaviour of the amplitudes at thresholds and pseudothresholds we use so called threshold and pseudothreshold functions (*cf.* Ref. [6]).

$$\varphi_{0i}(r) = [r-(m_0+m_i)^2]^{1/2},$$
$$\psi_{0i}(r) = [r-(m_0-m_i)^2]^{1/2}, \tag{2.3}$$

where $r = s, t, u$ for

$$i = 1,\ 2,\ 3.$$

On the physical Riemann sheet functions $\varphi_{0i}(r)$ and $\psi_{0i}(r)$ are positive for big positive real r. In the decay channel these functions are purely imaginary positive and the functions $\mathcal{R}_{0i}(r)$ defined by the formula

$$\mathcal{R}_{0i} = \varphi_{0i}(r)\, \psi_{0i}(r), \tag{2.4}$$

where $\mathcal{R} = \mathcal{S},\ \mathcal{T},\ \mathcal{U}$
$r = s, t, u$ for
$i = 1, 2, 3$ correspondingly,
are real negative.

In terms of these functions we express the kinematical behaviour of the decay transversity amplitudes. We find that the transversity amplitudes $T^d_{\tau_1\tau_2\tau_3,\tau_0}$ behave like

$$\begin{aligned} T^d_{\tau_1\tau_2\tau_3,\tau_0} &\sim \varphi_{01}(s)^{-\varepsilon_{12}(\tau_0-\tau_1)}\psi_{01}(s)^{-\varepsilon_{12}(\tau_0+\tau_1)}\times \\ &\times \varphi_{02}(t)^{\varepsilon_{23}\tau_2}\psi_{02}(t)^{-\varepsilon_{23}\tau_2}\varphi_{03}(u)^{\varepsilon_{31}\tau_3}\psi_{03}(u)^{-\varepsilon_{31}\tau_3} \\ &= \mathcal{S}_{01}^{-\varepsilon_{12}\tau_0}\left(\frac{\varphi_{01}(s)}{\psi_{01}(s)}\right)^{\varepsilon_{12}\tau_1}\left(\frac{\varphi_{02}(t)}{\psi_{02}(t)}\right)^{\varepsilon_{23}\tau_2}\left(\frac{\varphi_{03}(u)}{\psi_{03}(u)}\right)^{\varepsilon_{31}\tau_3}. \end{aligned} \tag{2.5}$$

In these formulae $\varepsilon_{ij} = \pm 1$. The meaning of these symbols is given below (formula (2.8)).

The derivation of Eq. (2.5) is given in Sec. 5.

For practical calculations we use here the rest system of particle 0 (the system in which $p_0 = (m_0, 0,0,0)$).

The following notation will be introduced: ω_i and $\vec{k}_i$ are the energy and the three--momentum of i-th particle ($i = 1, 2, 3$). In our convention all the $\vec{k}_i$'s lie in the xz plane.

In the rest system of particle 0 ω_i and k_i are

$$\omega_i = \frac{1}{2m_0}(m_0^2 + m_i^2 - r),$$

$$k_i = -\frac{\mathcal{R}_{0i}}{2m_0}, \tag{2.6}$$

where $r = s, t, u$, an $\mathcal{R}_{0i}$ is defined by Eq. (2.4).

The angles θ_{12}, θ_{23} and θ_{31} are the angles between $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ (Fig. 1). The cosines and sines of these angles can be expressed by the Mandelstam invariants in the following way

$$\begin{aligned} \cos\theta_{23} &= \frac{L_{23}}{\mathcal{T}_{02}\mathcal{U}_{03}}, & \sin\theta_{23} &= \frac{2m_0\sqrt{\Phi}}{\mathcal{T}_{02}\mathcal{U}_{03}}, \\ \cos\theta_{31} &= \frac{L_{31}}{\mathcal{U}_{03}\mathcal{S}_{01}}, & \sin\theta_{31} &= \frac{2m_0\sqrt{\Phi}}{\mathcal{U}_{03}\mathcal{S}_{01}}, \\ \cos\theta_{12} &= \frac{L_{12}}{\mathcal{S}_{01}\mathcal{T}_{02}}, & \sin\theta_{12} &= \frac{2m_0\sqrt{\Phi}}{\mathcal{S}_{01}\mathcal{T}_{02}}, \end{aligned} \tag{2.7}$$

where the numerators L_{ij} in the expressions for cosines of the decay angles are

$$L_{23} = (t+m_0^2-m_2^2)(u+m_0^2-m_3^2)-2m_0^2(m_0^2+m_1^2-m_2^2-m_3^2),$$
$$L_{31} = (u+m_0^2-m_3^2)(s+m_0^2-m_1^2)-2m_0^2(m_0^2+m_2^2-m_1^2-m_3^2),$$
$$L_{12} = (s+m_0^2-m_1^2)(t+m_0^2-m_2^2)-2m_0^2(m_0^2+m_3^2-m_1^2-m_2^2). \qquad (2.7')$$

The signs ε_{ij} of these numerators

$$\varepsilon_{ij} = \text{sgn}\, L_{ij} \qquad (2.8)$$

are the functions of the Mandelstam invariants.

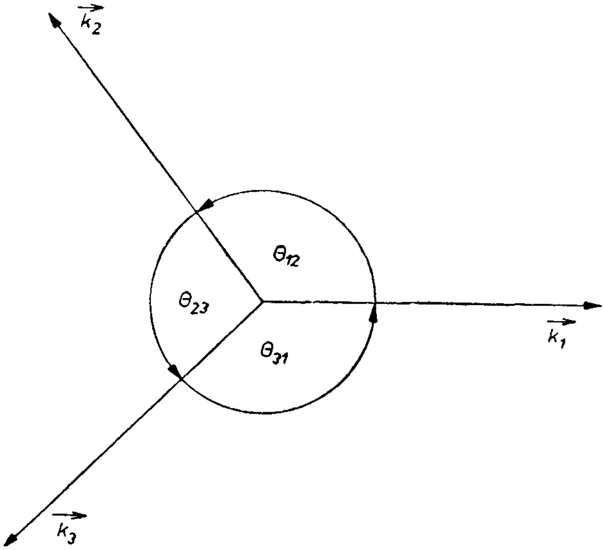

Fig. 1. Decay angles in the rest system of particle 0

Our definition of helicity frames is taken from Ref. [7]. For particle 0 we choose the helicity axis to be parallel to $\vec{k}_1$ (in the rest system of particle 0). For all the particles the y axis is orthogonal to the reaction plane: $y \,||\, -(\vec{k}_1 \times \vec{k}_2)$.

3. *Examples*

In this Section we discuss the possible applications of formula (2.1).

Our first example is the 3π decay of the natural parity meson V with the spin s_V.

$$V \to 3\pi.$$

The parity conservation condition for the three-body decay helicity amplitudes reads

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0} = \eta(-1)^{\sum_i (s_i+\lambda_i)} M^d_{-\lambda_1-\lambda_2-\lambda_3,-\lambda_0}, \qquad (3.1)$$

where $\eta = \eta_0 n_1 n_2 n_3$ is the product of intrinsic parities. This condition is analogous with the corresponding conditions for the two-body scattering helicity amplitudes (*cf.* Ref. [6]) and follows directly from the crossing relations between the scattering and decay helicity amplitudes [7].

For meson V $\eta_0 = (-1)^{s_V}$, we have the following parity conservation condition for the amplitudes $M^d_{\lambda_0}$

$$M^d_{\lambda_0} = (-1)^{2s_V+\lambda_0+1} M^d_{-\lambda_0} = (-1)^{\lambda_0+1} M^d_{-\lambda_0}. \tag{3.2}$$

This means that for $\lambda_0 = 0$

$$M^d_0 = -M^d_0 = 0. \tag{3.3}$$

Using the formula (2.1) we find that

$$M^d_{\lambda_0} \sim \sqrt{\Phi}^{\,|\lambda_0|} \tag{3.4}$$

and therefore (3.3) all the amplitudes vanish at the boundary of the physical region.

That is one of the well-known Zemach rules [13]. The other Zemach rules follow from the exchange symmetry of the final states.

The second example is the reaction

$$\left(\frac{3}{2}\right)^+ \to \left(\frac{1}{2}\right)^+, (0)^-, (0)^-.$$

Following Jackson and Hite [5] we take the general form of the amplitude to be

$$M = \bar{u}(p_1)\left\{\left[A_1 - i\gamma\left(\frac{p_3-p_2}{2}\right)B_1\right](p_3-p_2)_\mu + \right.$$

$$\left. + \left[A_2 - i\gamma\left(\frac{p_3-p_2}{2}\right)B_2\right](p_3+p_2)_\mu\right\}\gamma_5 U^\mu(p_0), \tag{3.5}$$

where $U^\mu(p_0)$ is the Rarita Schwinger wave function for particle 0 and $u(p_1)$ is a Dirac spinor for particle 1. The Dirac notation used here is analogous as that used in Ref. [5].

The helicity amplitudes are constructed by choosing the definite helicities for particles 0 and 1 in the rest system of particle 0. In order to simplify the kinematics we assume that particles 2 and 3 have the same mass.

After the reduction we get

$$M_{-1/2,\,3/2} = \frac{i\sqrt{2}\,\Phi}{\varphi_{01}(s)\,\psi_{01}(s)^2}\,B_1,$$

$$M_{1/2,\,3/2} = \frac{i\sqrt{2}\sqrt{\Phi}}{\varphi_{01}(s)}\left[A_1 - \frac{(m_0-m_1)}{\psi_{01}(s)^2}\,\frac{(t-u)}{2}\,B_1\right],$$

$$M_{-1/2,\,1/2} = \frac{i\sqrt{\Phi}}{\varphi_{01}(s)}\left[-\sqrt{\frac{2}{3}}\,A_1 + \sqrt{\frac{2}{3}}\,\frac{(3m_0(m_0-m_1)+\psi_{01}(s)^2)}{m_0\psi_{01}(s)^2}\,\frac{(t-u)}{2}\,B_1 + \right.$$

$$\left. + \frac{1}{\sqrt{6}}\,\frac{\varphi_{01}(s)^2}{m_0}\,B_2\right],$$

$$M_{1/2,\,1/2} = i\left\{\sqrt{\frac{2}{3}}\,\frac{(s+m_0^2-m_1^2)}{m_0\varphi_{01}(s)}\cdot\frac{(t-u)}{2}A_1+\right.$$

$$+\frac{1}{\sqrt{6}}\,\frac{\varphi_{01}(s)\,\psi_{01}(s)^2}{m_0}A_2+\sqrt{\frac{2}{3}}\,\frac{1}{\varphi_{01}(s)\,\psi_{01}(s)^2}\left[\Phi-\right.$$

$$\left.-\frac{(m_0-m_1)\,(s+m_0^2-m_1^2)}{m_0}\,\frac{(t-u)^2}{4}\right]B_1-$$

$$\left.-\sqrt{\frac{2}{3}}\,\frac{(m_0-m_1)}{m_0}\,\varphi_{01}(s)\,\frac{(t-u)}{4}\,B_2\right\}. \tag{3.6}$$

Formula (2.1) gives in this case the following behaviour of the three-body decay helicity amplitudes at the boundary of the physical region

$$M^d_{\lambda_1,\lambda_0}\sim\sqrt{\Phi}^{\,|\lambda_0-\lambda_1|} \tag{3.7}$$

which agrees with formulae (3.6).

For this case we can also check formula (2.5), that is the threshold and pseudothreshold behaviour of transversity amplitudes. Transversity amplitudes can be expressed as linear combinations of helicity amplitudes with numerical coefficients [11] (*cf.* Sec. 5).

For this reaction the non-vanishing transversity amplitudes are $T^d_{1/2,\,3/2}$, $T^d_{1/2,\,-1/2}$, $T^d_{-1/2,\,1/2}$ and $T^d_{-1/2,\,-3/2}$. The threshold and pseudothreshold behaviour of these amplitudes agrees with (2.5). The singular points are $s=(m_0\pm m_1)^2$.

Using Eq. (3.6) we obtain the following threshold and pseudothreshold behaviour

$$\begin{aligned}
T^d_{1/2;\,3/2}&\sim\varphi_{01}(s)^{-1}\,\psi_{01}(s)^{-2},\\
T^d_{-1/2;\,1/2}&\sim\varphi_{01}(s)^{-1},\\
T^d_{1/2;\,-1/2}&\sim\varphi_{01}(s),\\
T^d_{-1/2;\,-3/2}&\sim\varphi_{01}(s)\,\psi_{01}(s)^2
\end{aligned} \tag{3.8}$$

which can be compared with the behaviour following from (2.5)

$$T^d_{\tau_1,\,\tau_0}\sim\varphi_{01}(s)^{-(\tau_0-\tau_1)}\psi_{01}(s)^{-(\tau_0+\tau_1)}. \tag{3.9}$$

4. *Kinematical singularities at $\Phi(s,t)=0$*

We use the definition of the decay helicity amplitudes from Ref. [7], that is, we express these amplitudes in terms of the spinor amplitudes. The spinor amplitudes enjoy some important analyticity properties. We recall here the Joos expansion of the spinor amplitudes in terms of covariant polynomials [8]. The coefficients in this expansion were proved by Hepp [9] and Williams [10] to be free from kinematical singularities.

Using the Joos expansion of spinor amplitudes we write the helicity amplitudes in terms of Joos invariant amplitudes (*cf.* Ref. [6]) and extract the part singular at $\Phi(s,t)=0$.

In the frame where $\vec{k}_1 \| z$ amplitudes $M^d_{\lambda_1\lambda_2\lambda_3, \lambda_0}$ can be written

$$M^d_{\lambda_1\lambda_2\lambda_3, \lambda_0} = \sum d^{s_2}(\theta_{12})_{A_2 - \lambda_2} d^{s_3}(-\theta_{31})_{A_3 - \lambda_3} (\sin \theta_{12})^{|M_1|} \times$$
$$\times (\sin \theta_{31})^{|M_2|} F(s, t, u), \tag{4.1}$$

where $F(s, t, u)$ is kinematically regular at $\Phi(s, t) = 0$. The summation goes over A_2, A_3, M_1 and M_2 with

$$M_1 + M_2 = \lambda_0 - \lambda_1 + A_2 + A_3.$$

The terms singular at $\Phi(s, t) = 0$ are $\sin(\theta_{ij}/2)$ and $\cos(\theta_{ij}/2)$ since $\Phi(s, t) = 0$ implies $|\cos \theta_{ij}| = 1$.

Using explicit expressions for $d^s(\theta)_{A\lambda}$ and some simple relations between cosines and sines of $\theta_{ij}/2$ we find (*cf.* Appedix A) the following behaviour of the helicity decay amplitudes at the boundary of the physical region

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0} \sim \left(\sin \frac{\theta_{23}}{2}\right)^{|\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3|} \left(\sin \frac{\theta_{31}}{2}\right)^{|\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3|} \left(\sin \frac{\theta_{12}}{2}\right)^{|\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3|}. \tag{4.1'}$$

The derivation of formula (4.1′) is given in Appendix A.

5. *Kinematical singularities at thresholds and pseudothresholds*

We discuss the threshold and pseudothreshold singularities of transversity amplitudes [11] since for these amplitudes the singularities at thresholds and pseudotresholds are factorizable [6, 12].

The transversity decay amplitudes can be expressed as linear combinations of helicity amplitudes with numerical coefficients

$$T^d_{\tau_1\tau_2\tau_3, \tau_0} = D^{s_0}(R)_{\lambda_0\tau_0} D^{s_1}(R^*)_{\lambda_1\tau_1} D^{s_2}(R^*)_{\lambda_2\tau_2} D^{s_3}(R)_{\lambda_3\tau_3} M^d_{\lambda_1\lambda_2\lambda_3, \lambda_0}. \tag{5.1}$$

Here R is a rotation through $-\pi/2$ around the first axis, *i. e.* specified by Euler angles $\pi/2$, $\pi/2$, $-\pi/2$.

The well-known property of transversity amplitudes is the diagonal form of the crossing matrices [11]. The crossing relations for the helicity amplitudes between the two-body scattering channels and the three-body decay channel obtained in Ref. [7] can be rewritten for transversity amplitudes as

$$\begin{aligned}
T^d_{\tau_1\tau_2\tau_3, \tau_0} &= (-1)^{\Sigma s}(-1)^{s_0+s_1-s_2+s_3} \exp\left[-i\pi(\tau_0 - \tau_2 - \tau_3)\right] \times \\
&\quad \times \exp\left[-i(\tau_2\chi_2^s + \tau_3\chi_3^s)\right] T^s_{-\tau_0 - \tau_1, -\tau_2 - \tau_3} \\
&= (-1)^{\Sigma t}(-1)^{s_0-s_1+s_2+s_3} \exp\left[-i\pi(\tau_0 - \tau_1 - \tau_3)\right] \times \\
&\quad \times \exp\left[i(\tau_0\chi_0^t - \tau_1\chi_1^t - \tau_3\chi_3^t)\right] T^t_{-\tau_0 - \tau_2, -\tau_3 - \tau_1} \\
&= (-1)^{\Sigma u}(-1)^{s_0+s_1+s_2-s_3} \exp\left[-i\pi(\tau_0 - \tau_1 - \tau_2)\right] \times \\
&\quad \times \exp\left[i(\tau_0\chi_0^u - \tau_1\chi_1^u - \tau_2\chi_2^u)\right] T^u_{-\tau_0 - \tau_3, -\tau_1 - \tau_2}.
\end{aligned} \tag{5.2}$$

In these formulae factor $(-1)^{\Sigma}$ $(\Sigma = 0, 1)$ arises from the change of order of the indices in the amplitudes and depends on the kind of particles involved in the reaction (*cf.* Ref. [7]). The crossing angles χ are collected in the Appendix of Ref. [7].

The amplitudes $T^d_{\tau_1\tau_2\tau_3,\tau_0}$ have threshold and pseudothreshold singularities in s, t and u. From Eq. (2.6) one can guess that the singular points are

$$s = (m_0 \pm m_1)^2,$$

$$t = (m_0 \pm m_2)^2,$$

$$u = (m_0 \pm m_3)^2.$$

Since amplitudes $T^s_{-\tau_0-\tau_1,-\tau_2-\tau_3}$ are not singular at t and u thresholds and pseudothresholds, the singularities of $T^d_{\tau_1\tau_2\tau_3,\tau_0}$ at these points must be simultaneously the singularities of the factor $\exp\left[-i(\tau_2\chi_2^s+\tau_3\chi_3^s)\right]$ from the s-channel crossing relations (5.2).

Analogous arguments can be made for s and u singularities using the t-channel crossing relations and s and t singularities using the u-channel crossing relations. Each singularity at s, t and u thresholds and pseudothresholds appears thus in two independent crossing relations. This is an additional test of our results.

The kinematical behaviour of the expressions $\exp(i\chi)$ can be related to the behaviour of $\exp(i\theta_{ij})$. The kinematical singularities of the latter are discussed in Appendix B.

The method used here is to find some combinations φ of the angles χ and θ_{ij} such that the expressions $\exp(i\varphi)$ are not singular at certain thresholds and pseudothresholds. This problem is discussed in Appendix C.

From these relations and using (5.2) we find the kinematical behaviour of the transversity decay amplitudes at thresholds and pseudothresholds (2.5).

$$T^d_{\tau_1\tau_2\tau_3,\tau_0} \sim \varphi_{01}(s)^{-\varepsilon_{12}(\tau_0-\tau_1)}\psi_{01}(s)^{-\varepsilon_{12}(\tau_0+\tau_1)} \times$$

$$\times\, \varphi_{02}(t)^{\varepsilon_{23}\tau_2}\psi_{02}(t)^{-\varepsilon_{23}\tau_2}\varphi_{03}(u)^{\varepsilon_{31}\tau_3}\psi_{03}(u)^{-\varepsilon_{31}\tau_3}$$

$$= \mathcal{S}_{01}^{-\varepsilon_{12}\tau_0}\left(\frac{\varphi_{01}(s)}{\psi_{01}(s)}\right)^{\varepsilon_{12}\tau_1}\left(\frac{\varphi_{02}(t)}{\psi_{02}(t)}\right)^{\varepsilon_{23}\tau_2}\left(\frac{\varphi_{03}(u)}{\psi_{03}(u)}\right)^{\varepsilon_{31}\tau_3}. \tag{5.3}$$

APPENDIX A

Derivation of formulae (2.1)

Using explicit formulae for $d^s(\theta)_{\Lambda\lambda}$ we can rewrite (4.1) to get

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0} = \left(\sin\frac{\theta_{12}}{2}\right)^{|A_2+\lambda_2|+|M_1|}\left(\cos\frac{\theta_{12}}{2}\right)^{|A_2-\lambda_2|+|M_1|} \times$$

$$\times \left(\sin\frac{\theta_{31}}{2}\right)^{|A_3+\lambda_3|+|M_2|}\left(\cos\frac{\theta_{31}}{2}\right)^{|A_3-\lambda_3|+|M_2|} F'(s,t,u). \tag{A.1}$$

Since $\theta_{12}+\theta_{23}+\theta_{31} = 2\pi$, we can write the following identities:

$$\cos\frac{\theta_{31}}{2} = \sin\frac{\theta_{12}}{2}\sin\frac{\theta_{23}}{2}\frac{2\mathcal{T}_{02}}{\mathcal{S}_{01}+\mathcal{T}_{02}+\mathcal{U}_{03}} = \sin\frac{\theta_{12}}{2}\sin\frac{\theta_{23}}{2}\Lambda_{31},$$

$$\cos\frac{\theta_{12}}{2} = \sin\frac{\theta_{23}}{2}\sin\frac{\theta_{31}}{2}\frac{2\mathcal{U}_{03}}{\mathcal{S}_{01}+\mathcal{T}_{02}+\mathcal{U}_{03}} = \sin\frac{\theta_{23}}{2}\sin\frac{\theta_{31}}{2}\Lambda_{12},$$

$$\cos\frac{\theta_{23}}{2} = \sin\frac{\theta_{31}}{2}\sin\frac{\theta_{12}}{2}\frac{2\mathcal{S}_{01}}{\mathcal{S}_{01}+\mathcal{T}_{02}+\mathcal{U}_{03}} = \sin\frac{\theta_{31}}{2}\sin\frac{\theta_{12}}{2}\Lambda_{23}. \tag{A.2}$$

Functions Λ_{31}, Λ_{12} and Λ_{23} are not singular at $\Phi(s, t) = 0$. Using (A.2) we can rewrite (A.1) in the more symmetrical form:

$$M^{d}_{\lambda_1\lambda_2\lambda_3,\lambda_0} \sim \left(\sin\frac{\theta_{12}}{2}\right)^{|A_2+\lambda_2|+|M_1|+|A_3-\lambda_3|+|\lambda_0-\lambda_1+A_2+A_3-M_1|}\times$$

$$\times\left(\sin\frac{\theta_{23}}{2}\right)^{|A_2-\lambda_2|+|M_1|+|A_3-\lambda_3|+|\lambda_0-\lambda_1+A_2+A_3-M_1|}\times$$

$$\times\left(\sin\frac{\theta_{31}}{2}\right)^{|A_2-\lambda_2|+|M_1|+|A_3+\lambda_3|+|\lambda_0-\lambda_1+A_2+A_3-M_1|}. \tag{A.3}$$

Let

$$\mathcal{A} = |A_2+\lambda_2|+|M_1|+|A_3-\lambda_3|+|\lambda_0-\lambda_1+A_2+A_3-M_1|,$$

$$\mathcal{B} = |A_2-\lambda_2|+|M_1|+|A_3+\lambda_3|+|\lambda_0-\lambda_1+A_2+A_3-M_1|,$$

$$\mathcal{C} = |A_2-\lambda_2|+|M_1|+|A_3-\lambda_3|+|\lambda_0-\lambda_1+A_2+A_2-M_1|.$$

The minimal values of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are

$$\mathcal{A}_{\min} = |\lambda_0-\lambda_1-\lambda_2+\lambda_3|,$$

$$\mathcal{B}_{\min} = |\lambda_0-\lambda_1+\lambda_2-\lambda_3|,$$

$$\mathcal{C}_{\min} = |\lambda_0-\lambda_1+\lambda_2+\lambda_3|.$$

All the other values of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ differ from $\mathcal{A}_{\min}$, $\mathcal{B}_{\min}$ and $\mathcal{C}_{\min}$ by even positive integer.

Therefore the singular part of $M^{d}_{\lambda_1\lambda_2\lambda_3,\lambda_0}$ is

$$M^{d}_{\lambda_1\lambda_2\lambda_3,\lambda_0} \sim \left(\sin\frac{\theta_{12}}{2}\right)^{\mathcal{A}\min}\left(\sin\frac{\theta_{31}}{2}\right)^{\mathcal{B}\min}\left(\sin\frac{\theta_{23}}{2}\right)^{\mathcal{C}\min}. \tag{A.4}$$

APPENDIX B

The kinematical behaviour of $\exp(i\theta_{ij})$ at tresholds and pseudothresholds

The following equalities between L_{ij} (formula (2.7')) can be easily derived ($\varepsilon = \pm 1$)

$$L_{12}(s = (m_0+\varepsilon m_1)^2) = -L_{31}(s = (m_0+\varepsilon m_1)^2),$$

$$L_{23}(t = (m_0+\varepsilon m_2)^2) = -L_{12}(t = (m_0+\varepsilon m_2)^2),$$

$$L_{31}(u = (m_0+\varepsilon m_3)^2) = -L_{23}(u = (m_0+\varepsilon m_3)^2). \tag{B.1}$$

If we turn around the point $s = (m_0 \pm m_1)^2$ the following changes occur

$$\begin{cases} \theta_{12} \pm \pi \\ \theta_{31} \mp \pi \end{cases} \text{when } L_{12} \gtrless 0,\ L_{31} \lessgtr 0.$$

Turning around the point $t = (m_0 \pm m_2)^2$ implies

$$\begin{cases} \theta_{12} \pm \pi \\ \theta_{23} \mp \pi \end{cases} \text{when } L_{12} \gtrless 0,\ L_{23} \lessgtr 0.$$

Turning around the point $u = (m_0 \pm m_3)^2$

$$\begin{cases} \theta_{31} \pm \pi \\ \theta_{23} \mp \pi \end{cases} \text{when } L_{31} \gtrless 0,\ L_{23} \lessgtr 0.$$

From these dependences we can deduce the kinematical behaviour of $\exp(i\theta_{ij})$ at thresholds and pseudothresholds to be

$$\begin{aligned} \exp(i\theta_{12}) &\sim \varphi_{01}(s)^{-\varepsilon_{12}} \psi_{01}(s)^{-\varepsilon_{12}} \varphi_{02}(t)^{\varepsilon_{23}} \psi_{02}(t)^{\varepsilon_{23}}, \\ \exp(i\theta_{23}) &\sim \varphi_{02}(t)^{-\varepsilon_{23}} \psi_{02}(t)^{-\varepsilon_{23}} \varphi_{03}(u)^{\varepsilon_{31}} \psi_{03}(u)^{\varepsilon_{31}}, \\ \exp(i\theta_{31}) &\sim \varphi_{03}(u)^{-\varepsilon_{31}} \psi_{03}(u)^{-\varepsilon_{31}} \varphi_{01}(s)^{\varepsilon_{12}} \psi_{01}(s)^{\varepsilon_{12}}. \end{aligned} \tag{B.2}$$

The symbols ε_{ij} and the functions $\varphi_{0i}(r)$ and $\psi_{0i}(r)$ were introduced in Sec. 2.

APPENDIX C

Non-singular combinations of angles

The kinematical behaviour of the expressions $\exp(i\chi)$ in (5.2) can be related to the kinematical behaviour of $\exp(i\theta_{ij})$.

Using explicit expressions for the crossing angles χ from Ref. [7] we construct the combinations φ of these angles with the angles θ_{ij} such that $\exp(i\varphi)$ is not singular at certain thresholds and pseudothresholds.

These combinations are presented in Table I.

TABLE I

Non-singular combinations of angles

φ	not singular at
$\chi_0^t + \varepsilon\chi_1^t$	$s = (m_0 - \varepsilon m_1)^2$
$\theta_{12} + \chi_0^t$	$s = (m_0 \pm m_1)^2$
$\chi_0^u + \varepsilon\chi_1^u$	$s = (m_0 - \varepsilon m_1)^2$
$\theta_{31} - \chi_0^u$	$s = (m_0 \pm m_1)^2$
$\theta_{12} + \varepsilon\chi_2^s$	$t = (m_0 + \varepsilon m_2)^2$
$\theta_{23} + \varepsilon\chi_2^u$	$t = (m_0 - \varepsilon m_2)^2$
$\theta_{31} + \varepsilon\chi_3^s$	$u = (m_0 - \varepsilon m_3)^2$
$\theta_{23} + \varepsilon\chi_3^t$	$u = (m_0 + \varepsilon m_3)^2$

$\varepsilon = \pm 1$

Using Table I and formulae (B.2) we find the following behaviour of the expressions $\exp(i\chi)$

$$\exp [i(\chi_0+\chi_1)] \sim \varphi_{01}(s)^{2\varepsilon_{12}},$$

$$\exp [i(\chi_0-\chi_1)] \sim \psi_{01}(s)^{2\varepsilon_{12}},$$

$$\exp (i\chi_2) \sim \varphi_{02}(t)^{\varepsilon_{23}}\psi_{02}(t)^{-\varepsilon_{23}} = \left(\frac{\varphi_{02}(t)}{\psi_{02}(r)}\right)^{\varepsilon_{23}},$$

$$\exp (i\chi_3) \sim \varphi_{03}(u)^{\varepsilon_{31}}\psi_{03}(u)^{-\varepsilon_{31}} = \left(\frac{\varphi_{03}(u)}{\psi_{03}(u)}\right)^{\varepsilon_{31}}. \tag{C.1}$$

In these formulae indices s, t, u for the crossing angles are omitted since, as it can be checked, the behaviour of the expression $\exp(i\chi_i^r)$ ($r = s, t, u$; $i = 0...3$) depends only on the index "i".

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