ON THE SPINOR REPRESENTATIONS OF THE COMPLEX INHOMOGENEOUS LORENTZ GROUP. I

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We investigate the spinor representations of the complex Poincaré group in the case when the complex mass of "the particle" is different from zero.

The spinor representations of the complex Poincaré group are non-equivalent to the unitary representations, but they are equivalent to the unitary representations, when we restrict to the real Poincaré group.

1. Introduction

In this article we discuss the spinor representations of the complex inhomogeneous Lorentz group. The complex continuation of the covariant functions in the quantum field theory, and covariance under the complex homogeneous Lorentz group are guaranteed by the Bargmann-Hall-Wightman theorem. If we assume that the scattering amplitudes are covariant under the complex Poincaré group, we obtain automatically the results of quantum field theory such as the CPT theorem, crossing relations and the spin-statistics relation.

This point of view was proposed by Roffman [1]. We believe that the unitary representations of the complex Poincaré group and the non-unitary representations which reduce under the real Poincaré group to unitary representations have physical sense. The spinor representations which we discuss in this paper reduce to representations equivalent to unitary representations when we restrict to the real Poincaré group. We investigate the spinor representations in the case when the complex mass "particle" is different from zero.

The spinors which we introduce obey the Dirac type relation. In the simplest non-trivial case, we obtain the relation formally identical with the Dirac equation in the momentum representations, for complex mass and momentum of the "particle".

2. Complex inhomogeneous Lorentz group

The complex inhomogeneous Lorentz transformations are of the form

$$z^{\mu} \to z'^{\mu} = w^{\mu}+A^{\mu}x^{\nu}, \quad \mu, \nu = 0, 1, 2, \ldots \tag{1}$$

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where

$$A^T G A = G, \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This group, which we will denote $P(C)$, is also named complex Poincaré group.

The proper Poincaré group $P_+(C)$ is the connected component of the unity $P(C)$.

The $P_+(C)$ is the semi-direct product of two topological groups $T_4(C)$ and $L_+(C)$ (complex proper Lorentz group).

$T_4(C)$ is the Abelian group of four dimensional translations. We construct the universal covering group $\widetilde{P_+(C)}$ of $P_+(C)$.

With each four vector z^μ, we associate a 2×2 matrix

$$\underset{\sim}{z} = z^0\sigma_0 + z^k\sigma_k = \begin{pmatrix} z^0+z^3 & z^1-iz^2 \\ z^1+iz^2 & z^0-z^3 \end{pmatrix}. \tag{2}$$

Let SL(2, C) be the group of 2×2 complex matrices of determinant 1.

With each vector z and pair of the matrices (a, b), $a \in \mathrm{SL}(2, C)$, $b \in \mathrm{SL}(2, C)$ we associate the vector z' defined by

$$\underset{\sim}{z'} = a\underset{\sim}{z}b^+ = \widetilde{A(a, b)}\, z. \tag{3}$$

where

$$A(a, b) \in L_+(C).$$

$\widetilde{P_+(C)}$ is thus the set of elements (a, b, w) with the group law

$$(a_2, b_2, w_2)\,(a_1, b_1, w_1) = (a_2a_1, b_2b_1, w_2+a_2w_1b_2^+). \tag{4}$$

where $\underset{\sim}{w} = w \in T_4(C)$.

3. *Irreducible representations of the complex Poincaré group*

We denote by $U(A, w)$ or $U(a, b, w)$ the representations of the complex inhomogeneous Lorentz group.

Equation (4) implies the following multiplication law

$$U(a_2, b_2, w_2)\; U(a_1, b_1, w_1) = U(a_2a_1, b_2b_1, w_2+a_2w_1b_2^+).$$

In analogy with the Poincaré group we may find irreducible representations of the group $P_+(C)$ ([2], [3], [4]):

$$U(A, w) f(B_{r,s}) = e^{\frac{i}{2}(w\cdot r+\bar{w}\cdot s)}\, D(B_{r,s}^{-1} A B_{A_r^{-1}, A_s^{-1}}) f(B_{A_r^{-1}, A_s^{-1}}), \tag{5}$$

where $(A, w) \in P_+(C)$.

$B_{r,s}$ is the complex Lorentz transformation obeying:

$$B_{r,s}\overset{\circ}{r} = r, \quad B_{r,s}\overset{\circ}{s} = s. \tag{6}$$

where $\overset{\circ}{r}, \overset{\circ}{s}$ are the standard complex vectors, $R_{\overset{\circ}{r},\overset{\circ}{s}} = B_{r,s}^{-1} A B_{A_r^{-1}, A_s^{-1}}$ is the so-called Wigner rotation which belongs to the stationary group of $\overset{\circ}{r}, \overset{\circ}{s}$

$$R_{\overset{\circ}{r},\overset{\circ}{s}}\overset{\circ}{r} = \overset{\circ}{r}, \quad R_{\overset{\circ}{r},\overset{\circ}{s}}\overset{\circ}{s} = \overset{\circ}{s} \tag{7}$$

$D(R_{\overset{\circ}{r},\overset{\circ}{s}})$ is an irreducible representation of the stationary group $\overset{\circ}{r}, \overset{\circ}{s}$.

$T_4(C)$ has irreducible representations labelled by two complex four vectors r, s.

When $r = \bar{s}$, the representation of the $T_4(C)$ is an irreducible unitary representation in one-dimensional Hilbert space. In this paper we consider the case $r = \bar{s}$ and

$$r^2 = (r^0)^2 - (\vec{r})^2 = \alpha^2 \tag{8}$$

is different from zero. It is therefore convenient to fix the standard positions of $\overset{\circ}{r}, \overset{\circ}{s}$.

$$\overset{\circ}{r} = \alpha(1, 0, 0, 0), \quad \overset{\circ}{s} = \bar{\alpha}(1, 0, 0, 0). \tag{9}$$

The stationary group of the $\overset{\circ}{r}$ $\overset{\circ}{s}$ is isomorphic with the complex orthogonal group SO(3, C) in the three dimensions.

It is given by pairs of matrices

$$(a, (a^+)^{-1}), \quad a \in SL(2, C). \tag{10}$$

The SO(3, C) group is locally isomorphic with the SL(2, C) group.

The representations of the $P_+(C)$ are then given by operator $U(A, w)$

$$U(A, w)f(B_r) = e^{i\,\mathrm{Re}(w \cdot r)} D(B_r^{-1} A B_{A^{-1}r}) f(B_{A^{-1}r}) \tag{11}$$

where $B_r = B_{r,\bar{r}}$.

4. *Finite dimensional representations of the complex Lorentz group $L_+(C)$*

The finite dimensional (non-unitary) representations of the $L_+(C)$ can be obtained as a tensor product of the spinor representations of two SL(2, C) representations

$$D^{JK}(a, b) = D^J(a) \otimes D^K(b), \tag{12}$$

where $J = (j_1, j_2)$, $K = (k_1, k_2)$.

$$j_1, j_2, k_1, k_2 = 0, \tfrac{1}{2}, 1 \ldots$$

The representations $D^J(a)$ and $D^K(b)$ are the ordinary spinor representations of SL(2, C) ([5], [6]).

The representations $D^J(a)$ and $D^K(b)$ can be written as:

$$D^J(a) = D^{(j_1, j_2)}(a) = D^{(j_1, 0)}(a) \otimes D^{(0, j_2)}(a).$$

$$D^K(b) = D^{(k_1, k_2)}(b) = D^{(k_1, 0)}(b) \otimes D^{(0, k_2)}(b). \tag{13}$$

In particular, we are interested in the representations $D^{JK}(a, b)$ which are irreducible under SO(3, C).

If we adopt

$$b = (a^{+})^{-1}. \tag{14}$$

Then corresponding to (12), (13, (14), we must have

$$D^{JK}(a, (a^{+})^{-1}) = D^{(j_1,0)}(a) \otimes D^{(0,j_2)}(a) \otimes D^{(0,k_1)}(a) \otimes D^{(k_2,0)}(a)$$
$$= \sum_{r=|j_1-k_2|}^{j_1+k_2} \oplus \sum_{s=|j_2-k_1|}^{j_2+k_1} \oplus D^{rs}(a), \tag{15}$$

where we have used

$$D^{(j,0)}(a) = D^{(0,j)}((a^{+})^{-1}). \tag{16}$$

Equation (16) implies that the following representations of $L_+(C)$

$$D^{(0,j)(0,k)}(a, b),\ D^{(j,0)(k,0)}(a, b) \tag{17}$$

are irreducible under SO(3, C).

In analogy in with (15) we may reduce the representation of the $L_+(C)$ under the SL(2, C) subgroup.

$$D^{JK}(a, a) = D^{(j_1,0)}(a) \otimes D^{(0,j_2)}(a) \otimes D^{(k_1,0)}(a) \otimes D^{(0,k_2)}(a) = \sum_{r=|j_1-k_1|}^{j_1+k_1} \otimes \sum_{s=|j_2-k_2|}^{j_2+k_2} \otimes D^{rs}(a). \tag{18}$$

5. Spinor representations of the complex inhomogeneous Poincaré group

We define two spinor functions:

$$\varphi(r) = D^{(j,0)(k,0)}(B_r)f(B_r),$$

and

$$\hat{\varphi}(r) = D^{(0,j)(0,k)}(B_r)f(B_r) \tag{19}$$

where $D^{(j,0)(k,0)}(B_r)$, $D^{(0,j)(0,k)}(B_r)$ are irreducible finite dimensional representations of the $L_+(C)$ group.

The spinor functions $\varphi(r), \hat{\varphi}(r)$ transform under the transformations of $P_+(C)$ correspondingly

$$U(A, w)\,\varphi(r) = e^{i\,\mathrm{Re}(w\cdot r)}D^{(j,0)(k,0)}(A)\,\varphi(A^{-1}r),$$
$$U(A, w)\,\hat{\varphi}(r) = e^{i\,\mathrm{Re}(w\cdot r)}D^{(0,j)(0,k)}(A)\,\hat{\varphi}(A^{-1}r). \tag{20}$$

The spinor functions $\varphi(r)$, $\hat{\varphi}(r)$ are not independent, since they obey the relation

$$\varphi(r) = D^{(j,0)(k,0)}(B_r)\,D^{(0,j)(0,k)}(B_r^{-1})\,\hat{\varphi}(r). \tag{21}$$

We may use (3), (4) to calculate

$$D^{(j,0)(k,0)}(B_r)\,D^{(0,j)(0,k)}(B_r^{-1}). \tag{22}$$

We write

$$B_r = (a_r, b_r). \tag{23}$$

and after some algebraic manipulations we obtain

$$D^{(j,0)(k,0)}(B_r) \cdot D^{(0,j)(0,k)}(B_r^{-1}) = D^{(j,0)}(a_r a_r^+) \otimes D^{(k,0)}(b_r b_r^+). \tag{24}$$

With the help of the expression (24) we write (21) as

$$\varphi(r) = D^{(j,0)}(a_r a_r^+) \otimes D^{(k,0)}(b_r b_r^+)\hat{\varphi}(r). \tag{25}$$

This is the first type of the generalized Dirac equation for the $P_+(C)$ group.

We obtain the second type of the generalized Dirac equation if we take

$$\varphi(r) = D^{(j,0)(k,0)}(B_r) f(B_r) \tag{26}$$

and

$$\tilde{\varphi}(r) = D^{(0,k)(0,j)}(B_r) f(B_r).$$

The generalized Dirac equation for the $\varphi, \tilde{\varphi}$ spinor functions may thus be written

$$\varphi(r) = D^{(j,0)(k,0)}(B_r) D^{(0,k)(0,j)}(B_r^{-1})\tilde{\varphi}(r) \tag{27}$$

which after some manipulations, may be written as

$$\varphi(r) = D^{(j,0)}(\underset{\sim}{r}/\alpha) \otimes D^{(k,0)}(\underset{\sim}{r}^+/\bar{\alpha})\tilde{\varphi}(r). \tag{28}$$

In the simplest nontrivial case $j = 1/2$, $k = 0$ we obtain the relation

$$\varphi(r) = \underset{\sim}{r}/\alpha\ \tilde{\varphi}(r). \tag{29}$$

If we introduce the four spinor

$$\psi = (\varphi(r), \tilde{\varphi}(r))$$

we may rewrite (29) in the more symmetric form

$$(r^\mu \gamma_\mu - \alpha)\psi = 0$$

where γ_μ are the usual Dirac matrices.

This is the Dirac equation for complex momentum r^μ and complex mass α.

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