Vol. B1 (1970) ACTA PHYSICA POLONICA Fasc. 4

ON THE SPINOR REPRESENTATIONS OF THE COMPLEX INHOMOGENEOUS LORENTZ GROUP. II

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(*Received May 14, 1970*)

We construct the spinor representation of the complex Poincaré group in the case when the complex mass of the "particle" is equal to zero.

1. *Introduction*

In this paper we discuss the spinor representations of the complex Poincaré group for "massless particles". The spinor representations which we consider can be defined only for one particular case which correspond to unitary finite-dimensional representations of the stationary groups Z_2.

In this article we construct three types of the spinor functions.

2. *The definition and properties of the Z_2 group*

We may write the representations of the complex inhomogeneous Lorentz group as [1]:

$$U(A, w)f(B_r) = e^{i\,\mathrm{Re}\,(w\cdot r)}D(B_r^{-1}AB_{A^{-1}r})f(B_{A^{-1}r}), \tag{1}$$

where $B_r^{-1}AB_{A^{-1}r}$ belongs to the stationary group of standard momentum $\mathring{r}$.

If we assume that $r^2 = (\mathring{r})^2 = 0$, the stationary group of the standard momentum $\mathring{r} = (1, 0, 0, 1)$ contains the matrices

$$(a, b) = \left(\begin{pmatrix} \lambda & \lambda^{-1}z \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \begin{pmatrix} \bar{\lambda}^{-1} & \bar{\lambda}\bar{w} \\ 0 & \bar{\lambda} \end{pmatrix}\right) = g(\lambda, z, w) \tag{2}$$

where λ, z, w are complex numbers and the multiplication law is the following:

$$g(\lambda, z, w)\, g(\mu, r, s) = g(\lambda\mu, z+\lambda^2 r, w+\lambda^{-2}s). \tag{3}$$

This defines the Z_2 group.

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The subgroup of all elements of the form $g(1, z, w)$ is an invariant Abelian subgroup $T_2(C)$ isomorphic to the complex two-dimensional translation group.

The subgroup of all elements $g(\lambda, 0, 0)$ is an Abelian subgroup isomorphic to the SO(2, C) complex orthogonal two-dimensional group.

The translation subgroup $T_2(C)$ of Z_2 has irreducible unitary representations labelled by two complex numbers σ_1, σ_2. This representation is defined by:

$$T^{\delta}(\lambda, z, w) = e^{i\delta \operatorname{Re}(z\sigma_1 + w\sigma_2)} \tag{4}$$

where $\delta = \overline{\delta}$ is a real number.

We construct the irreducible representations Z_2, by the standard method ([2], [3]):

1. σ_1, σ_2 are different from zero.

Then the irreducible representation of the Z_2 is given by the formula:

$$T^{\delta}(\lambda, z, w) f(\sigma_1, \sigma_2) = e^{i\delta \operatorname{Re}(z\sigma_1 + w\sigma_2)} f(\lambda^2 \sigma_1, \lambda^{-2}\sigma_2). \tag{5}$$

2. $\sigma_1 = 0,\ \sigma_2 = 0$.

Then the representations are related with the representations of homogeneous SO(2, C) group, which have the form:

$$T(\lambda, z_1, z_2) f(\mu, \nu) = \lambda^{\mu-1}\bar{\lambda}^{\nu-1} f(\mu, \nu), \tag{6}$$

where μ, ν are the complex parameters.

The difference $\mu - \nu = n$ must be an integer number. We may introduce a new complex number

$$\varrho = \mu + \nu,$$

and rewrite the irreducible representation of the Z_2 group rewrite in a more convenient form:

$$T(\lambda, z_1, z_2) f(\varrho, n) = |\lambda|^{\varrho-2} e^{in\varphi} f(\varrho, n), \tag{7}$$

where $\varphi = \arg \lambda$.

3. The spinor representations of complex Poincaré group in the case $r^2 = (r^0)^2 - (\vec{r})^2 = 0$

In this case the spinor representations exist only for $\sigma_1 = 0,\ \sigma_2 = 0, \varrho = 2$.

We introduce a new quantum number, $\gamma = \dfrac{n}{2}$ $(\gamma = 0, \pm\frac{1}{2}, \pm 1 \ldots)$ and we consider the SO(2, C) group, which correspond to rotations about "z-axis", as a subgroup of the SO(3, C).

We construct the spinor representations as follows ([4]): we take the boost B_r in the form (a_r, b_r) and we use the identities:

1. $$D^{(|\mu|, 0)(|\nu|, 0)}_{\mu A, \nu B}(a, b) = \delta_{\mu A}\delta_{\nu B} D^{(|\mu|, 0)(|\nu|, 0)}_{\mu\mu, \nu\nu}(a, b) \tag{8}$$

for $\mu, \nu > 0$, $\mu, \nu = 0, \pm\frac{1}{2}, \pm 1, \ldots$ $(a, b) \in Z_2$.

2. $$D^{(|\mu|,0)(|\nu|,0)}_{\mu A,\, B\nu}(a,b) = \delta_{\mu A}\delta_{\nu B}D^{(|\mu|,0)(|\nu|,0)}_{\mu\mu,\, \nu\nu}(a,b) \tag{9}$$

for $\mu > 0$, $\nu < 0$, $(a, b) \in Z_2$ or symmetric case.

3. $$D^{(|\mu|,0)(|\nu|,0)}_{A\mu,\, B\nu}(a,b) = \delta_{\mu A}\delta_{\nu B}D^{(|\mu|,0)(|\nu|,0)}_{\mu\mu,\, \nu\nu}(a,b) \tag{10}$$

for $\mu < 0$, $\nu < 0$, $(a, b) \in Z_2$.

We may define six types of the spinors of the $P_+(C)$ which correspond to 1, 2, 3 cases:

1. $$\varphi^{++}_{AB}(r) = D^{(|\mu|,0)}_{A,-\mu}(a_r\sigma_2)D^{(|\nu|,0)}_{B,-\nu}(b_r\sigma_2)f(B_r)$$
$$\hat{\varphi}^{++}_{AB}(r) = D^{(0,|\mu|)}_{A,-\mu}(a_r\sigma_2)D^{(0,|\nu|)}_{B,-\nu}(b_r\sigma_2)f(B_r). \tag{11}$$

2. $$\varphi^{+-}_{AB}(r) = D^{(|\mu|,0)}_{A,-\mu}(a_r\sigma_2)D^{(|\nu|,0)}_{B\nu}(b_r)f(B_r)$$
$$\hat{\varphi}^{+-}_{AB}(r) = D^{(0,|\mu|)}_{A,-\mu}(a_r\sigma_2)D^{(|\nu|,0)}_{B\nu}(b_r)f(B_r). \tag{12}$$

3. $$\varphi^{--}_{AB}(r) = D^{(|\mu|,0)}_{A,\mu}(a_r)D^{(|\nu|,0)}_{B,\nu}(b_r)f(B_r)$$
$$\hat{\varphi}^{--}_{AB}(r) = D^{(0,|\mu|)}_{A,\nu}(a_r)D^{(0,|\nu|)}_{B,\nu}(b_r)f(B_r). \tag{13}$$

The transformation law of the spinor functions (11), (12), (13) can be deduced from (1).

We obtain, after some simple algebraic manipulations, the result:

$$U(A,w)\,\varphi_{AB}(r) = e^{i\,\mathrm{Re}\,(w\cdot r)}D^{(|\mu|,0)(|\nu|,0)}_{AA',\, BB'}(A)\varphi_{A'B'}(A^{-1}r) \tag{14}$$
$$U(A,w)\hat{\varphi}_{AB}(r) = e^{i\,\mathrm{Re}(w\cdot r)}D^{(0,|\mu|)(0,|\nu|)}_{AA',\, BB'}(A)\hat{\varphi}_{A'B'}(A^{-1}r).$$

The spinor functions $\varphi_{AB}(r)$, $\hat{\varphi}_{AB}(r)$ are not independent since that they obey relations which are completly analogous to the Weyl equations in momentum representation.

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