GENERAL RELATIVISTIC FLUID SPHERES. III. A SIMULTANEOUS SOLVING OF TWO EQUATIONS

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Several new solutions of the gravitational field equations for space filled with matter are given. They are obtained under the following assumptions: (*a*) spherically symmetric distribution of a perfect fluid, (*b*) static gravitational field, (*c*) canonical Schwarzschild coordinates. These assumptions are the same as in the two preceding parts of the work; the procedure of deriving the solutions is modified to the extent that instead of dealing with one differential equation for one unknown function we have to deal with two such equations. Only those solutions are presented which have such a simple form that a study of general features of relativistic stellar models may be easily performed with their help. Some of the solutions (being of an especially simple form) are examined in more detail; asymptotic equations of state of the ultrarelativistic matter in the central region of highest density are given.

1. *Outline of approach*

This paper is the third in a series dealing with deriving new exact solutions of the system of gravitational field equations, under the simple assumption of a spherically symmetric, static distribution of a perfect fluid. The system of equations given by the formulae (2.4) ... (2.7) of paper I[1] was reduced to a homogeneous linear equation of the second order in $y = e^{\nu/2}$ in paper I where several exact solutions have been derived. In Section 8 of paper I and in paper II another possibility of reducing the abovementioned system of equations was considered: that of dealing with a first order inhomogeneous linear equation for the function $z = e^{-\lambda}$.

In paper I we have tried out various possible expressions for $e^{-\lambda}$ which could guarantee us a simple integration of the differential equation for $e^{\nu/2}$. In paper II we have considered some simple expressions for e^{ν} for which it has been possible to present the solution of the differential equation for $e^{-\lambda}$ in terms of elementary functions. Now, it is possible to apply an

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[1] The first two parts of this paper (Kuchowicz 1968*a* and 1968*b*) are denoted, respectively, by the symbols I and II.

apparently more difficult approach, namely that of dealing with two differential equations for two functions at once.

If the differential equation (8.1) from paper I is given in the following notation:

$$z'+f(r)z = g(r) \tag{1.1}$$

then its solution is

$$e^{-\lambda} = Ce^{-F(r)}+e^{-F(r)}\int^{r} g(r')e^{F(r')}dr' \tag{1.2}$$

with

$$F(r) = \int^{r} f(r')dr'.$$

We must take into account the fact that the function $f(r)$ is given by the first of the Equations (1.2) of paper II. With the substitution $y = e^{\nu/2}$ this becomes the following second order differential equation for y:

$$r^2y'' - \left(1+\frac{1}{2}fr\right)ry' - \left(1+\frac{1}{2}fr\right)y = 0. \tag{1.3}$$

The symbol $f(r)$ denotes the same function in the three equations given above. Instead of dealing with one differential equation only, as in the preceding parts of this paper, we have solved simultaneously the two Equations (1.1) and (1.3) with the same function $f(r)$. We must add, besides, that the function $g(r)$ which appears in Eq. (1.1) is not completely arbitrary but depends in the following way on r (through the intermediary of $y(r)$ and its derivative $y'(r)$),

$$g(r) = -\frac{2y(r)}{r^2y'(r)+ry(r)}. \tag{1.4}$$

It may seem that the arbitrary function $f(r)$ is superfluous. It is not quite so since we may choose such a form of it that the two equations (1.1) and (1.3) may be at the same time integrated in terms of elementary functions. In the following, we look for solutions of this system of equations with various simple choices of $f(r)$ of power form *etc.*

2. *Solution for* $f = 2D/r$

Eq. (1.3) has now the form:

$$r^2y''-(1+D)ry'-(1+D)y = 0 \tag{2.1}$$

which is exactly the same when rewritten in terms of the independent dimensionless variable $x = r/r_b$. Here r_b denotes the radius of the fluid sphere we deal with, and the variable x is confined to the interval (0, 1). Explicit form of the solution depends on the sign of the discriminant

$$\Delta = D^2+8D+8. \tag{2.2}$$

We distinguish thus the three following cases:

I. positive Δ which occurs for $D < -2\sqrt{2}-4 \approx -6.82$, and for $D > 2\sqrt{2}-4 \approx -1.18$:

$$y = C_1 r^{\alpha} + C_2 r^{\beta} \tag{2.3}$$

with

$$\alpha, \beta = \frac{1}{2}(2+D\pm\sqrt{\Delta}).$$

II. $\Delta = 0$, *i. e.* $D = \pm 2\sqrt{2}-4$:

$$y = r^{1+\frac{D}{2}}(C_1 + C_2 \ln r). \tag{2.4}$$

III. negative Δ, which occurs for $-2\sqrt{2}-4 < D < 2\sqrt{2}-4$:

$$y = r^{1+\frac{D}{2}}\left[C_1 \cos\left(\frac{\sqrt{-\Delta}}{2}\ln r\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{2}\ln r\right)\right]. \tag{2.5}$$

C_1 and C_2 denote here arbitrary integration constants while D may be regarded as a parameter that will be adjusted later to give integrable expressions for $e^{-\lambda}$. In the following we shall deal with such expressions corresponding to positive Δ, as these are to be handled with in the most easy way.

2.1. Some explicit solutions for $e^{-\lambda}$ in case of $\Delta > 0$

When we insert the expressions for $f(r)$ and for $g(r)$ (in terms of our solution (2.3) and its derivative) into Eq. (1.2), we arrive at the following general formula:

$$e^{-\lambda} = Cr^{-2D} - \frac{C_1}{B_1 D} + \frac{2}{\sqrt{\Delta}}\left(\frac{B_2}{B_1}C_1 - C_2\right) r^{-2D} \int \frac{v^{\frac{2D}{\sqrt{\Delta}}-1}}{B_1 v + B_2}\, dv \tag{2.6}$$

where under the sign of the integral the auxiliary variable $v = r^{\sqrt{\Delta}}$ has been introduced. C is here an arbitrary integration constant while the constants B_1 and B_2 are expressed in dependence on the constants C_1, C_2, and the parameter D:

$$B_1 = C_1\left(2 + \frac{D}{2} + \frac{\sqrt{\Delta}}{2}\right)$$

$$B_2 = C_2\left(2 + \frac{D}{2} - \frac{\sqrt{\Delta}}{2}\right). \tag{2.7}$$

The integration in Eq. (2.6) will be carried out explicitly in some simple cases.

2.1.1. $2D/\sqrt{\Delta} = 1/2$. This case corresponds to the following value of the parameter D:

$$D = \frac{1}{15}(4+2\sqrt{34}) \approx 1.044. \tag{2.8}$$

We have

$$y = C_1 r^{\alpha} + C_2 r^{\beta} \tag{2.9}$$

with

$$\alpha' = \frac{1}{3}(5+\sqrt{34}) \approx 3.63, \quad \beta = \frac{1}{5}(3-\sqrt{34}) \approx -0.566$$

and the following values of the integration constants (which are obtained from the standard set of boundary conditions given in the Appendix to paper I):

$$C_1 = \frac{a(1+2\beta)-2\beta}{8D\sqrt{1-a}\,r_b^{\alpha}}, \quad C_2 = \frac{2\alpha-a(1+2\alpha)}{8D\sqrt{1-a}\,r_b^{\beta}}. \tag{2.10}$$

The parameter $a = r_s/r_b$ denotes the ratio of the Schwarzschild radius of the fluid sphere to its geometrical radius r_b; it cannot exceed unity. We call this parameter the mass concentration. We see at once that the integration constant C_1 is always positive, while the other constant, C_2, is positive only for

$$a < \frac{2}{11}(\sqrt{34}-1) \approx 0.878. \tag{2.11}$$

Depending on the sign of C_2 we arrive at different exact formulae giving $e^{-\lambda}$.

2.1.1.1. Positive C_2. For values of mass concentration fulfilling the inequality (2.11) the following expression for $e^{-\lambda}$ is valid:

$$e^{-\lambda} = -\frac{1}{D(1+\alpha)} + r^{-2D}\Bigg\{C - \frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}}\sqrt{\frac{C_2}{C_1}}\,\mathrm{arctg}\left[\sqrt{\frac{(1+\alpha)\,C_1}{(1+\beta)\,C_2}}\,r^{2D}\right]\Bigg\}. \tag{2.12}$$

From the boundary conditions for a standard fluid sphere we have the following value of the integration constant C:

$$C = \Bigg\{1-a+\frac{1}{D(1+\alpha)}+\frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}}\sqrt{\frac{2\alpha-a(1+2\alpha)}{a(1+2\beta)-2\beta}}\times$$

$$\times\,\mathrm{arctg}\left[\sqrt{\frac{(1+\alpha)\,[a(1+2\beta)-2\beta]}{(1+\beta)\,[2\alpha-a(1+2\alpha)]}}\right]\Bigg\}\,r_b^{2D}. \tag{2.13}$$

The fact that this constant is always positive is of a high importance for the proper behaviour of the metric function $e^{-\lambda}$ which near the centre goes to infinity like Cr^{-2D} In order to save space we do not give here the explicit formulae for the dependence of ϱ and of p on the radial variable r; these expressions can be easily obtained by inserting our solutions (2.9) and (2.12) into the formulae (1.3) from paper II.

2.1.1.2. $C_2 = 0$. This case corresponds to the unique value of mass concentration

$$a = \frac{2(\sqrt{34}-1)}{11} \approx 0.878. \tag{2.14}$$

Since the solution is of an exceptionally simple form:

$$e^{-\lambda} = Cr^{-2D} - \frac{1}{D(1+\alpha)} \tag{2.15}$$

with

$$C = \left[1 - a + \frac{1}{D(1+\alpha)}\right] r_b^{2D} \tag{2.16}$$

the explicit formulae for matter density and pressure are simple, too:

$$8\pi\varrho = \frac{(2D-1)C}{r^{2D+2}} + \frac{1+D(1+\alpha)}{D(1+\alpha)r^2} = \frac{13-\sqrt{34}}{20r^2}\left(\frac{r_b}{r}\right)^{\frac{4}{15}(2+\sqrt{34})} + \frac{\sqrt{34}-1}{4r^2}, \tag{2.17}$$

$$8\pi p = \frac{C(1+2\alpha)}{r^{2D+2}} - \frac{1+2\alpha+D(1+\alpha)}{D(1+\alpha)r^2} = \frac{5+\sqrt{34}}{4r^2}\left\{\left(\frac{r_b}{r}\right)^{\frac{4}{15}(2+\sqrt{34})} - 1\right\}.$$

Both the density and pressure are decreasing now monotonously in value with increasing r, and are always positive definite. When we consider a very small neighbourhood of the centre of the sphere, in each of the equations (2.17) the first term is the leading term, and the following equation of state is obtained asymptotically in the central region:

$$p = \frac{1}{3}(11+2\sqrt{34})\varrho \approx 7.55\,\varrho. \tag{2.18}$$

This relation is at variance with the standard assumption concerning sound velocity in the fluid interior[2].

Provided we would like to accept the usual condition: $p \leqslant \varrho$, the validity of the solution presented in this subsection would be restricted to the following spherical layer:

$$0.78\, r_b \leqslant r \leqslant r_b \tag{2.19}$$

and at the internal boundary $r_i = 0{,}78\, r_b$ it should be matched to another internal solution corresponding *e. g.* to the equation of state $p = \varrho$. The difficulties with obtaining exact solutions which are physically meaningful at every point are known, and one should not wonder at having a solution that is applicable only in some part of the fluid sphere[3].

2.1.1.3. Negative C_2. For mass concentrations a exceeding the value given by Eq. (2.14) we have

$$e^{-\lambda} = -\frac{1}{D(1+\alpha)} + r^{-2D}\left\{C - \frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}}\sqrt{-\frac{C_2}{C_1}}\ln\Phi(r)\right\} \tag{2.20}$$

[2] There should be $v_{\text{sound}} \leqslant c$.

[3] This remark applies also to a major part of other exact solutions derived in this paper.

where

$$\Phi(r) = \frac{(1+\beta)C_2-(1+\alpha)C_1r^{4D}+2\sqrt{-C_1C_2(1+\alpha)(1+\beta)r^{2D}}}{(1+\beta)C_2+(1+\alpha)C_1r^{4D}}$$

and

$$C = \left[1-a+\frac{1}{D(1+\alpha)}+\frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}}\sqrt{\frac{a(1+2\alpha)-2\alpha}{a(1+2\beta)-2\beta}}\ln\Phi(r_b)\right]r_b^{2D}.$$

2.1.2. $2D/\sqrt{\Delta} = 1/3$. This case corresponds to the following value of the parameter D:

$$D = \frac{4+2\sqrt{74}}{35} \approx 0.606. \tag{2.21}$$

The function $e^{\nu/2} \equiv y$ is given by Eq. (2.9) with the following values of α and β:

$$\alpha = \frac{1}{5}(7+\sqrt{74}) \approx 3.120, \qquad \beta = \frac{1}{7}(5-\sqrt{74}) \approx -0.515.$$

The function $e^{-\lambda}$ is given below:

$$e^{-\lambda} = -\frac{1}{D(1+\alpha)}+r^{-2D}\left\{C-\frac{2C_2^{1/3}}{C_1^{1/3}}\frac{1}{(1+\alpha)^{4/3}(1+\beta)^{2/3}}\times\right.$$

$$\left.\times\left[\frac{3}{2}\ln\frac{\sqrt[3]{C_1(1+\alpha)}\,r^{2D}+\sqrt[3]{C_2(1+\beta)}}{\sqrt[3]{C_1(1+\alpha)\,r^{6D}+C_2(1+\beta)}}-\sqrt{3}\,\mathrm{arctg}\,\frac{\sqrt{3}\,r^{2D}}{r^{2D}-2\sqrt[3]{\frac{C_2(1+\beta)}{C_1(1+\alpha)}}}\right]\right\} \tag{2.22}$$

with

$$C = \left\{1-a+\frac{1}{D(1+\alpha)}+\frac{2}{(1+\alpha)^{4/3}(1+\beta)^{2/3}}\times\right.$$

$$\times\sqrt[3]{\frac{2\alpha-a(1+2\alpha)}{a(1+2\beta)-2\beta}}\left[\frac{3}{2}\ln\frac{\sqrt[3]{(1+\alpha)[a(1+2\beta)-2\beta]}+\sqrt[3]{(1+\beta)[2\alpha-a(1+2\alpha)]}}{\sqrt[3]{(\alpha-\beta)(2-a)}}-\right.$$

$$\left.\left.\sqrt{3}\,\mathrm{arctg}\,\frac{\sqrt{3}}{1-2\sqrt[3]{\frac{(1+\beta)[2\alpha-a(1+2\alpha)]}{(1+\alpha)[a(1+2\beta)-2\beta]}}}\right]\right\}r_b^{2D}.$$

The two other integration constants are given by Eq. (2.10). Formulae for matter density and pressure are given below only for a very specific value of mass concentration:

$$a = \frac{2}{13}(\sqrt{74}-3) \approx 0.862 \tag{2.23}$$

when the metric functions have a very simplified form from the beginning since $C_2 = 0$. This sub-case is very similar to that studied under point 2.1.1.2. The whole term in square brackets in Eq. (2.22) vanishes, and the matter density and pressure are given by formulae resembling Eq. (2.17):

$$8\pi\varrho = \frac{29-3\sqrt{74}}{28r^2}\left(\frac{r_b}{r}\right)^{\frac{4+2\sqrt{74}}{35}} + \frac{\sqrt{74}-3}{4r^2}$$

$$8\pi p = \frac{7+\sqrt{74}}{4r^2}\left[\left(\frac{r_b}{r}\right)^{\frac{4+2\sqrt{74}}{35}} - 1\right]. \tag{2.24}$$

Asymptotically in the neighbourhood of the centre we obtain also a troublesome equation of state like Eq. (2.18); the numerical factor on the right-hand side is now *ca.* 31.4. Under the assumption $p \leqslant \varrho$ our solution is again valid in a spherical layer matching the surface.

2.1.3. $2D/\sqrt{\Delta} = -n$. If n denotes a natural number, the integral in Eq. (2.6) may be expressed in terms of a finite number of elementary functions for the two following series of values of the parameter D:

$$D_{\pm} = \frac{-4n^2 \pm 2n\sqrt{2(n^2+4)}}{n^2-4}, \qquad n = 3, 4, \ldots \tag{2.25}$$

It must be mentioned here why we do not consider smaller values of n. For $n = 0$ we have immediately $D = 0$, and it can be shown that our solution is just given by the formulae (5.9) and (5.10) of Wyman's generalization of Tolman's solution VI (Wyman 1949). In case of $n = 2$ our solution is reduced to the special solution given under point 6.2 in paper II. For $n = 1$ we may use the formulae to be given below only with the plus sign in Eq. (2.25); there does not exist the second parameter D_- in this case (as its sign would be incompatible with the sign convention in the condition: $2D/\sqrt{\Delta} = -1$).

With $n \geqslant 3$, the values of $D_{\pm}$ are restricted to the following intervals:

$$-1 > D_+ > 2\sqrt{2}-4, \qquad -2\sqrt{2}-4 > D_- > -14 \tag{2.26}$$

i. e. they are always negative. The metric function $e^{-\lambda}$ is given by the general formula:

$$e^{-\lambda} = Cr^{-2D} - \frac{2n}{D[4n+(n-2)D]} +$$

$$+ \frac{8n^2 r^{-2D}}{4n+(n-2)D}\left\{\sum_{k=1}^{n} \frac{(-1)^{k-1}}{n+1-k}\cdot\left[\frac{C_1}{C_2}\right]^{k-1} \frac{[4n+(n-2)D]^{k-1}}{[4n+(n+2)D]^{k}}\, r^{\frac{2D}{n}(n+1-k)} +\right.$$

$$\left. +(-1)^n\left[\frac{C_1}{C_2}\right]^n \frac{[4n+(n-2)D]^n}{[4n+(n+2)D]^{n+1}} \ln\left(\frac{C_1[4n+(n-2)D]}{2n} + \frac{C_2[4n+(n+2)D]\, r^{\frac{2D}{n}}}{2n}\right)\right\}. \tag{2.27}$$

It goes to a finite value in the centre:

$$\frac{2n[-4n+(6-n)D]}{D[4n+(n-2)D]^2}.$$

Now, if Eq. (2.27) should be valid everywhere inside the sphere, this finite value should be positive, and this gives the following condition:

$$D_- > \frac{4n}{6-n}, \; n > 6 \tag{2.28}$$

which is fulfilled for $n \leqslant 14$. In dealing with the series D_- given by Eq. (2.25) we are thus restricted to its members corresponding to $n \leqslant 14$; no restrictions of such type occur in case of the second series D_+. It should be added that though in general the values of the parameter D from Eq. (2.25) are irrational numbers, just for $n = 14$ we obtain two rational numbers: $D_- = -7/6$, and $D_- = -7$.

2.2. A general solution corresponding to $\Delta = 0$, i. e. $D = \pm 2\sqrt{2} - 4$

By inserting Eq. (2.4) into (1.4) and (1.2) we obtain

$$e^{-\lambda} = Cr^{-2D} - \frac{1}{D(1+\alpha)} +$$

$$+ \frac{2}{(1+\alpha)^2} \exp\left[-2D\left(\frac{C_1}{C_2} + \frac{1}{1+\alpha}\right)\right] r^{-2D} \mathrm{Ei}\left[2D\left(\frac{C_1}{C_2} + \frac{1}{1+\alpha}\right) + 2D \ln r\right] \quad (2.29)$$

where Ei(x) denotes the integral exponential function (defined in section 4 of paper II), and $\alpha = 1 + D/2 = \pm\sqrt{2} - 1$.

The three integration constants which appear in formulae (2.4) and (2.29) are expressed in terms of the mass concentration a and geometrical radius r_b:

$$C_1 = \left[\sqrt{1-a} + \frac{2\alpha - a(1+2\alpha)}{2\sqrt{1-a}} \ln r_b\right]$$

$$C_2 = \frac{a(1+2\alpha) - 2\alpha}{2\sqrt{1-a}}$$

$$C = \left[\frac{3 \mp \sqrt{2}}{4} - a - e^{-f(a)} \mathrm{Ei}\,[f(a)]\right] r_b^{2D} \quad (2.30)$$

where

$$f(a) = 4\,\frac{4 \mp 2\sqrt{2} + a(-3 \pm \sqrt{2})}{2 \mp 2\sqrt{2} + a(\pm 2\sqrt{2} - 1)}.$$

The upper signs correspond to $D_+ = 2\sqrt{2} - 4 \approx -1.18$, the lower signs — to $D_- = -2\sqrt{2} - 4 \approx -6.82$. Matter density and pressure are given below:

$$8\pi\varrho = \frac{D(1+\alpha)+1}{D(1+\alpha)r^2} - \frac{2}{(1+\alpha)^2 r^2\left[\frac{C_1}{C_2} + \frac{1}{1+\alpha} + \ln r\right]} +$$

$$+(2D-1)\,Cr^{-2D-2} + \frac{2D-2}{(1+\alpha)^2} \exp\left[-2D\left(\frac{C_1}{C_2} + \frac{1}{1+\alpha}\right)\right] \times$$

$$\times r^{-2D-2} \mathrm{Ei}\left[2D\left(\frac{C_1}{C_2} + \frac{1}{1+\alpha} + \ln r\right)\right]$$

$$8\pi p = -\frac{D(1+\alpha)+1+2\alpha}{D(1+\alpha)\,r^2} - \frac{2C_2}{(C_1 + C_2 \ln r)r^2}\left(\frac{1}{D(1+\alpha)} + Cr^{-2D}\right) + C(1+2\alpha)r^{-2D-2} +$$

$$+\frac{1+2\alpha}{(1+\alpha)^2}\exp\left[-2D\left(\frac{C_1}{C_2}+\frac{1}{1+\alpha}\right)\right]r^{-2D-2}\mathrm{Ei}\left[2D\left(\frac{C_1}{C_2}+\right.\right.$$

$$\left.\left.+\frac{1}{1+\alpha}+\ln r\right)\right]\left\{1+\frac{2C_2}{(1+2\alpha)(C_1+C_2\ln r)}\right\}. \tag{2.31}$$

The solution corresponding to D_- seems to be of a rather low value due to its unphysical behaviour (*e. g.* negative value of $e^{-\lambda}$ in the centre where the pressure is going to minus infinity). In case of the second solution (with $D_+ \approx -1.18$) the following equation of state is asymptotically valid in a sufficiently small vicinity of the centre:

$$p=\frac{1}{7}(2\sqrt{2}-1)\,\varrho \approx 0.271\,\varrho. \tag{2.32}$$

This equation, though describing matter of the highest density, is compatible not only with the standard assumption dealt with under point 2.1.1.2. but also with the much stronger condition $p \leqslant 1/3\varrho$ which is considered frequently in case of relativistic superdense fluids (Zeldovich and Novikov 1967).

3. *Solution for* $f = 2A - \frac{2}{r}$

With this substitution for f we obtain from (1.3):

$$r^2y''-Ar^2y'-Ary=0. \tag{3.1}$$

The one solution of this equation that will be used in this section is

$$y=Bre^{Ar} \tag{3.2}$$

where B is an integration constant. The other linearly independent solution is not elementary enough to be used in Eq. (1.4) and (1.2). This is, however, no trouble since to maintain the necessary number of adjustable constants in our general solution it is sufficient to treat the parameter A as if it were the second integration constant. The third integration constant C appears in the expression for $e^{-\lambda}$:

$$e^{-\lambda}=Cr^2e^{-Ar}+\frac{1+Ar}{2}+\frac{A^2r^2}{2e^4}e^{-2Ar}\,\mathrm{Ei}\,(2Ar+4)-\frac{5}{4}A^2r^2e^{-2Ar}\mathrm{Ei}\,(2Ar). \tag{3.3}$$

The density and pressure are:

$$8\pi\varrho=\frac{1}{2r^2}-\frac{A}{r}-\frac{A^3r}{2Ar+4}+\frac{5A^2}{4}+C(2Ar-3)\,e^{-2Ar}+$$

$$+\frac{A^2}{2e^4}e^{-2Ar}(2Ar-3)\,\mathrm{Ei}\,(2Ar+4)+\frac{5}{4}A^2e^{-2Ar}(3-2Ar)\,\mathrm{Ei}\,(2Ar)$$

$$8\pi p=\frac{1}{2r^2}+\frac{5A}{2r}+A^2+\frac{A^2}{e^4}(3+Ar)e^{-2Ar}\,\mathrm{Ei}\,(2Ar+4)-$$

$$-5A^2\left(\frac{3}{4}+\frac{Ar}{2}\right)e^{-2Ar}\,\mathrm{Ei}\,(2Ar)+2ACre^{-2Ar}. \tag{3.4}$$

From the boundary conditions we find the integration constants:

$$A = \frac{3a-2}{2(1-a)r_b}, \qquad B = \frac{\sqrt{1-a}}{r_b} \exp\left[\frac{2-3a}{2(1-a)}\right],$$

$$C = \left\{\frac{4-9a+4a^2}{4(1-a)} \exp\left(\frac{3a-2}{1-a}\right) + \frac{3a-2}{8(1-a)^2}\left[\frac{5}{2}\,\mathrm{Ei}\left(\frac{3a-2}{1-a}\right) - \right.\right.$$
$$\left.\left. - \frac{1}{e^4}\,\mathrm{Ei}\left(\frac{2-a}{1-a}\right)\right]\right\} r_b^{-2}. \tag{3.5}$$

All our expressions are valid under the assumption

$$A \neq 0, \; i.\ e.\ a \neq 2/3.$$

When $A = 0$, we have the equation $y'' = 0$, hence $y = C_1 r + C_2$, and this is just the solution studied under point 6.2 in paper II.

It follows from Eq. (3.4) that as we approach the centre of the sphere, the equation of state of matter goes in the asymptotic limit in the most rigid form: $p = \varrho$.

4. *Concluding remarks*

This paper presented a continuation of the author's previous efforts devoted to a derivation of exact solutions of Einstein's equations. The solutions are in many cases of such a simple form that they provide a means of studying general features of relativistic stellar models with sufficient ease. These solutions were obtained under the three simplifying assumptions: (a) spherical symmetry, (b) static gravitational field, and (c) canonical Schwarzschild coordinate system. Since it seems that it is no more easy to obtain further solutions under all these assumptions, we shall resign from either of them in our future investigations of the problem.

REFERENCES

Kuchowicz, B., *Acta Phys. Polon.*, **33**, 541 (1968*a*).
Kuchowicz, B., *Acta Phys. Polon.*, **34**, 131 (1968*b*).
Wyman, M., *Phys, Rev.*, **75**, 1930 (1949).
Zeldovich, Ya. B., Novikov, I. D., *Relativistic Astrophysics* (in Russian), Moscow 1967.