PHENOMENOLOGY OF DOUBLE REGGE POLES

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The problem of the experimental consequences of the assumption of the existence of double Regge trajectories is discussed from the point of view of the resonances region as well as the high energy region.

Some arguments in favour of such a hypothesis are presented. The finite energy sum rules and Veneziano representation for the double poles are written down and briefly discussed.

1. *Introduction*

In this paper we wish to re-examine the problem of a possible existence of double Regge trajectories. It is well known that there is no physical principle which forbids the double poles to appear. The only requirement is that the stable particles should correspond to the single poles; we are however, free in ascribing the order of the pole to the resonance state.

In such a situation the problem of double poles may be looked at from two points of view. Firstly, we may ask whether a dynamical model can be invented in which double poles would appear. Secondly, we may look for possible experimental consequences of the hypothesis that a given unstable particle is situated on a double Regge trajectory. We do not intend to deal with the first problem and here we wish to give only a brief comment. As is well known Bell and Goebel [1] presented a potential model in which double poles were generated. This was done by introducing to the Schrödinger equation for the S-wave scattering a potential function of such a type that two regions of trapping of the scattered particle existed. Then, by an adjustment of the parameters it was possible to show that double poles may indeed appear. It is clear that such potential can be easily realized if the tail of the interaction were described by a certainYukawa-like attractive force which, however, for smaller r would be masked by the centrifugal force. Then, assuming that for very small r another attractive interaction appears which behaves at the origin in a more singular way than the centrifugal force, we obtain the second region of trapping. We see moreover that in such a case the two regions of trapping join for $l \leqslant 0$, so that the double poles can be obtained

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only for $l \geqslant 1$. Since all the known stable mesons have spins 0, such a model would be sufficient to save us from double poles in that energy region where they cannot occur.

There are, however, also other possible mechanisms of generating double poles, like that discussed by Guralnik and Hagen [2], who used the relativistic wave equation with different coupling schemes, but with a source of the type $1/r$. This problem has been discussed also by Eden and Landshoff [3] in the framework of the perturbation theory.

A frequently used argument against the double poles is that if the analyticity in the coupling constant existed we would be able by increasing the strength of the interaction to pull double poles into the region below the threshold so that they would appear as stable particles. It is not so, however, since we are allowed to introduce into the residue function a factor depending in the trajectory $\alpha(s)$, which vanishes for all the values of l lesser than 0. This would again mean that no double poles appear with $l = 0$ which (for bosons) is completely satisfactory. In fact we used to introduce such cancelling factors for negative s values at least to avoid the ghost states and also sometimes to improve the experimental fits. It would not be more difficult to use them also for $s > 0$. A person who would say that even in such a case double "stable" poles for $l \geqslant 1$ are possible should first explain why there are no stable mesons with spins larger than 0.

Summing up, we are of the opinion that the existence of double poles depends only on dynamics and is not the question of the general principles. As such the question should now be solved by the phenomenological analysis of the data. It is just what we would like to discuss in this paper.

The problem of experimental consequences of double poles and in particular of the double Regge poles has been already partially examined by Gatto [4] and by Kreps and Moffat [5]. For the sake of completeness we shall quote in this paper also certain formulas derived by those authors, and particularly by Gatto.

In fact the most attractive argument for the existence of double poles is of course the A_2 meson [6]. It is clear that the double pole hypothesis is not a unique way to explain the data. This problem has been reviewed by Morrison [7], where another hypothesis has been put forward, namely that in the region of the A_2 peak two resonances exist, one of $J^P = 2^+$ and one $J^P = 1^+$. The main argument for such an assumption is a non-typical energy behaviour of the total cross-section for the process $\pi^- + p \rightarrow A_2^- + p$ which is of the type $(p_{\mathrm{LAB}})^{-0.5}$, whereas for similar processes the cross-section behaves like $(p_{\mathrm{LAB}})^{-1.5}$. However, as discussed by Morrison, there are also certain arguments for the existence of one 2^+ meson only, which in that case should be a double pole. The strongest support for that is the splitting of the peak which can be explained best by assuming a double pole fit [6]. One could also take into account that the branching ratio for the three observed decays of A_2, that is into $\varrho\pi$, $\eta\pi$ and $K\bar{K}$ do not change appreciably with the energy. As we show below, the different peak structure for $K\bar{K}$ and $\varrho\pi$ decays [8] can be easily explained by the double pole model.

Some arguments in favour of the hypothesis that the ϱ meson is a double pole have been also presented. The first of them is the good agreement of the so called "dipole fit" with the experimental data for the electromagnetic nucleon form-factors [5], [9]. Moreover, as shown by Kreps and Moffat [5] the non-vanishing polarization in the $\pi^- p$ charge exchange scattering can be most easily explained assuming that the ϱ trajectory is in fact double. The non-existence of the double peak structure is obviously not an argument against such an idea.

In such a situation one can even speculate [10] that all the mesons are situated on double Regge trajectories, which does not necessarily mean that they all correspond to double poles, as certain cancelling factors may appear in the residues[1].

After this introduction we devote Section 2 to a general description of double Regge poles. In Section 3 we discuss the Breit-Wigner formula and Argand diagrams, and in Section 4 we arrive at some high energy aspects of the double Regge poles model. In the Section 5 we present a short discussion of the finite energy sum rules for the double Regge poles and show the possibility of writing down a Veneziano-like representation for them.

2. *Double Regge poles*

Our basic assumption is that a scattering amplitude corresponding to the total baryonic number $B = 0$ has double poles as its only singularities in the complex λ-plane (λ, if non-negative integer, is called l). More precisely, we should say that all the singularities of the amplitude are such that in their neighbourhood it can be developped into the Mac Laurent series ending with the minus second power term,

$$f(\lambda, s) \sim \frac{f_2(s)}{(\lambda-\alpha(s))^2} + \frac{f_1(s)}{\lambda-\alpha(s)} + f_0(s) + \ldots \tag{2.1}$$

with $f_2(s)$ not identically equal to zero.

The notion of the signature τ can be introduced in a standard way. Having performed the Sommerfeld-Watson transformation we find (apart from the background integral)

$$f(k_s, \cos\theta_s) = \frac{1}{k_s}\sum_i \left\{\frac{X_{\alpha_i}R_{1i}}{\sin\pi\alpha_i} + \frac{R_{2i}X'_{\alpha_i}}{\sin\pi\alpha_i} - \frac{\pi R_{2i}X_{\alpha_i}\cos\pi\alpha_i}{\sin^2\pi\alpha_i}\right\}, \tag{2.2}$$

where

$$X_{\alpha i}(\cos\theta_s) = P_{\alpha i}(-\cos\theta_s) + \tau P_{\alpha i}(\cos\theta_s) \tag{2.3}$$

and $R_{2i}(k_s)$, $R_{1s}(k_s)$ stand for the residue functions. $X'_{\alpha i}$ denotes the derivative of $X_{\alpha i}$ witb respect to λ taken at $\lambda = \alpha_i$.

It is now easy to find the contribution of a double Regge pole to a given partial wave. We get

$$f_l(k_s) = \frac{1}{k_s}\left[\frac{\overline{R_1}}{(\alpha+l+1)^2(\alpha-l)} - \frac{R_2}{(\alpha+l+1)(\alpha-l)^2}\right](1+\tau(-1)^l), \tag{2.4}$$

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[1] Another interesting piece of information concerning double Regge poles is provided by the recent calculation due to Halliday [11]. He shows that the double poles are able to generate themselves through a bootstraplike mechanism and, moreover, that the condition of such selfconsistency is $\alpha(0) = 0.46$, which is surprisingly close to $\alpha(0)$ for the "classical" 1^- and 2^+ trajectories. We are indebted to Dr Bassetto for calling our attention to this result.

where $\overline{R}_1 = (\alpha+l+1)R_1 - R_2$. We notice in passing that the model constructed here differs from that introduced by Kreps and Moffat [5], who obtained the double pole by differentiating the single pole term with respect to α assuming that the residue functions are α-independent. This led them to a certain relation between the functions R_1 and R_2.

There are at least two problems connected with the formulae (2.2) and (2.4) which we wish to discuss here. The first of them is the validity of the factorization theorem for the residue functions R_1 and R_2. To discuss this let us consider an n-channel problem described by a certain S-matrix. This matrix can be diagonalized using a non-singular orthogonal matrix U, $S = \tilde{U}S_{\Delta}U$, where S_{Δ} has only diagonal elements. We assume in an analogy to the simple pole case that the leading singularity appears in only one eigenamplitude. This leading singularity is a pole of the second order. Assuming that we decide that the R_2 function factorizes.

Now we have *a priori* three possibilities for the next-to-the-leading term, that is for the pole of the first order. It may appear in the same eigenamplitude which also contains the double pole, or in another (yet only one) eigenamplitude, or in two or more eigenamplitudes. In the third case the residue function R_1 does not factorize. In the first and second case the principle of factorization holds, but the two cases differ with respect to the properties of the ratio R_1/R_2. In the first case this ratio is fixed once for ever for a given pole and does not depend on the channel to which the pole is coupled. Contrary, in the second case the ratio might differ in general for two different channels. This is however, impossible since, as we shall see later, a double pole contribution cannot be made unitary without a single pole term.

The second problem, which is certainly of great importance for the model, is whether all the mesons can be situated on a double Regge trajectories, and what cancelling factors are in fact necessary to get from the model the same spectrum of the particles as the experimental one. It has to be noticed that the signature factor in (2.4) provides us with a single zero only, so that it reduces a given double pole into a single one situated at the wrong signature point. In general such poles could correspond to the mesons coupled to the given two-particle channel through electromagnetic and/or weak interactions only. This would imply, however, that the residue function R_2 is proportional to the electromagnetic or weak coupling constant at least at the wrong signature point. This is certainly not interesting from the point of view of this paper, so we shall assume that the residues at wrong signature points have another zero and that the trajectory at that point decouples. Even then, however, there is still the problem of 0^- mesons, which, again with the accuracy up to the strong interactions, are stable and have to be described by single poles. This means that the R_2 function has at least a simple zero at $\alpha = 0$. What is most important to notice, is that even if all mesons situated on a given trajectory were single poles, the conclusions from the point of view of high energy physics would be different if the trajectory were of the second order.

In what follows we assume that below the threshold the functions R_1, R_2 and α are purely real.

3. *Breit–Wigner formula*

Starting from the formula (2.4) and assuming that Im α is small and not rapidly changing with s when Re α is close to an integer l we can arrive at the Breit–Wigner formula for the double pole,

$$f_l^{\mathrm{B-W}} \equiv \frac{1}{2i}\left(\eta_l e^{2i\delta_l}-1\right) = \frac{\chi_1}{s_r - s - \frac{i}{2}\Gamma} + \frac{\chi_2}{\left(s_r - s - \frac{i}{2}\Gamma\right)^2}. \tag{3.1}$$

The formula (3.1), as is well known, leads to a non-exponential decay law [12].

A priori, both constants χ_1 and χ_2 may be complex. The only requirement, which could be important here, is the unitarity condition

$$|f_l|^2 \leqslant \mathrm{Im}\, f_l. \tag{3.2}$$

Let us introduce dimensionless variable x and constants c_1, c_2:

$$x = \frac{2(s_r - s)}{\Gamma}, \quad c_1 = \frac{2\chi_1}{\Gamma}, \quad c_2 = \frac{2\chi_2}{\Gamma}. \tag{3.3}$$

Then

$$f_l^{\mathrm{B-W}} = \frac{c_1}{x-i} + \frac{c_2}{(x-i)^2}. \tag{3.4}$$

Clearly

$$\begin{aligned}
|f_l|^2 &= \frac{1}{(x^2+1)^2}\left[|c_1|^2(1+x^2)+|c_2|^2+2\mathrm{Re}\,(c_2^*c_1)x+2\,\mathrm{Im}\,(c_2^*c_1)\right],\\
\mathrm{Im}\, f_l &= \frac{1}{(x^2+1)^2}\left[\mathrm{Re}\, c_1\,(1+x^2)+\mathrm{Im}\, c_1 x(x^2+1)+\mathrm{Im}\, c_2(x^2-1)+2\,\mathrm{Re}\, c_2 x\right],\\
\mathrm{Re}\, f_l &= \frac{1}{(x^2+1)^2}\left[\mathrm{Re}\, c_1 x(1+x^2)-\mathrm{Im}\, c_1(x^2+1)+\mathrm{Re}\, c_2(x^2-1)-2\,\mathrm{Im}\, c_2 x\right].
\end{aligned} \tag{3.5}$$

The first point we would like to discuss is a possible appearance of the two peak structure of $|f_l|^2$. It exists if the polynomial

$$\mathrm{Re}\,(c_2^*c_1)(1-3x^2) - x(1+x^2)|c_1|^2 - 2x|c_2|^2 - 4x\,\mathrm{Im}\,(c_2^*c_1) \tag{3.6a}$$

has three real roots. It is easy to check, that the necessary (but not sufficient) condition for that is

$$3\,\mathrm{Re}^2\,(c_2^*c_1) > |c_1|^2\left[|c_1|^2+2|c_2|^2+4\,\mathrm{Im}\,(c_2^*c_1)\right]. \tag{3.6b}$$

The double peak structure implies then a certain relation between the phases and relative magnitudes of c_1 and c_2, which, as we shall see, is not always fulfilled. In particular, we expect that in both cases $|c_1| \gg |c_2|$ and $|c_2| \gg |c_1|$ a single peak structure will emerge.

Let us now examine the unitarity condition (3.2). It is fulfilled for a certain x, if for that x

$$\begin{aligned}
x^3\,\mathrm{Im}\, c_1 + x^2(\mathrm{Im}\, c_2 + \mathrm{Re}\, c_1 - |c_1|^2) + x\,[\mathrm{Im}\, c_1 + 2\mathrm{Re}\, c_2 - 2\,\mathrm{Re}\,[(c_2^*c_1)] - \\
- |c_1|^2 - |c_2|^2 + \mathrm{Re}\, c_1 - \mathrm{Im}\, c_2 - 2\,\mathrm{Im}\,(c_2^*c_1) \geqslant 0.
\end{aligned} \tag{3.7}$$

The condition (3.7) can be fulfilled for all x, if one of the two possibilities holds, either:

$$\mathrm{Im}\, c_1 = 0,\ \mathrm{Im}\, c_2 + \mathrm{Re}\, c_1 = |c_1|^2,\ \mathrm{Re}\, c_2 = \mathrm{Re}\,(c_2^* c_1) \tag{3.8a}$$

and

$$|c_2|^2 + 2\,\mathrm{Im}\, c_2 + 2\,\mathrm{Im}\,(c_2^* c_1) \leqslant 0 \tag{3.8b}$$

(if in (3.8b) the equality holds we get a purely elastic amplitude for all x), or

$$\mathrm{Im}\, c_1 = 0,\ \mathrm{Im}\, c_2 > \mathrm{Re}\, c_1(\mathrm{Re}\, c_1 - 1),$$
$$(\mathrm{Re}\, c_2)^2 \leqslant (\mathrm{Re}\, c_1 - \mathrm{Re}^2\, c_1 + \mathrm{Im}\, c_2)(\mathrm{Re}\, c_1 - \mathrm{Im}\, c_2). \tag{3.9}$$

In the first case, as is easy to check, we obtain

$$\mathrm{Re}\, c_2 = 0,\ \mathrm{Im}\, c_2 = \mathrm{Re}\, c_1(\mathrm{Re}\, c_1 - 1),\ 0 \leqslant \mathrm{Re}\, c_1 \leqslant 2, \tag{3.10}$$

which means, that one of the zeros of the polynomial (3.6a) will be situated at $x = 0$. We will call such a case "symmetric".

If elastic unitarity holds, then either (1) $\mathrm{Im}\, c_2 = 0$, $\mathrm{Re}\, c_1 = 1$ (which corresponds in fact to a single pole with purely elastic coupling), or (2) $\mathrm{Im}\, c_2 = \mathrm{Re}\, c_1 = 2$. We see that the purely elastic wave in the region of a resonance has to show a symmetric double peak structure. The distance between the two maxima is equal to $\Delta \equiv x_{\max}(2) - x_{\max}(1) = 2$ (Fig. 1).

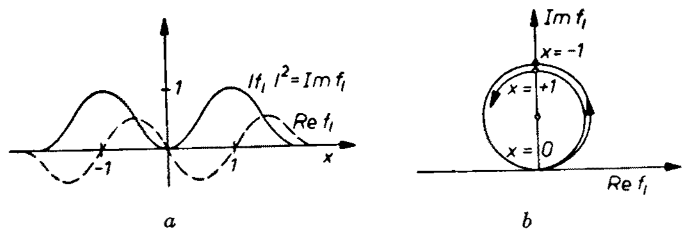

Fig. 1. The energy dependence of $|f_l|^2$, $\mathrm{Im}\, f_l$ and $\mathrm{Re}\, f_l$ is shown in Fig. 1*a* in the case of a purely elastic double pole. In Fig. 1*b* the Argand diagram for that case is presented

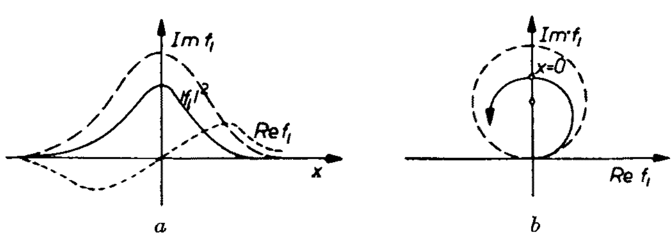

Fig. 2. The same as in the Fig. 1 but for the inelastic double pole, assuming that $0 < \mathrm{Re}\, c_1 < 2 - \frac{1}{\sqrt{2}}$

In the inelastic case we may get both single and double peak structures, depending on the value of $\mathrm{Re}\, c_1$. If

$$0 < \mathrm{Re}\, c_1 < 2 - \frac{1}{\sqrt{2}}\,,$$

the peak is single. It should be noticed that the double peak structure does not necessarily correspond to more than one maximum of $\mathrm{Im}\, f_l$. This may be seen on Figs 2–5. It is interesting to notice that the inelastic cross-section proportional to $\mathrm{Im}\, f_l - |f_l|^2$, shows always only one maximum centered at $x = 0$.

An even more interesting case is described by the conditions (3.9). Now $\mathrm{Re}\, c_2$ does not necessarily vanish which means that the point $x = 0$ is not an extremum of $|f|^2$ (an

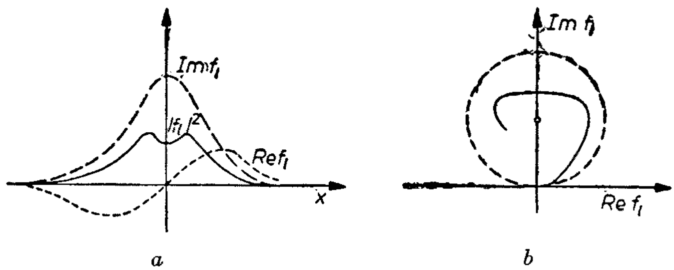

Fig. 3. The same as in the Fig. 2, but assuming that $2 - \frac{1}{\sqrt{2}} < \mathrm{Re}\, c_1 < \frac{4}{3}$

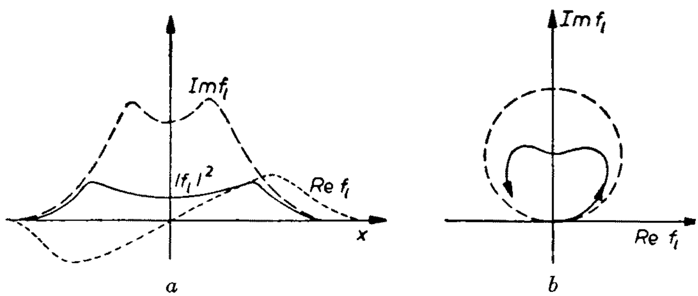

Fig. 4. The same as in the Fig. 2, but assuming that $\frac{4}{3} < \mathrm{Re}\, c_1 < \frac{3}{2}$

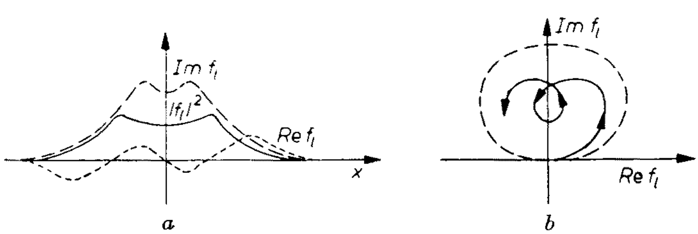

Fig. 5. The same as in the Fig. 2, but assuming that $\frac{3}{2} < \mathrm{Re}\, c_1 < 2$

"asymmetric" case). A complete discussion of the conditions (3.9) in terms of the three parameters is useless, and we limit ourselves to a special case, which is $\mathrm{Re}\, c_1 = 1$, $\mathrm{Re}\, c_2 = \mathrm{Im}\, c_2 = \frac{1}{2}$. In that case, as can be seen from Fig. 6, we get a clearly asymmetric structure for the elastic cross-section, showing only a trace of the second peak. Moreover, the main maximum is shifted by about 0.6 Γ (which may well correspond to 60–80 MeV) from the position $s = s_r$. If we now look at the inelastic cross-section, we see that it shows a very symmetric double peak structure. We see that in general the peak in two different channels may have completely different shapes. In particular, it is possible to observe a double peak

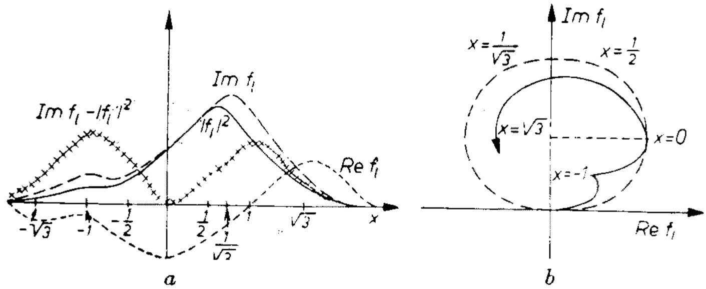

Fig. 6. The same as in the Fig. 2, but in the "asymmetric" case (see text)

structure in one channel and a single peak structure in another. This fact strikingly resembles the situation encountered in the A_2 meson decay, where two peaks are observed in the $\varrho\pi$ decay channel and only one in the $K\bar{K}$ decay channel.

Of course the examples discussed above do not necessarily correspond to reality and the conditions we have imposed to get them are certainly too strong. We demanded the unitarity condition to be fulfilled at all the x values which is not physically necessary. In fact this condition should hold only above the threshold, that is for x lesser than a certain positive value x_0. In such a case solutions with Im $c_1 > 0$ can also be constructed. Moreover, we should not claim that the Breit–Wigner form of amplitude (3.1) is valid for all $x > x_0$, but only in the vicinity of the resonance. Then also the solutions with Im $c_1 < 0$ could be allowed. In this case no qualitatively new effects appear as compared to our previous discussion.

The last comment we would like now to make concerns the narrow width approximation for the double poles. It is clear that in such an approximation we should arrive finally to a stable particle with a zero width. However, a stable particle has to be a single pole. This means that in the narrow width approximation we should consider the χ_2 coupling constant as proportional to Γ and vanishing with it.

4. Double Regge poles and high energy behaviour

Many of the formulae of this Section have been already obtained by Gatto [4]. Nevertheless we quote them here partially for the sake of completeness and partially to use them in further discussion.

Let us introduce two factors connected to the signature τ (we change $s \leftrightarrow t$):

$$\xi_i = \frac{1+\tau \exp\left[-i\pi\alpha_i(t)\right]}{\sin \pi\alpha_i}, \quad \eta_i = \frac{\pi(\tau+\cos\pi\alpha_i)}{\sin^2 \pi\alpha_i}. \tag{4.1}$$

Then, introducing the invariant amplitude $A(s, t, u) = 8\pi\sqrt{t} f(k_t, \cos\theta_t)$ and taking the high energy limit in the formula (2.2) we get

$$A(s, t, u) = \sum_i s^{\alpha_i}[\beta_{1i}(t)\xi_i(t) + \beta_{2i}(t)(\xi_i(t)\ln s - \eta_i(t))] \tag{4.2}$$

since

$$\frac{sX_\alpha}{d\alpha} \Rightarrow \ln s \cdot s^{\alpha_i}[1 + \tau \exp(-i\pi\alpha_i)] - i\tau s^{\alpha_i}\pi \exp(-i\pi\alpha_i).$$

It should be noticed that the amplitude (4.2) is not proportional to the signature factor ξ, so that the phase of the amplitude depends not only on the trajectory α but also on the residue functions β.

It is now easy to calculate the differential cross-section for a process dominated by the exchange of a double Regge pole. The formula reads

$$\frac{d\sigma}{dt} = \frac{s^{2[\alpha(t)-1]}}{\sin^2\pi\alpha}[(1+\tau^2+2\tau\cos\pi\alpha)(\beta_1+\beta_2\ln s)^2 +$$

$$+\beta_2^2\frac{\pi^2(1+\tau\cos\pi\alpha)^2}{\sin^2\pi\alpha} - \pi\tau\beta_2\frac{(1+\tau\cos\pi\alpha)^2}{\sin\pi\alpha}(\beta_1+\beta_2\ln s)] \tag{4.3}$$

so that the total elastic cross-section behaves asymptotically like $s^{2[\alpha(0)-1]}\ln s$ assuming that β's are slowly varying functions of their argument.

The total cross-section for the scattering from a given initial state is dominated by the contribution coming from the exchange of the Pomeron. Assuming it is a double pole we would get an asymptotic behaviour of the total cross-section proportional to ln s, which, even if not excluded by the present data and consistent with the Froissart bound, seems not to be plausible. If the Harari [13] conjecture concerning the dynamical origin of Pomeron were true, we would have a basis for treating this singularity in a distinguished way as compared to other "mesonic" trajectories. In such a case the asymptotic behaviour of the total cross-section might be still constant.

If we calculate the ratio of the Real to the Imaginary part of a one-pole dominated amplitude

$$x \equiv \frac{\text{Re}\,A_\alpha}{\text{Im}\,A_\alpha} = \frac{1+\tau\cos\pi\alpha}{\sin^2\pi\alpha}\frac{[(\beta_1+\beta_2\ln s)\sin\pi\alpha - \pi\tau\beta_2]}{-\tau(\beta_1+\beta_2\ln s)} \tag{4.4}$$

we see that this ratio in general tends to a constant except for the case of the Pomeron exchange dominated processes, when $\chi \to (\ln s)^{-1}$. This means that the limit 0 is approached rather slowly.

It is also easy to calculate the value of the polarization for such processes like $\pi - N$ scattering. It is now in general different from 0 even for a one pole exchange,

$$\frac{d\sigma}{dt}P \sim \left[\frac{1+\tau\cos\pi\alpha}{\sin^2\pi\alpha}\beta_1^{(1)}\beta_2^{(2)} - \beta_2^{(1)}\beta_1^{(2)})\right] s^{2[\alpha(t)-1]} \tag{4.5}$$

where the upper index denotes two different spin channels. It is clear that P does not vanish if the ratio β_1/β_2 can change from one channel to the other, which excludes the possibility that only one of the eigenamplitudes is singular, as discussed in Section 2. We also see that the polarization as compared to the differential cross-section vanishes only like $(\ln s)^{-2}$. Indeed, the present data for the polarization of the $\pi - N$ charge exchange do not show any strong s dependence [14].

The lack of full factorizability has its consequence in the fact that relations like $(\sigma_{12})^2 = \sigma_{11}\sigma_{22}$ are now absent, if not in the region of the ultrahigh energies where the $\ln s$ term dominates.

The sense — nonsense mechanism does not change very much as compared to the single pole case, with the only exception that the two residue functions have to be treated independently. It means, *e.g.*, that they may choose different mechanisms at the points $\alpha = 0, \ -1, \ -2, \ \ldots$

Although we postpone a detailed numerical analysis of the model to the forthcoming paper we wish to give here a brief example of how the model works. It is well known that the phenomenological fits to the data do not fully confirm two basic assumptions often made in a theoretical analysis, namely the straightlinearity and degeneracy of the leading meson trajectories. In particular the A_2 [15], ϱ and ω [16] trajectories are best fitted by rather curved lines dispersed at $t = 0$ between approximately 0.3–0.6. We shall now present a numerical argument that if the data are interpreted in terms of double poles, the trajectories can be fitted by straight lines in a quite large region of t values. By saying that, we do not intend to convince anybody that the trajectories are straight lines everywhere, since they may be curved by unitarity corrections. We only wish to raise the problem, what should be the first approximation and what may be left aside for the moment as a correction.

Let us take as an example the A_2 trajectory. The fit obtained by Phillips [15]

$$\alpha_{A_2} = -1 + \frac{1.35}{1-0.34\,t}$$

gives $\alpha_{A_2}(0) = 0.35$, which is quite far from the degeneracy with, say, α_ϱ. But let us assume the trajectory to be double. In such a case *e.g.* the differential cross-section for the forward scattering of the process dominated by the A_2 exchange (like $\pi + N \to \eta + N$) will be given approximately by (we set $\alpha_{A_2}(0) = \frac{1}{2}$, $\tau = +1$ and $\gamma = \beta_2(0)/\beta_1(0)$):

$$\frac{d\sigma}{dt} \sim \frac{1}{s}\,[2(1+\gamma \ln s)^2 + \pi^2\gamma^2 - \pi\gamma(1+\gamma \ln s)], \tag{4.6}$$

whereas the observed energy behaviour is, as follows from the quoted fit, of about $s^{-1.3}$ (we assume here as a rough simplification the elastic type kinematic relations). At first sight logarithmic factors in the nominator of the formula (4.6) make the situation even worse. It is, however, not so bad. What we need in fact is that the factor in the square bracket in (4.6) be a decreasing function of s. This requirement is fulfilled if

$$\ln\left(\frac{s}{s_0}\right) < \frac{\pi\gamma - 4}{4\gamma}.$$

Taking the scaling factor $s_0 = 1$ GeV^2 and γ small negative we see that the above inequality holds in a very large region of s values (*e.g.* for $\gamma = -0.2$ it holds for $s \gtrsim e^6$ GeV^2 and for $\gamma = -0.1$ for $s \gtrsim e^{10}$ GeV^2), certainly larger than that covered by present experimental data. To be more specific let us assume $\gamma = -0.1$. Then, if *e.g.* for $s = 10$ GeV^2 we have $\frac{d\sigma}{dt} \sim \text{const} \cdot 0.15$, we get for $s = 50$ GeV^2 $\frac{d\sigma}{dt} \sim$ (the same constant) $\cdot\, 0.02$.

One may get almost the same decrease taking $\frac{d\sigma}{dt} \sim s^{-1.3}$. This example, even if oversimplified, shows that the linear trajectory for A_2 degenerate with the ϱ trajectory can be reconciled with the data. Moreover one should take into account a possible displacement of the observed position of the resonance maximum as was discussed in Sec. 3.

5. *Finite energy sum rules and Veneziano representation*

In this Section we come to a short discussion of the finite energy sum rules and a possible way to write down a Veneziano-like representation for the amplitude of the process dominated by the double Regge poles.

The derivation of the FESR follows in exactly the same way as for the single poles [17]. We start with the superconvergent amplitude $\bar{f} = f - R$, where R contains by definition all the double poles which have $\alpha > -1$ (at a given point t). We may then write

$$\int_0^\infty \mathrm{Im}\,\bar{f}(\nu)d\nu = 0 = \int_0^\infty [\mathrm{Im}\,f(\nu) - \sum \nu^\alpha(\beta_1 + \beta_2 \ln \nu)]d\nu,$$

which finally leads us to the result

$$\int_0^N \mathrm{Im}\,f(\nu)d\nu = \sum \frac{N^{\alpha+1}}{\alpha+1}\left[\beta_1 + \beta_2\left(\ln N - \frac{1}{\alpha+1}\right)\right], \tag{5.1}$$

where the sum contains the contribution of all the double Regge poles. Assuming that for some poles $\beta_2 \equiv 0$, we may say that (5.1) contains also all single Regge poles which might be present in a given case.

Having written the FESR for the double poles we may easily repeat the calculation done by Mohapatra [18] to show that trajectories of such poles cannot be straightlinear if the number of trajectories is finite. The only (obvious) difference between the result obtained by us and by Mohapatra is that in our case the elastic width behaves as a constant when $s \to \infty$, instead of vanishing like $(\ln s)^{-1}$. This result, however, has a doubtful meaning, due to the fact that the elastic processes are dominated by the Pomeron exchange, and the Pomeron singularity, as we mentioned in Sec. 4, probably should not be treated as a double pole.

Let us now discuss the possibility of writing down the Veneziano representation for the double poles [19]. An obvious complication is that the Veneziano amplitude deals with purely real trajectories which then correspond to the zero width resonances. As we mentioned in Sec. 3, in such a case double poles should reduce to single ones. The problem may be stated in such a way: is it possible to have a typical double Regge pole behaviour in the high energy region in the crossed channel having at the same time only single poles in the direct channel. We show that this is indeed possible.

In order to construct the Veneziano representation let us first notice that the function

$$B_{22}(\alpha, \beta) \equiv \int_0^1 x^{-\alpha-2} \ln x(1-x)^{-\beta-2} \ln (1-x) dx \tag{5.2}$$

has the following properties:

(1) It is totally crossing-symmetric under the exchange $s \leftrightarrow t$;

(2) It has double poles as its only singularities for $\alpha = -1, 0, 1, 2 \ldots$

(3) For $\alpha \to \infty$ it behaves like

$$(-\alpha)^{\beta}[C_1(\beta)+C_2(\beta) \ln (-\alpha)] \tag{5.3}$$

where $C_2(\beta) = -C_1'(\beta)$.

Assuming straightlinear real trajectories, $\alpha \equiv \alpha(s) = \alpha_0+\alpha_1 s$; $\beta \equiv \alpha(t)$, we get a typical double Regge pole behaviour. The only problem is to reduce the double poles into single ones not spoiling at the same time the behaviour of (5.3). This may be done by multiplying the B_{22} by a certain factor $L(\alpha, \beta)$, which would give single zeros for α integer and real being at the same time bounded at infinity by a constant. The zeros should, however, be absent for a complex α thus giving a double pole back, if Im α and, in the consequence, Γ were not equal to zero. A good example of such a factor L is

$$L(\alpha, \beta) = \sin \pi\alpha \cdot \sin \pi\beta. \tag{5.4}$$

To remove the pole at $\alpha = -1$ we should consider two cases, depending on whether $\alpha = 0$ is a right or a wrong signature point. In the first case $\alpha = -1$ is a wrong signature point which means that the pole is cancelled by the additional zero contained in the signature factor. However, in the second case we should modify the representation (5.2) using instead of it the function $B_{22}(\alpha-1, \beta-1)$. This function possesses poles only for $\alpha = 0, 1, 2, \ldots$, but behaves at the infinity like $(-\alpha)^{\beta-1}$. This means that the function in such a case should contain a linearly rising factor so that it might be written as $L(\alpha, \beta) = (\alpha+n) \sin \pi\alpha \cdot$ $\cdot (\beta+m) \sin \pi\beta$, where n and m are free parameters.

It is now easy to write down the formula for double poles exchange in one channel and single poles in another. It reads

$$B_{21}(\alpha, \beta) \equiv \int_0^1 x^{-\alpha-1} \ln x(1-x)^{-\beta-2} dx \tag{5.5}$$

if the double poles appear in the s channel, and single — in the t channel. The B_{21} function behaves

$$\begin{aligned} &\text{for} \quad \alpha \to \infty \quad \text{like} \quad (-\alpha)^{\beta} \\ &\text{for} \quad \beta \to \infty \quad \text{like} \quad (-\beta)^{\alpha} C_1(\alpha)+C_2(\alpha) \ln (-\beta)]. \end{aligned} \tag{5.6}$$

The final step consists in writing the full formula for the exchange of the double + single poles, as postulated in Sec. 2. (see (2.1)). We write

$$\bar{B}_{22} \equiv \int_0^1 x^{-\alpha-2}(1-x)^{-\beta-2}[A_1 x + A_2 \ln x][B_1(1-x) + B_2 \ln (1-x)]dx. \qquad (5.7)$$

Setting A_2 and/or B_2 equal to zero we get a single pole exchange in the first and/or the second channel. The cancelling factors should be, of course, properly introduced here.

To calculate the residuum at a given pole for the amplitude (5.2) it is sufficient to notice that $B_{22}(\alpha, \beta) = \frac{d}{d\alpha}\frac{d}{d\beta} B(\alpha-2, \beta-2)$. Since the B function has polynomial residua at the poles, the same will be also true for the B_{22} functions. However, the cancelling factors of the type (5.4) cannot be polynomials since in the contrary case they would spoil the asymptotic behaviour. We are thus led necessarily to the appearance of the ancestors. This difficulty is closely connected in its origin to the fact that we are dealing with real trajectories rejecting the unitarity condition, which forces us to reduce double poles into single ones.

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