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SOME REMARKS ABOUT TWO TRIALS OF MACH'S PRINCIPLE REALIZATION

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J. A. Wheeler's interpretation of Mach's Principle and generalization of orthodox General Relativity found by C. Brans and R. H. Dicke are briefly presented. Critical analysis shows that neither Wheeler's nor the Dicke-Brans theory have much in common with what we are entitled to call Mach's Principle.

In the recent scientific literature the tendency reappears to incorporate the so-called Mach's Principle (MP) into the frame of relativistic physics. This can be achieved in two different ways: (a) one can interprete MP in such a way that Einstein's field equations contain it automatically, (b) the field equations can be generalized so that theyselves become Machian. The first way was chosen by Wheeler [8], the second by Dicke and Brans [2]. In this paper their theories will be briefly presented and some critical remarks appended. The mathematical notation used by the cited Authors will be retained.

I. WHEELER'S THEORY

Wheeler made an attempt to formulate the boundary condition problem for the field equations in a somewhat new manner and proposed to understand MP as a selection rule of these solutions which are consistent with the beforehand assigned boundary conditions.

1. *Boundary conditions as the "Thin-Sandwich" problem*

Wheeler's modification of the well known boundary conditions method consists in the following programme: *A.* one has to translate the 4-dimensional formulation of General Relativity (GR) into the $(3+1)$-dimensional formulation, *i. e.* to resolve the spacetime into a set of spacelike hypersurfaces numerated by the parameter x^0; *B.* to determine, as boundary conditions, the internal 3-geometry ${}^{(3)}G$ of a spacelike hypersurface σ and the rate of change of this geometry, $\partial^{(3)}G/\partial x^0$, with respect to x^0 (the so-called "Thin-Sandwich" method); *C.* the constraints equations hold automatically.

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A. The resolution of the 4-geometry into (3+1)-formulation depends on the direction of the vector $\vec{n}$ normal to σ at the considered point and on the external curvature tensor K^p_q at the same point. Having these quantities the metrical tensor can be resolved into six potentials defining the internal geometry on σ, and four potentials determining the external geometry:

$$^{(4)}g_{\mu\nu} = \begin{pmatrix} N_s N^s - N^2 & N_k \\ N_i & ^{(3)}g_{ik} \end{pmatrix}. \tag{1}$$

Following this decomposition the Riemannian curvature tensor $R^\alpha_{\beta\gamma\delta}$ also resolves into two "physical components". Knowing the internal geometry on σ, N and N^i, one can rebuild the whole spatio-temporal 4-geometry. Arnowitt, Deser and Misner [1] have shown that as the field momentum it is appropriate to assume the quantity:

$$\pi^{ij} = \sqrt{^{(3)}g}(^{(3)}g^{ij}\mathrm{Tr}\, K - K^{ij}). \tag{2}$$

B. Calculating: $J = J_m + J_g$ (where: J_m — action integral for matter, J_g — action integral for the field, all in (3+1)-formulation), $\lim_{\Delta x^0 \to 0} J/\Delta x^0$, and using the following abbreviations:

$$K_{ij} = \gamma_{ij}/N$$

$$\gamma_{ij} = \frac{1}{2}(N_{i/j} + N_{j/i} - \partial^{(3)}g_{ij}/\partial x^0)$$

$$\gamma_2 = (\mathrm{Tr}\,\gamma)^2 - \mathrm{Tr}\,\gamma^2$$

we obtain the action principle for the "Thin-Sandwich" problem:

$$J = \int\{N/2(2\varepsilon - {}^{(3)}R) + (\gamma_2/2N) + N_k S^k\}\sqrt{^{(3)}g}\, d^3x = \text{extremum} \tag{3}$$

where:

$\varepsilon = k \cdot$ density of energy,
$S^k = k \cdot$ density of energy flow,
$k = 8\pi\gamma/c^4 = \text{const}$ (here γ — Newtonian constant of gravitation).

C. From (3) we easily get the four constraints equations:

$$\begin{aligned} ^{(3)}R + (\mathrm{Tr}\, K)^2 - \mathrm{Tr}\, K^2 &= 2\varepsilon \\ (K^k_i - \delta^k_i \,\mathrm{Tr}\, K)_{/k} &= S_i \end{aligned} \tag{4}$$

by which our "Thin-Sandwich" boundary conditions are held automatically.

2. *Mach's Principle as a selection rule*

Wheeler considers MP as "a boundary condition to select allowable solutions of Einstein's equations from physically inadmissible solutions". According to him MP may be expressed in the following way: "The specification of a sufficiently regular closed 3-geometry at two immediately succeeding instants, and the density and flow of mass-energy, is to determine the geometry of spacetime, past present and future, and thereby the inertial properties of every inertial test particle." [8], (p. 369).

In order to realize MP in the above formulation, one must:

A. Take as the initial data a closed and regular (*i.e.* without singularities) 3-geometry ${}^{(3)}G$, the rate of its change in time $\partial^{(3)}G/\partial x^0$, density and flow of mass-energy in this closed and regular hypersurface. The closure is needed to avoid difficulties with boundary conditions in solving Eq. (4).

B. Eq. (4) may be solved *e.g.* by the way of a power series development. "... one can start one integration after another of the equations for the N_i-each with a different choice of the coefficients in the power series expressions used to start the integration — until at last one obtains an everywhere-regular solution." [8], (p. 363).

C. Having N_i and N we calculate the external curvature tensor K^p_q and field momentum π^{ij} which *ipso facto* satisfy the constraints equatoins.

D. Now, using Einstein's field equations, we determine the entire spatio-temporal 4-geometry.

E. In this 4-geometry geodesics are "well determined" which is "equivalent to knowing inertial properties of every infinitesimal test particle". In this precisely — according to Wheeler's opinion — consists the realization of MP in GR.

Demanding that initial hypersurface must be closed imposes a great restriction on the physically admissible class of models. In this point Wheeler makes an ingenious dodge. He proves that every open space with Schwarzschild's metric may be regarded "as a piece — perhaps a very large piece, but still only a piece — of a closed geometry. It is cut off short of infinity and joined as smoothly to other geometries as to build up a finite but unbounded 3-space." [8], (p. 370).

3. *Critical remarks*

1. Wheeler's formulation of MP has not very much in common with Mach's original ideas. According to Wheeler not only spacetime geometry but *a priori* given boundary conditions as well, determine the inertial properties of any particles. These boundary conditions are absolute — and therefore antimachian — elements of the theory.

2. Solving Eq. (4) we have to suppose the correctness of the theorem on existence and uniqueness of solutions. This theorem, however, is not proved up to date. It is demonstrated by Wheeler in a very special case for Friedmann's model only.

3. Even if we suppose the above discussed theorem to be correct, nevertheless it is not necessarily possible — and most likely it would be even completely impossible — to rebuild the entire everywhere regular 4-geometry from the initial data. In the process of rebuilding of 4-geometry singularities may appear. The following theorems indicate this clearly enough:

Theorem 1: If spacetime M has the following properties: (*a*) for every timelike vector ξ^α: $R_{\alpha\beta}\xi^\alpha\xi^\beta \geqslant 0$; (*b*) M has a compact Cauchy surface S; (*c*) on S: $R_{\alpha\beta}\xi^\alpha\xi^\beta > 0$, for every timelike or null vector; then M is timelike incomplete.

Theorem 2: If spacetime M has the following properties: (*a*) for every timelike vector

ξ^α: $R_{\alpha\beta}\xi^\alpha\xi^\beta \geqslant 0$; (*b*) M contains a compact spacelike 3-submanifold S; (*c*) the divergence of the vector field generated by S is strictly positive on S; then M is timelike incomplete.

Theorem 3: If spacetime M has the following properties: (*a*) for every timelike vector ξ^α:$R_{\alpha\beta}\xi^\alpha\xi^\beta \geqslant 0$, equality holds for: $R_{\alpha\beta} = 0$; (*b*) M contains a compact spacelike 3-submanifold S; (*c*) there is a point P on S having no horizon relative to S; then M is either timelike incomplete, or flat.

Theorems (1) and (2) are proved by Hawking, theorem (3) by Geroch (see [6]).

Explanations of terms: *1*) All above stated theorems deal with the so-called geodesic completness. We will call a half-geodesic a geodesic curve with one endpoint. A spacetime is timelike (spacelike or null) complete, if an affine parameter on every timelike (spacelike or null) half-geodesic assumes arbitrarily large values.

2) Inequality $R_{\alpha\beta}\xi^\alpha\xi^\beta \geqslant 0$, for every timelike vector ξ^α, gives, after taking into considerations the fields equations with energy-momentum tensor for a perfect fluid:

$$\varrho + p \geqslant 0 \quad \text{and} \quad \varrho + 3p \geqslant 0.$$

To violate these conditions (so-called energy conditions) fluid of density 1 g/cm^3 and a pressure of minus 10^{15} atmospheres is required. Consequently every "reasonable" matter must be subject to these limitations.

3) Let us be given the 4-geometry M and its spacelike submanifold S. If every timelike and null curve without endpoints intersects S once and only once, S is called a Cauchy surface.

4) Let us be given the 4-geometry M and its spacelike submanifold S. If ξ^α is the unit tangent vector field to the congruence of geodesics emanating normally from S, then ξ^α is called vector field generated by S.

5) Let us be given the spacetime M and its compact spacelike 3-submanifold S and let P be a point on S. If there exists a timelike curve without endpoints which intersects S, but which either fails to enter the past light cone of P, or fails to enter the future light cone of P, we say there is horizon at P relative to S.

The geodesic incompletness appears to be a necessary (but not sufficient) condition of the existence of singularities. (Sometimes geodesic incompletness is simply identified with the definition of singularities.) Therefore the theorems (1)–(3) deal — at least indirectly — with the existence of singularities. Even if one does not identify the incompletness with singularity, the above stated theorems contradict Wheeler's argumentation. Namely incompletness does not allow any unique determination of geodesics (and consequently the inertial properties of every test particle) in the whole spacetime.

Three foregoing theorems refer to the closed world models (compactness of space). And just closure was postulated by Wheeler to determine the geometry "well"!

Suppose we are given not the whole spacetime M, but only the portion U of M in a neighbourhood of S. Let the condition (*a*) and (*c*) of the theorems (2) and (3) be verified on U. Because S is not necessarily a Cauchy surface, in general the entire 4-dimensional spacetime will not be uniquely determined by U. From theorems (2) and (3) it follows that even in such a case spacetime M has to be timelike incomplete. In Wheeler's "Thin-Sandwich" method hypersurfaces, in which initial data are given, appear to be Cauchy surfaces, and therefore theorem (1) contradicts Wheeler's argumentation too.

Some conditions of theorems (1)–(3) are quite natural (*e.g.* the energy condition), but in every theorem there is at least one condition which does not have to be fulfiled in the actual universe. For instance condition (*c*) of theorem (3) holds in the oscillating Friedmann's model, but does not necessarily hold for other cosmological models. For this reason the above stated theorems do not exclude Wheeler's argumentation completely, though strongly point against it.

4. According to Wheeler the demand of the closure of the initial hypersurface is based on the analogy with Einstein's static world model. In this model closure makes stability possible. Stability plus uniformity of matter distribution give "the universal standard of rest". Thus in a certain sense it may be stated that the matter distribution in each point determines the local inertial system. In a similar manner in the Robertson-Walker models one can determine a local inertial frame by means of "the standard of avarge motion" (possible owing to the so-called Weyl's postulate). But when dealing with models without such a great degree of symmetry, a similar procedure is completely excluded.

The above remarks make it clear that the "Thin-Sandwich" method may be considered, at most, as an interesting working hypothesis of MP mathematical formulation.

II. THEORY OF DICKE AND BRANS

1. Scalar-tensor theory of gravitation

Dicke states precisely his starting point: "The types of field theories of gravitation which we would like to consider are those for which the only geometrical concepts introduced *a priori* are those of a differentiable 4-dimensional manifold with neither a metric nor affine connection defined. One would hope to find metrical properties after the dynamical problem is solved." [3], (p. 212). The programme is truly Machian!

The tensorial character of the gravitational field leads to Einstein's general theory. The vectorial theory of Sciama [7] — owing precisely to its vectorial formulation — remains only an approximative theory. According to Dicke a true Machian theory ought to be a scalar--tensor one.

Scalars formed from the curvature tensor, as falling off more rapidly than r^{-1} from the mass source, are beyond interest. A new scalar field φ must be introduced. Its main task will be to vary the gravitational "constant".

According to Dicke and Brans (D–B) an invariant action principle generalized in a desired way has the following form:

$$\delta \int [\varphi R + (16\pi/c^4)L - \omega(\varphi_{,i}\, \varphi^{,i}/\varphi)] \sqrt{-g}\, d^4x = 0 \tag{5}$$

ω is a new dimensionless constant (the connection constant). In any reasonable theory the value of ω must be of the order of unity. φ has the dimension: [mass] · [length]$^{-3}$ · [time]^2. Varying (5) with respect to φ and $\varphi_{,i}$ we obtain the wave equation for φ:

$$2\omega\varphi^{-1}\Box\varphi - (\omega/\varphi^2)\, \varphi^{,i}\varphi_{,i} + R = 0 \tag{6}$$

where $\Box$ is the generally covariant d'Alembertian. Varying the components of the metric tensor and its first derivatives in (5) gives the desired field equations:

$$R_{ij} + \frac{1}{2} R g_{ij} = (8\pi\varphi^{-1}/c^4) T_{ij} + (\omega/\varphi^2)\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2} g_{ij}\varphi_{,k}\varphi^{,k}\right) + \varphi^{-1}(\varphi_{,i;j} - g_{ij}\Box\varphi). \quad (7)$$

Performing a contraction in (7) and combining it with (6), we receive:

$$\Box\varphi = \frac{8\pi}{(3+2\omega)c^4} T. \quad (8)$$

D–B found solutions of (7) and (8) parallel to the approximate solution for weak fields and to the external Schwarzschild solution of GR. From this we can infer that the gravitational shift is equal to that of orthodox relativity, but the expression for light deflection gives:

$$\frac{3+2\omega}{4+2\omega} \cdot (\text{value of GR})$$

and for perihelion motion:

$$\frac{4+3\omega}{6+3\omega} \cdot (\text{value of GR}).$$

Is the new theory really Machian? The answer depends on the boundary conditions for the equations of the scalar field. In the general case the problem remains unsolved. D–B found the solution of (8) only for the static shell of mass M and radius R emboded in an ampty universe. Choosing "Machian" boundary conditions and introducing retarded Green's function η, we find the solution of (8) to be:

$$\varphi(x_0) = \frac{8\pi}{(3+2\omega)c^4} \int \eta T \sqrt{-g}\, d^4x \quad (9)$$

or in approximation:

$$\varphi(x_0) = \frac{M}{Rc^2}. \quad (10)$$

D–B write: "Note that this equation states that φ at the point x_0 is determined by an integral over the mass distribution, with each mass element contributing a wavelet which propagates to the point x_0. This is just the interpretation of Mach's Principle desired." [2], (p. 255).

2. *Critical remarks*

1. The leading motive in creation of the new theory for D–B was the desire of the *MP* realization. The Authors understand this principle in the maximalistic way—as the postulate of removing all absolute elements from the physical concept of space. This remained just a declaration. In practice the new theory differs from the orthodox relativity by the introduction of the scalar field only. Owing to this field the mathematical apparatus of the new theory becomes very complicated, but its adventages are not so great. The Founders of the new theory succeeded only in realizing the MP narrowly understood as the condition (10).

2. D–B believe that Machian properties of their theory appear in the Eq. (8): the scalar field is determined by the contracted energy-momentum tensor. But this is an appearance only. Let us assume the vanishing energy-momentum tensor, Eq. (8) then has evident non-trivial solutions. In the empty universe there remains the scalar field as the new absolute element!

3. From the solution of scalar field equations for the massive static shell in the empty universe it follows that inside of this shell the condition (10) is fulfiled. D–B believe this demonstrates the consistency of their theory with MP. But "it may be impossible — as they write — to construct such a static massive shell in a universe empty except for the shell, without giving matter nonphysical properties. This is not meant to imply a practical limitation of real materials, but rather a fundamental limitation on the stress-energy tensor of matter." [2], (p. 253).

4. Making the "constant" γ variable is only an apparent generalization. Instead of γ in the D–B theory a new "absolute" constant ω appears. Besides that its value remains unknown.

5. The weak point of the new theory is the lack of any laboratory effects demonstrating the existence of the scalar field. According to Dicke: "...the scalar interaction, if it exists, is expected to be very weak and... (assuming that for all fundamental particles: $m = m_0 f(\varphi)$, f is the same for all particles) masquarades like gravitation, being almost indistinguishable from true gravitation." [5], (p. 209).

6. The expected difference between GR and D–B theories in the two relativistic tests (light deflection and perihelion motion) depends on the value of the ω constant. Operating with this value one may make the difference between both theories larger or smaller adapting it to the actual measurements. Difference between old and new cosmology depends on the value of ω too. Assuming reasonable values for ω these differences are practically undetectible. *E.g.* for $\omega = 6$ there is only 2% difference between both theories.

7. Dicke believes that the evolution of stars and galaxies is sensitive to the gravitational "constant" variability. It would thus be possible to verify the new theory in the field of astrophysics. Dicke writes: "...it is difficult to see why globular clusters 25.10^9 years old should exist in an evolutionary universe of the close type for which the age should be less than 9 billion years. It is also strange that certain galactic clusters should be older than the first heavy elements, as determined by uranium dating, even though these clusters have a very sizable amount of heavy element content. It is also strange that the evolutionary age of the galactic system, as determined by its rate of converting gas into stars, is at best only half the age of the globular clusters, a constituent part of the galaxy." [4], (p. 286). All these discrepances disappear, when the new theory is applied to astrophysical problems (calculations was made for: $\omega = 4{,}5$ and $\omega = 6$). One must, however, remember that Dicke's argumentation is based on uncertain theoretical and observational data concerning the stars and galaxies evolution.

I cannot see any reasons for believing the D–B theory to be superior to the conventional form of GR.

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