MACH'S PRINCIPLE AND DIFFERENTIABLE MANIFOLDS

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A more complete list of different formulations pretending to play the role of the so-called Mach's Principle is given and some theorems about the possibility of their realization are proved. A physical theory using for the geometrization of the gravitation a geometry based on the concept of differentiable manifold cannot be entirely Machian.

The postulate demanding a mutual dependence between certain physical magnitudes (such as inertia) and a spatio-temporal structure of the universe is commonly known as Mach's Principle (MP). There is no need to recall the great influence of this postulate on Einstein's work during the period of the origin of General Relativity (GR). Up to now, from time to time, some papers appear, authors of which try to realize Machian ideas in the field of relativistic cosmology (see: [5], [10], [16]), but unsuccessfully. MP strongly resists its mathematization.

In scientific articles one can find quite different formulations pretending to play the role of Mach's original postulate. An often cited, but very incomplete, list of such "Machian Principles" was made up by Pirani [12]. Setting the concepts in order being the first step to achieve any serious results, it would be very useful to give — if possible — the full list of different MP formulations. This should permit, I hope, to arrive at certain conclusions on the possibility of realizing MP.

1. The different formulations of MP

The most vague MP formulation seems to be the following statement of Bertotti:

(I) "... we may call 'Machian Property' a connection between the local dynamics and the structure of the universe as a whole." [3], (p. 179).

"Machian Property" is sometimes expressed as a postulate demanding connections between the local motion and the mass-energy distribution in the universe or between the local motion and the motion considered globally. According to Klein:

(II, 1) "... only motion with the respect to material bodies — the word taken in its daily-life sense — has a physical meaning." [11], (p. 293).

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(II, 1)-formulation states — a rather trivial statement — that any frame of reference should be connected with material bodies. As we shall demonstrate later on, the realization of even such a minimalistic postulate is bound with some important restrictions.

The local inertial frame of reference may be determined dynamically (observing the motion of the Foucault pendulum or of the gyroscope) or kinematically (measuring the angular velocity of Earth with respect to fixed stars). Both measurements give exactly the same results. Casual coincidence seems extremely improbable and any influence of the local inertial system on the motion of the stars and of the galaxies is difficult to imagine. One, therefore, must conclude that

(II, 2) "the local inertial frame is determined by some average of the motion of the distant astronomical objects." [4], (p. 28).

The same idea may be expressed in a somewhat different manner:

(II, 3) the motions equivalent kinematically are equivalent dynamically. [12], (p. 199).

According to Pirani only the following formulation has empirically non-empty contents:

(II, 4) "The local reference frames in which Newton's laws are approximately valid (without the introduction of Coriolis or centrifugal forces) are those frames which are approximately non-rotating relative to the distant stars." [12], (p. 199).

It is shown by Pirani that in GR one cannot consider (II, 4) as a fundamental law but rather as an experimental result, only approximately confirmed by the theory. [12].

The next formulations deal not with the inertial frames, but directly with the inertia of material bodies.

(III, 1) The magnitude of inertia of any test body depends on the masses of the universe and their distribution.

The weak postulate (III, 1) holds in GR. We can see it clearly *e. g.* in Thiring-Lens effect. Most often, the following statement is used as MP:

(III, 2) The magnitude of the inertia of any test body is entirely determined by the masses of the universe and by their distribution. [4], (p. 29)

or in Wheeler's formulation:

(III, 2a) "The geometry of spacetime and therefore the inertial properties of every infinitesimal test particle are determined by the distribution of energy flow through all space." [16], (p. 365).

The foregoing ideas may be considered as the postulate of relativity of inertia:

(III, 3) "In a consistent theory of relativity there can be no inertia relatively to 'space', but only an inertia of masses relatively to one another. If, therefore, I have a mass at a sufficient distance from all other masses in the universe, its inertia must fall to zero." (Einstein) [7].

This opinion of Einstein suggests that a test body in empty space has no determined inertia. Here a new problem — I think a non trivial one — appears, namely, what is to be understood under the term "empty space" on ground of GR.

The well known Gödel's model [9] has two interesting — from our point of view — properties: (*a*) the mass-energy distribution is the same as in Einstein's static universe; (*b*) the matter rotates relative to the so called "compass of inertia" (the tangent to the path of a free test particle, when it is given an initial radial velocity), or *vice versa*. Property (*a*) means that in GR, MP in the (III, 2)-formulation is not fulfiled: for the same $T^{\mu\nu}$ there are two different solutions. In order to exclude this anti-machian situation, one may demand that:

(III, 4) "The bulk matter (or 'fixed stars') of the universe determines the compass of inertia, nad the two cannot rotate relative to each other." [1], (p. 377).

MP is often expressed as a formal prescription of its realization. Assumption that physical concepts should have an operational meaning leads to the following formulation:

(IV, 1) "Space devoid of all matter should be devoid of physical structure and the concept of the structure of a physical space should have meaning only when the space contains matter." [6], (p. 4).

The concept of "physical structure" is a rather vague one. It ought to be specified further. (IV, 1) in Einstein's version takes the form:

(IV, 2) "*G*-field is entirely determined by the masses of bodies. Mass and energy, according to conclusions of special relativity, are essentially equivalent; formally energy is described by the symmetric energy tensor, *i. e.* *g*-field is defined and determined by the energy-matter tensor." [8], (p. 241).

But we may ask further, what does: "The metric field is d e t e r m i n e d by the masses" mean? And here one can postulate:

(IV, 3) For the empty space ($T^{\mu\nu} = 0$) the field equations should have no solutions, except the flat (euclidean) ones.

Prescription (IV, 3) holds in "the three-dimensional GR" (two space-like and one time--like dimensions), in which the curvature tensor has the same number of independent components as the Ricci tensor (see [13] and [14]). (IV, 3) is sometimes expressed in the slighty modified form:

(IV, 3a) In the absence of matter spacetime necessarily should be Minkowskian [12], (p. 199).

In order to determine the metric field one may state stronger demands:

(IV, 4) "In the absence of matter there should be no geometric structure to spacetime, *i. e.* with $T^{\mu\nu} = 0$ there should be no solutions to the field equations" [15], (p. 230).

In GR we are dealing with partial differential field equations. The matter distribution cannot be, therefore, uniquely determined without the imposition of suitable boundary conditions. Introducing the so-called "cosmological constant" Einstein was able to show:

(IV, 5) (*a*) that there exists a solution with uniform density of matter and space curved in such a way that although unbounded it is finite. It abolished infinity where all difficulties with boundary conditions arose; (*b*) he thought, though mistakenly, that for positive values of cosmological constant field equations have no solutions for $T^{\mu\nu} = 0$.

These two points, according to Einstein's early opinion, incorporate MP into the frame of GR. The above suggestion of Einstein is nowadays sometimes expressed in the following manner:

(IV, 6) Our real world is described only by these solutions of Einstein's equations which are closed in the spacial dimensions. [2].

MP by some authors is formulated in a very special shape demanding variability of certain physical "constants".

(V, 1) The constant of gravitation contains informations about the structure and evolution of the universe, *i. e.* it changes depending on the distribution of matter in space and time in the universe. [4], (p. 29).

In connection with (V, 1) Dicke's remarks seem to be important: "According to MP, it should be only the matter distribution of the universe that determines the acceleration of the earth toward the sun and simple dimensional arguments are sufficient to determine the acceleration for the universe of some definite type. For example, a flat universe having some definite form of time-dependence for its expansion parameter can be characterized by two parameters, its present matter density ϱ, and its present age T. On dimensional grounds, therefore, the acceleration of the earth toward the sun would be expected to be (for $M_S \ll T^3 c^3$ and $R_S \ll cT$):

$$a \cong \frac{M_s}{R_s^2} \cdot \frac{1}{\varrho T^2} \,. \tag{a}$$

The dependence upon the mass and distance of the sun (M_S and R_S) is necessary for Newton's gravitational law to be satisfied. The dependence on the density and age of the universe is such as to give the right dimensions for acceleration. Combining Eq. (a) with the more usual expression, gives for the gravitational constant expression:

$$\gamma \sim \frac{1}{\varrho T^2} \,. \tag{b}$$

It should be noted that an expression such as Eq. (b) seems to imply one of at least two things. Either the gravitational constant is dependent upon the mass distribution or the mass distribution is fixed by the requirement that (b) be satisfied with a fixed γ. Inasmuch as it is the inertial reaction that is being presumed, under MP, to be dependent upon the matter distribution of the universe,

(V, 2) it would be the inertial mass of a particle which might be expected to vary with position, not the gravitational constant.

With this interpretation Eq. (b) is satisfied by changing the unit of mass used to express ϱ." [6], (p. 33–34).

Postulate (V, 1) may be extended to other physical constants:

(V, 3) All physical constants are determined by the "material contents" of the universe, *i. e.* the values of all physical constants contain some informations about structure and evolution of the universe.

All foregoing formulations of MP suggest the following, most maximalistic, postulate:

(VI) The physical theory should be devoid of any absolute elements. All physical magnitudes (such as: all constants, quantity of different kinds of elementary particles, their masses, charges and so on) should be defined by the "material contents" of the universe.

We cannot forsee, whether this, rather methodological, postulate will ever be fulfilled or not, yet it seems almost certain that the tendency to its realization does point out the direction of research in contemporary relativistic physics.

2. Some theorems about possibility of MP realization

Theorem 0: The differentiable manifold implies geometrical structure, namely the affine space locally tangent.

Proof: Let us consider the n-dimensional differentiable manifold of class C^l, *i. e.* the triple $\{X, \theta, D\}$, where: X — an arbitrary set, θ — a collection of subsets of X obeying the axioms of connected Hausdorff topological spaces, D — a family of real-valued functions defined on X and satisfying:

(*i*) if: *a*) f is a real-valued function defined on X;
b) to every point $p \in X$ there corresponds a neighbourhood U of p;
c) in every point $q \in U$ a function $g \in D$ is defined such that, for all $q \in U$: $f(q) = g(q)$,
then: $f \in D$;

(*ii*) if: *a*) $g^1, g^2, \dots g^k \in D$. where k is positive integer;
b) f is real-valued function of class C^l defined on R^k, where R^k—k-dimensional number space,
then: $f(g^1, g^2, \dots g^k) \in D$;

(*iii*) to every point $p \in X$ corresponds a neighbourhood U of p and n functions: $x^1, x^2, \dots x^n \in D$ such that:

a) the mapping:

$$U \ni q \to (x^1(q), x^2(q), \dots x^n(q)) \in R^n$$

is a homeomorphism of U onto a subset of R^n;

b) to every $f \in D$ corresponds a function F defined on R^n, of class C^l, such that:

$$f(q) = F(x^1(q), x^2(q), \dots x^n(q))$$

supposing that $q \in U$.

(For the definitions of used concepts see [15].)

Let $p \in X$ be a point of the differentiable manifold and $\boldsymbol{u}$ a linear mapping of D into R (R — the set of real numbers) such that, for any f, $g \in D$:

$$\boldsymbol{u}(fg) = f(p)\,\boldsymbol{u}(g) + g(p)\,\boldsymbol{u}(f) \tag{1}$$

then $\boldsymbol{u}$ is called a vector tangent to manifold at point p. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors tangent to manifold at p, a — a real number, and function $f \in D$. We define:

$$(\boldsymbol{u}+\boldsymbol{v})\,(f) = \boldsymbol{u}(f)+\boldsymbol{v}(f) \tag{2}$$

$$(a\boldsymbol{u})\,(f) = a(\boldsymbol{u}(f)) \tag{3}$$

(2) and (3) are clearly the linear mappings D to R satisfying (1). We will call the set of all vectors tangent to manifold at $p \in X$ vector space (over the field R) tangent to differentiable manifold at $p \in X$, and denote by T_p. The formulae (1)–(3) guarantee that elements of T_p satisfy the axioms for the affine space. This ends the proof.

From the theorem 0, which appears to be a purely geometrical one, follows directly:

Theorem 1: A physical theory, using for the geometrization of the gravitation any possible geometry based on the concept of the differentiable manifold, and identifying — in any manner — the physical structure with the geometrical one, cannot be Machian in the sense of (IV, 1).

Theorem 2: A physical theory, using for the geometrization of the gravitation any possible geometry based on the concept of the differentiable manifold, and identifying — in any manner — the local inertial frame of reference with the affine local coordinates system, cannot be Machian in the sense of (II, 2).

Proof: Let $\{X, \theta, D\}$ be a differentiable manifold and $\{x^i\}$ — a system of coordinates in the neighbourhood U of a point $p \in X$. Let us further consider in the same point p the tensors $\xi^i_{(j)} (j = 1, 2, \dots n)$ having in $\{x^i\}$ coordinates:

$$\xi^i_{(j)} = \delta^i_j. \tag{4}$$

Vectors $\boldsymbol{e}_k$, in the tangent space T_p, corresponding to tensors (4), form an affine local system of coordinates (in the point $p \in X$). To the resolution of any tensor in the local coordinate system $\{x^i\}$ (in the differentiable manifold) corresponds (in the tangent space T_p):

$$\boldsymbol{\xi} = \xi^i \boldsymbol{e}_i. \tag{5}$$

We see that coordinates of the tensor in the manifold (at $p \in X$) behave as coordinates of the vector in the tangent space T_p. Local affine coordinate system in the differentiable manifold is, therefore, in a quite natural way, induced by the concept of the tangent space, without introducing any external — in reference to the concept of differentiable manifold — elements. This remark completes our proof.

According to theorem 2 the local inertial frame cannot be entirely determined by something external in reference to the concept of the differentiable manifold (*e. g.* by the random motion of distant astronomical objects), but nothing prevents the local inertial frame from being partially determined by some external factors.

Having in mind that the motion of the test particle must be referred — in any manner — to the local inertial frame defined in the proof of the theorem 2, one can state:

Theorem 3: A physical theory, mentioned in theorem 2, cannot be Machian in the sense of (II, 1).

Theorem 4: A physical theory, using for the geometrization of the gravitation any possible geometry based on the concept of the differentiable manifold, and prescribing to measure the inertia of the test particle in reference to the local inertial frame, cannot be Machian in the sense of (III, 2).

Indeed, if the inertial frame — using Einstein's terminology — may be defined relatively to the space (*i. e.* to the differentiable manifold) and not necessarily relatively to the matter (as was demonstrated above), then inertia can be defined relatively to the space too, and not necessarily relatively to the matter.

All assumptions of theorems 1–4 are satisfied in GR. It is, therefore, obvious that Einstein's theory of gravity cannot be Machian in the senses stated above. Our theorems, moreover, suggest clearly enough that changing the differentiable manifold in the Riemannian space (*e. g.* by introducing the metrical field on the affine space tangent to the manifold) automatically excludes the possibility of realization of MP in Einstein's (IV, 2)-formulation.

The theorems 1–4 concern not only GR but a very extensive class of possible physical theories. Our consideration suggests that in order to incorporate Mach's ideas into theoretical physics, one must radically change some concepts lying in the basis of modern geometry.

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