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# RARITA-SCHWINGER FIELDS IN GENERAL RELATIVITY*

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An investigation is made of the possibility of extending a previous calculation of the motion of Fermions in a linearized gravitational field to particles of higher spin. The Rarita-Schwinger equations or arbitrary half-integral spin are discussed. It is found that the $R-S$ equations for spin $> \frac{3}{2}$ become self-inconsistent in the presence of gravitational field with other than constant Riemann curvature. For spin $\frac{3}{2}$ the equations are seen to be consistent in the presence of an exterior gravitational field in regions where the sources vanish. The equations for the trajectory of a spin $\frac{3}{2}$ particle in such a situation are calculated and take the same form as in the spin $\frac{1}{2}$ case with the appropriate generator of Lorentz transformations appearing in the spin tensor. The deflection of high energy spin $\frac{3}{2}$ particles passing near a massive body is computed. Although the corresponding equations for the trajectory of a spin 1 particle can be shown to have the same form as the spin $\frac{1}{2}$, $\frac{3}{2}$ cases, it is doubtful whether the result will hold for spins $> \frac{3}{2}$, except in fields of constant Riemann curvature.

## 1. *Introduction*

The motion of spinning particles in gravitational fields has been studied by many authors, both from the point of view of classical spinning bodies [1], [2], [3] and from the point of view of Dirac spinors in curved space [4], [5], [6]. By considering the classical equations of motion, Papapetrou [1] has shown that bodies which possess higher moments than the monopole moment (mass), *e.g.* rotating bodies, do not, in general, follow geodesics of the metric. Additional force terms appear in the equations of motion expressing the interaction of the higher moments with the Riemann tensor of the gravitational field. Pagels [4] has proposed comparable spin interaction force terms which he arrives at by considering the invariance properties of transformations between states of spin $\frac{1}{2}$ in a space with gravitation present. He further shows from relativistic invariance arguments that massless particles whose spin is constrained to point along the direction of motion, must follow null geodesics.

In previous works [7], [8] the author has made a connection between these calculations and the treatment of the covariant Dirac equation in general relativity. It was shown that results equivalent to those of Pagels [4] could be derived, at least for linearized gravitational

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fields, from the covariant Dirac equation by means of a WKB-like approximation in which all $\hbar$'s in the theory are limited to zero except for the $\hbar$ in the magnitude of the spin angular momentum. It is the purpose of the present paper to examine the extension of this result to fields of higher spin.

The achievement of this goal is made difficult by the complexity of the formulations of theories of higher spins [9], [10], [11], [12], [13]. These theories are generally so complex that it is not possible to make general statements about fields of arbitrary spin. Instead, it is necessary to treat the fields for each different spin value separately. One case which at first seems to avoid this difficulty is the Rarita-Schwinger formalism [12], [14] for half--integral spins, and it is this formalism that we consider in this paper. Unfortunately, the Rarita-Schwinger formalism falls victim to a second malady common to theories of higher spin. This arises from the fact that, in addition to the dynamical field equations of the theory, one must introduce auxilliary conditions to project out only those field components which refer to the desired spin. A familiar example of this is the vanishing of the divergence of the massive vector field which removes the zero spin field component from the theory. In the case of most higher spin theories, when interactions with other fields are included, the auxilliary conditions and the field equations become inconsistent with one another. An example of this is the Bargmann-Wigner formulation [13], [15] in the presence of an electromagnetic (or gravitational) field. To a large extent, as we will see, the Rarita-Schwinger formalism also shares this defect. It turns out that for spins $> \frac{3}{2}$ the field equations and the auxilliary conditions are consistent only for special gravitational fields. Even in the case of spin $\frac{3}{2}$ the equations are not completely consistent [16]. It is necessary to restrict ourselves to a discussion of the motion in an external gravitational field, ignoring the contributions to the field of the particle itself. Also we must consider motion only in regions outside the sources of the external gravitational field. Within these limitations, however, we are able to show that the equations of the trajectories of particles of spin $\frac{3}{2}$ are quite analogous to those for spin $\frac{1}{2}$. It is necessary only to replace the Fermion spin tensor in the theory with the appropriate spin $\frac{3}{2}$ tensor. It is easy to show that the same prescription works for the vector meson as well. If one makes certain assumptions about the field equations for higher spin fields, the prescription can be extended even farther, though for spin $> \frac{3}{2}$ these assumptions are of very doubtful validity, if the space has non-constant Riemannian curvature.

We mention in passing that we are considering only spin-orbit type interactions of the particles. We are ignoring here the so-called "spin-spin" interactions which arise in certain formulations of the covariant Dirac theory [17], [18] and which lead to non-symmetric terms in the affine connections, and so forth. Such effects, if present, would appear in addition to those which we discuss.

In section 2 we develop the covariant form of the Rarita-Schwinger equations for arbitrary half integral spins and examine their consistency. In section 3 a particular solution for the free particle spin $\frac{3}{2}$ equations is discussed. In section 4 we apply our WKB-like approximation to the covariant spin $\frac{3}{2}$ equations to derive the equations of motion for a particle in a gravitational field, and apply this to the calculation of the deflection of such particles passing at high energy near a massive body. The results and their extension to higher spin fields are discussed in section 5.

## 2. *The Rarita-Schwinger equations*

The Rarita-Schwinger equations for free particles of spin $s+\frac{1}{2}$, with $s$ an integer, are [12][1]

$$\left(\gamma^{\mu}\partial_{\mu}-\frac{m}{\hbar}\right)\psi_{\mu_1\mu_2\ldots\mu_s}=0 \tag{2.1}$$

where $\psi_{\mu_1\mu_2\ldots\mu_s}$ is symmetric in the $s$ tensor indices $\mu_i$ and contains one suppressed Dirac spinor index, and the $\gamma^{\mu}$ are the usual Dirac matrices. We also need the auxilliary conditions

$$\gamma^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_s}=0. \tag{2.2}$$

Multiplying Eq. (2.1) on the left by $\gamma^{\mu_1}$ and employing Eq. (2.2) and the Clifford algebra of the Dirac matrices

$$\{\gamma^{\mu},\gamma^{\nu}\}=2\eta^{\mu\nu} \tag{2.3}$$

produces the additional relations

$$\partial^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_s}=0 \tag{2.4}$$

Eq. (2.2) and Eq. (2.4) together assure that $\psi_{\mu_1\mu_2\ldots\mu_s}$ will have just enough remaining degrees of freedom to describe a particle of spin $s+\frac{1}{2}$.

To extend these equations to a Riemannian space we employ the usual "minimum coupling" prescription[2]. First we replace the Dirac matrices $\gamma^{\mu}$ by their covariant counterparts $\tilde{\gamma}^{\mu}(x)$ where

$$\{\tilde{\gamma}^{\mu}(x),\tilde{\gamma}^{\nu}(x)\}=2g^{\mu\nu}(x). \tag{2.5}$$

The covariant Dirac matrices are related to the usual ones by the "vierbein fields" $e^{\mu}_{\alpha}(x)$ by

$$\tilde{\gamma}^{\mu}(x)=e^{\mu}_{\alpha}(x)\gamma^{\alpha} \tag{2.6}$$

where

$$e^{\mu}_{\alpha}(x)e^{\nu}_{\beta}(x)\,\eta^{\alpha\beta}=g^{\mu\nu}(x). \tag{2.7}$$

As the second step of the prescription we replace all partial derivatives by covarian derivatives. The covariant derivative of a quantity with one spinor index and a series o tensor indices is given by

$$\begin{aligned}\nabla_{\mu}A^{\varrho\sigma\ldots}_{\alpha\beta\ldots}&=\partial_{\mu}A^{\varrho\sigma\ldots}_{\alpha\beta\ldots}+i\Gamma_{\mu}A^{\varrho\sigma\ldots}_{\alpha\beta\ldots}+\\&+\Gamma^{\varrho}_{\mu\lambda}A^{\lambda\sigma\ldots}_{\alpha\beta\ldots}+\Gamma^{\sigma}_{\mu\lambda}A^{\varrho\lambda\ldots}_{\alpha\beta\ldots}-\\&-\Gamma^{\lambda}_{\mu\alpha}A^{\varrho\sigma\ldots}_{\lambda\beta\ldots}-\Gamma^{\lambda}_{\mu\beta}A^{\varrho\sigma\ldots}_{\alpha\lambda\ldots}+\ldots\end{aligned} \tag{2.8}$$

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[1] We use the following notation and conventions: The metric used has the signature $(+,-,-,-)$. Units are chosen such that $c=1$. Ordinary partial derivatives are denoted by a comma or by $\partial_{\mu}$; covariant derivatives are denoted by a semicolon or by $\nabla_{\mu}$. Square brackets [ ] denote commutators and braces { } denote anti-commutators. Greek indices run from 0 to 3, Latin from 1 to 3.

[2] For a more detailed discussion of the quantities used in the covariant Dirac theory see References [4], [6] and [19].

where the $\Gamma^{\alpha}_{\mu\nu}$ are the usual Christoffel symbols and the $\Gamma_{\mu}$ are the Fock-Ivanenko coefficients or spin connections. For linearized gravitational fields we may take

$$\Gamma_{\mu} = -\frac{1}{4} h_{\mu\alpha,\beta}\sigma^{\alpha\beta}, \tag{2.9}$$

where we have written our metric in the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu}+h_{\mu\nu}(x) \tag{2.10}$$

and neglect orders of $h_{\mu\nu}$ higher than the first and where

$$\sigma^{\alpha\beta} \equiv \frac{i}{2}[\gamma^{\alpha}, \gamma^{\beta}] \equiv \frac{i}{2}[\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha}]. \tag{2.11}$$

According to our prescription, then, the covariant Rarita-Schwinger equations are

$$\left(\tilde{\gamma}^{\mu}\nabla_{\mu}-\frac{m}{h}\right)\psi_{\mu_1\mu_2\ldots\mu_s} = 0 \tag{2.12}$$

and

$$\tilde{\gamma}^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_s} = 0. \tag{2.13}$$

Multiplying Eq. (2.12) on the left by $\tilde{\gamma}^{\mu_1}$ and using Eqs (2.5) and (2.13) and the fact that

$$\nabla_{\mu}\tilde{\gamma}^{\nu} = 0 \tag{2.14}$$

we find, in analogy with Eq. (2.4), that

$$\nabla^{\mu_1}\psi_{\mu_1\mu_2\ldots\mu_s} = 0. \tag{2.15}$$

We must now investigate whether our covariant Rarita-Schwinger equations are still consistent. To do this we follow Cohen [16]. We will employ the following relations

$$R^{\varrho}_{\lambda\mu\nu}\tilde{\gamma}^{\lambda}\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu} = -2R^{\varrho}_{\mu}\tilde{\gamma}^{\mu} \tag{2.16}$$

where the Riemann tensor is

$$R^{\varrho}_{\lambda\mu\nu} = \Gamma^{\varrho}_{\lambda\mu,\nu}-\Gamma^{\varrho}_{\lambda\nu,\mu}+\Gamma^{\sigma}_{\lambda\mu}\Gamma^{\varrho}_{\sigma\nu}-\Gamma^{\sigma}_{\lambda\nu}\Gamma^{\varrho}_{\sigma\mu} \tag{2.17}$$

and the Ricci tensor is

$$R^{\varrho}_{\mu} = g^{\nu\lambda}R^{\varrho}_{\lambda\mu\nu} \tag{2.18}$$

we also have

$$R_{\alpha\beta\mu\nu}\tilde{\gamma}^{\alpha}\tilde{\gamma}^{\beta}\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu} = -2R \tag{2.19}$$

where

$$R = g^{\mu}_{\varrho}R^{\varrho}_{\mu}. \tag{2.20}$$

It is further possible to show that

$$[\nabla_{\mu}, \nabla_{\nu}]\psi_{\mu_1\mu_2\ldots\mu_s} = \frac{1}{4}R_{\varrho\sigma\mu\nu}\tilde{\gamma}^{\varrho}\tilde{\gamma}^{\sigma}\psi_{\mu_1\mu_2\ldots\mu_s}-$$

$$-R^{\varrho}_{\mu_1\mu\nu}\psi_{\varrho\mu_2\ldots\mu_s}-R^{\varrho}_{\mu_2\mu\nu}\psi_{\mu_1\varrho\ldots\mu_s}-\ldots-R^{\varrho}_{\mu_s\mu\nu}\psi_{\mu_1\mu_2\ldots\varrho}. \tag{2.21}$$

Starting with our Rarita-Schwinger field equation (2.12) we write

$$\tilde{\gamma}^{\mu_1}\left(\tilde{\gamma}^{\mu}\nabla_{\mu}+\frac{m}{\hbar}\right)\left(\tilde{\gamma}^{\nu}\nabla_{\nu}-\frac{m}{\hbar}\right)\psi_{\mu_1\mu_2\ldots\mu_s}=0. \tag{2.22}$$

Then by Eqs (2.5), (2.13) and (2.21) we have

$$\tilde{\gamma}^{\mu_1}\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu}\left[\frac{1}{4}R_{\varrho\sigma\mu\nu}\tilde{\gamma}^{\varrho}\tilde{\gamma}^{\sigma}\psi_{\mu_1\mu_2\ldots\mu_s}-R^{\varrho}_{\mu_1\mu\nu}\psi_{\varrho\mu_2\ldots\mu_s}-R^{\varrho}_{\mu_2\mu\nu}\psi_{\mu_1\varrho\ldots\mu_s}-\ldots-R^{\varrho}_{\mu_s\mu\nu}\psi_{\mu_1\mu_2\ldots\varrho}\right]=0. \tag{2.23}$$

By equations (2.13), (2.16) and (2.19) and the relation

$$\tilde{\gamma}^{\mu_1}\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu}=\gamma^{\mu}\tilde{\gamma}^{\nu}\tilde{\gamma}^{\mu_1}+2g^{\mu_1\mu}\tilde{\gamma}^{\nu}-2g^{\mu_1\nu}\tilde{\gamma}^{\mu} \tag{2.24}$$

this becomes

$$2R^{\varrho}_{\mu}\tilde{\gamma}^{\mu}_{\varrho\mu_1\ldots\mu_s}+4g^{\mu_1\nu}\tilde{\gamma}^{\mu}R^{\varrho}_{\mu_2\mu\nu}\psi_{\mu_1\varrho\ldots\mu_s}+\ldots+4g^{\mu_1\nu}\tilde{\gamma}^{\mu}R^{\varrho}_{\mu_s\mu\nu}\psi_{\mu_1\mu_2\ldots\varrho}=0. \tag{2.25}$$

The first term of this expression vanishes if the Ricci tensor has the form

$$R^{\varrho}_{\mu}=\lambda g^{\varrho}_{\mu}. \tag{2.26}$$

The remaining terms will vanish only if the space has constant Riemannian curvature

$$R^{\varrho}_{\mu\nu\varrho\sigma}=-K(g_{\mu\varrho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\varrho}). \tag{2.27}$$

This reproduces the result of Buchdahl [19] concerning the compatibility of the equations of fields of spin $> 3/2$ in the two spinor formalism. Since for $s = 1$ only the first term of Eq. (2.25) arises, we conclude that our covariant Rarita-Schwinger equations are suitable to describe the motion of a particle with spin 3/2 in a gravitational field of the form of Eq. (2.26), that is, in the case in which we ignore the gravitational field produced by the particle itself and in regions where the stress energy tensor of the matter giving rise to the gravitational fields vanishes. For spin $> 3/2$, however, the field equations are only valid for a very restricted class of spaces.

In the next section, then, we will calculate the equations of motion for a particle of spin $\frac{3}{2}$ in an external gravitational field. It should be pointed out that Cohen [16] derives a set of spin $\frac{3}{2}$ equations consistent for any gravitational field, but these reduce to ours in the case we are considering.

### *3. Flat space solutions of spin $\frac{3}{2}$ field equations*

In this section we wish to discuss a particular form for free solutions of the spin $\frac{3}{2}$ R—S equations

$$\left(\gamma^{\mu}\partial_{\mu}-\frac{m}{\hbar}\right)\psi_{\lambda}=0, \tag{3.1a}$$

$$\gamma^{\mu}\psi_{\mu}=0. \tag{3.1b}$$

A positive energy "vector-spinor" solution representing a particle at rest with its spin oriented in the $+z$ direction is given by the "product" of a four vector and a Dirac spinor

$$\psi_{\lambda}=\varepsilon_{\lambda}u_{+}(0) \tag{3.2}$$

where the vector part is

$$\varepsilon_\lambda = \frac{1}{\sqrt{2}}(0, i, -1, 0) \tag{3.3}$$

and the spinor is

$$u_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{3.4}$$

a specific meaning for the term "spin orientation" is given at the end of this section. From this we obtain the corresponding solution for a particle at rest with its spin oriented in the direction of a unit vector $\hat{n}$, with

$$(n_1, n_2, n_3) = (\sin\theta\cos\Phi, \sin\theta\sin\Phi, \cos\theta) \tag{3.5}$$

by performing a rotation on the expressions (3.3) and (3.4). We find

$$\psi_\lambda(\theta, \Phi) = \varepsilon_\lambda(\theta, \Phi)u_+(\theta, \Phi)$$

where the vector is

$$\varepsilon_\lambda(\theta, \Phi) = \frac{1}{\sqrt{2}}(0, i\cos\theta\cos\Phi+\sin\Phi, i\cos\theta\sin\Phi-\cos\Phi, -i\sin\theta) \tag{3.6}$$

and the spinor is

$$u_+(\theta, \Phi) = \begin{Vmatrix} \cos\frac{\theta}{2}e^{-i\frac{\Phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\Phi}{2}} \\ 0 \\ 0 \end{Vmatrix}. \tag{3.7}$$

Finally, to obtain the corresponding solution for a particle moving with momentum $p$ and energy $E$ in the $+z$ direction we perform a Lorentz transformation to a frame moving in the $-z$ direction with velocity $v = \frac{p}{E}$. The result is

$$\psi_\lambda(p_\mu, \theta, \Phi) = \varepsilon_\lambda(p_\mu, \theta, \Phi)u_+(p_\mu, \theta, \Phi)e^{\frac{i}{\hbar}p_\mu x^\mu} \tag{3.8}$$

where

$$\varepsilon_\lambda(p_\mu, \theta, \Phi) = \frac{1}{\sqrt{2}}\left(\frac{ip}{m}\sin\theta, i\cos\theta\cos\Phi+\sin\Phi, i\cos\theta\sin\Phi-\cos\Phi, -\frac{iE}{m}\sin\theta\right) \tag{3.9}$$

and where

$$u_+(p_\mu, \theta, \Phi) = \sqrt{\frac{E+m}{2m}} \left\| \begin{array}{c} \cos\frac{\theta}{2} e^{-i\frac{\Phi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\Phi}{2}} \\ \frac{p}{E+m} \cos\frac{\theta}{2} e^{-i\frac{\Phi}{2}} \\ \frac{-p}{E+m} \sin\frac{\theta}{2} e^{i\frac{\Phi}{2}} \end{array} \right\| . \tag{3.10}$$

It is straightforward to verify directly that this solution satisfies the equations (1a, b). Like a Dirac spinor, our vector-spinor is determined by its four momentum and the rest frame spin orientation.

We define the adjoint vector spinor

$$\bar{\psi}_\lambda = \varepsilon^*_\lambda \bar{u}_+ = \varepsilon^*_\lambda u^\dagger_+ \gamma^\circ \tag{3.11}$$

since

$$\bar{u}_+ u_+ = 1 \tag{3.12}$$

and since

$$\eta^{\mu\nu} \varepsilon^*_\mu \varepsilon_\nu = -1 = \varepsilon^{*\mu} \varepsilon_\mu \tag{3.13}$$

we have

$$\bar{\psi}^\lambda \psi_\lambda = -1. \tag{3.14}$$

The spin tensor for a vector field is

$$(S^{\mu\nu})_{\varrho\sigma} = i(\delta^\mu_\varrho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\varrho). \tag{3.15}$$

In what follows we will be interested in the quantities

$$S^{\mu\nu} \equiv \frac{\bar{\psi}^\varrho (S^{\mu\nu})_{\varrho\sigma} \psi^\sigma}{\bar{\psi}^\lambda \psi_\lambda} = -i(\bar{\psi}^\mu \psi^\nu - \bar{\psi}^\nu \psi^\mu). \tag{3.16}$$

A simple calculation shows these quantities to be, for our solution of the R—S equations,

$$S^{12} = n_3; \; S^{31} = \frac{E}{m} n_2; \; S^{23} = \frac{E}{m} n_1,$$

$$S^{01} = \frac{p}{m} n_2; \; S^{02} = \frac{-p}{m} n_1; \; S^{03} = 0. \tag{3.17}$$

Similarly, the spin tensor for a spinor field is

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]. \tag{3.18}$$

and we will also need the quantities

$$\Sigma^{\mu\nu} \equiv \frac{\bar{\psi}^{\lambda}\sigma^{\mu\nu}\psi_{\lambda}}{\bar{\psi}^{\lambda}\psi_{\lambda}} = \bar{u}_{+}\sigma^{\mu\nu}u_{+}. \tag{3.19}$$

One can quickly show that, for our vector-spinor solution,

$$\Sigma^{\mu\nu} = S^{\mu\nu}. \tag{3.20}$$

Finally, if one defines the spin operator for our vector spinor

$$(\mathscr{S}^{\mu\nu})_{\varrho\sigma\alpha'\beta'} \equiv (S^{\mu\nu})_{\varrho\sigma}\delta_{\alpha'\beta'} + \frac{1}{2}\,\eta_{\varrho\sigma}(\Sigma^{\mu\nu})_{\alpha'\beta'} \tag{3.21}$$

where the primed indices are Dirac spinor indices, and if one writes

$$\mathscr{S}^1 \equiv S^{23};\ \mathscr{S}^2 \equiv S^{31};\ \mathscr{S}^3 \equiv S^{12} \tag{3.22}$$

then we have

$$\eta^{\sigma\tau}(\hat{n}\cdot\vec{\mathscr{S}})_{\varrho\sigma\alpha'\beta'}\psi^{(\theta,\Phi)}_{\tau\beta'} = \frac{3}{2}\,\psi_{\varrho\alpha'}(\theta,\Phi) \tag{3.23}$$

where

$$\psi_{\varrho\alpha'}(\theta,\Phi) = \varepsilon_{\varrho}(\theta,\Phi)u_{+\alpha'}(\theta,\Phi). \tag{3.24}$$

It is in this sense that we say that in the rest frame the spin of our particle is oriented in the direction of the unit vector $\hat{n}$.

## 4. *Spin $\frac{3}{2}$ equations of motion*

In this chapter and in what follows we will adopt harmonic coordinate conditions which for the linearized theory have the form

$$h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}h^{\alpha}_{\alpha} = 0. \tag{4.1}$$

We now begin with our covariant Rarita-Schwinger field equation (2.12) for spin $\frac{3}{2}$, that is, with only one tensor index. This can be iterated to give a second order equation

$$\tilde{\gamma}^{\mu}\Delta_{\mu}\tilde{\gamma}^{\nu}\Delta_{\nu}\psi_{\lambda} = \frac{m^2}{\hbar^2}\,\psi_{\lambda}. \tag{4.2}$$

We now write this out in full, keeping only first orders in gravitational field strengths:

$$\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu}\psi_{\lambda,\mu\nu} + \gamma^{\mu}\tilde{\gamma}^{\nu}{}_{,\mu}\psi_{\lambda,\nu} + i\gamma^{\mu}\Gamma_{\mu}\gamma^{\nu}\psi_{\lambda,\nu} + i\gamma^{\mu}\gamma^{\nu}\Gamma_{\nu}\psi_{\lambda,\mu} +$$

$$+ i\gamma^{\mu}\gamma^{\nu}\Gamma_{\nu,\mu}\psi_{\lambda} - \gamma^{\mu}\gamma^{\nu}\Gamma^{\varrho}_{\nu\lambda,\mu}\psi_{\varrho} - \gamma^{\mu}\gamma^{\nu}\Gamma^{\varrho}_{\nu\lambda}\psi_{\varrho,\mu} - \gamma^{\mu}\gamma^{\nu}\Gamma^{\varrho}_{\mu\lambda}\psi_{\varrho,\nu} - \frac{m^2}{\hbar^2}\,\psi_{\lambda} = 0. \tag{4.3}$$

In harmonic coordinates, defined by Eq. (4.1), and using the expression (2.8) for the linearized Fock-Ivanenko coefficients, we can show that

$$\gamma^{\mu}\Gamma_{\mu} = -\frac{i}{8}\gamma^{\mu}\eta^{\alpha\beta}h_{\alpha\beta,\mu}. \tag{4.4}$$

We may also write [19]

$$\tilde{\gamma}^{\nu}{}_{,\mu} = e_{\alpha}^{\nu}{}_{,\mu}\gamma^{\alpha} = -\frac{1}{2}\eta^{\nu\gamma}h_{\alpha\gamma,\mu}\gamma^{\alpha}. \tag{4.5}$$

Then using the relation

$$\gamma^{\mu}\gamma^{\nu} = \eta^{\mu\nu} - i\sigma^{\mu\nu} \tag{4.6}$$

as well as the harmonic coordinate conditions and the absence of gravitational sources in the region we are considering, reduces Eq. (4.3) to

$$g^{\mu\nu}\psi_{\lambda;\mu\nu} + \frac{i}{2}\sigma^{\mu\alpha}\eta^{\nu\gamma}h_{\alpha\gamma,\mu}\psi_{\lambda,\nu} + \frac{i}{2}\sigma^{\mu\nu}R^{\varrho}_{\lambda\nu\mu}\psi_{\varrho}.$$

$$-2\eta^{\mu\nu}\Gamma^{\varrho}_{\mu\lambda}\psi_{\varrho,\nu} - \frac{m^2}{\hbar^2}\psi_{\lambda} = 0 \tag{4.7}$$

where the linearized Riemann tensor is

$$R^{\varrho}_{\lambda\nu\mu} = \Gamma^{\varrho}_{\nu\lambda,\mu} - \Gamma^{\varrho}_{\mu\lambda,\nu} = \frac{1}{2}\eta^{\varrho\sigma}(h_{\nu\sigma,\lambda\mu} - h_{\nu\lambda,\mu\sigma} + h_{\mu\lambda,\nu\sigma} - h_{\mu\sigma,\lambda\nu}). \tag{4.8}$$

We now wish to discuss the solutions of this wave equation in a WKB-like approximation. We will limit all $\hbar$'s in the theory to zero except for those of the spin. The effect of this is to allow us to go over from the motion of waves to a discussion of trajectories of particles, without turning off the effect of the spin on the trajectories. To implement the approximation we take the Rarita-Schwinger field to be of the form

$$\psi_{\lambda}(x) = A_{\lambda}(x)e^{i\frac{m}{\hbar}S(x)} \tag{4.9}$$

where $A_{\lambda}(x)$ is a vector-spinor like $\psi_{\lambda}$ and $S(x)$ is a real scalar function. Substitution of (4.9) into (4.7) yields

$$g^{\mu\nu}\left(\frac{\hbar^2}{m^2}A_{\lambda,\mu\nu} + \frac{i\hbar}{m}S_{,\nu}A_{\lambda,\mu} + \frac{i\hbar}{m}S_{,\mu\nu}A_{\lambda} + \frac{i\hbar}{m}S_{,\mu}A_{\lambda,\nu} - S_{,\mu}S_{,\nu}A_{\lambda}\right) -$$

$$-\frac{1}{2}\frac{\hbar}{m}S_{,\nu}\eta^{\nu\gamma}\sigma^{\mu\alpha}h_{\alpha\gamma,\mu}A_{\lambda} + \frac{i\hbar^2}{2m^2}\sigma^{\mu\nu}R^{\varrho}_{\lambda\nu\mu}A_{\varrho} - 2\frac{i\hbar}{m}(S_{,\nu}\eta^{\mu\nu}\Gamma^{\varrho}_{\lambda\mu})A_{\varrho} - A_{\lambda} = 0 \tag{4.10}$$

clearly the term $g^{\mu\nu}\psi_{\lambda,\mu\nu}$ of Eq. (4.7) contains no spin interactions. Thus in the contributions from this term we set $\hbar = 0$, while keeping, for the time being, the other $\hbar$ terms. We have

$$(g^{\mu\nu}S_{,\mu}S_{,\nu}+1) + \frac{1}{2}\frac{\hbar}{m}S_{,\nu}\eta^{\nu\gamma}h_{\alpha\gamma,\mu}\left[\frac{\bar{\psi}^{\lambda}\sigma^{\mu\alpha}\psi^{\lambda}}{\bar{\psi}^{\sigma}\psi_{\sigma}}\right] +$$

$$+2i\frac{\hbar}{m}S_{,\nu}\eta^{\mu\nu}\Gamma^{\varrho}_{\mu\lambda}\left[\frac{\bar{\psi}^{\lambda}\psi_{\varrho}}{\bar{\psi}^{\sigma}\psi_{\sigma}}\right] - \frac{i}{2}\frac{\hbar^2}{m^2}R^{\varrho}_{\lambda\nu\mu}\left[\frac{\bar{\psi}^{\lambda}\sigma^{\mu\nu}\psi_{\varrho}}{\bar{\psi}^{\sigma}\psi_{\sigma}}\right] = 0. \tag{4.11}$$

This equation represents a Hamilton-Jacobi equation for the trajectories of the spin $\frac{3}{2}$ particles. When we go over to the second order, geodesic form of the equations of motion in the usual way [8], [21] we find, to first order in gravitational fields,

$$\frac{D}{D\tau} u^{\sigma} + 2i \frac{\hbar}{m} \eta^{\sigma\alpha} u^{\beta} R_{\lambda\tau\alpha\beta} \left[ \frac{\bar{\psi}^{\lambda} \psi^{\tau}}{\bar{\psi}^{\gamma} \psi_{\gamma}} \right] + \frac{\hbar}{2m} R^{\sigma}_{\beta\nu\varrho} u^{\beta} \left[ \frac{\bar{\psi}_{\lambda} \sigma^{\nu\varrho} \psi^{\lambda}}{\bar{\psi}^{\gamma} \psi_{\gamma}} \right] +$$

$$+ i \frac{\hbar^2}{m^2} \eta^{\sigma\alpha} R_{\lambda\tau\mu\nu,\alpha} \left[ \frac{\bar{\psi}^{\lambda} \sigma^{\nu\mu} \psi^{\tau}}{\bar{\psi}^{\gamma} \psi_{\gamma}} \right] = 0. \tag{4.12}$$

where $\frac{D}{D\tau}$ represents covariant proper time differentiation along the path of the particle:

$$\frac{D}{D\tau} A^{\cdots}_{\cdots} = (\nabla_{\mu} A^{\cdots}_{\cdots}) \frac{dx^{\mu}}{d\tau}. \tag{4.13}$$

Consistent with our linearized approximation for gravitational fields we may take the quantities in Eq. (4.12) which involve the R—S wave functions to be just those flat space quantities calculated in section 3. We thus write

$$\frac{D}{D\tau} u^{\sigma} + \frac{\hbar}{m} u^{\beta} R^{\sigma}_{\beta\lambda\tau} \left( S^{\lambda\tau} + \frac{1}{2} \Sigma^{\lambda\tau} \right) - \frac{\hbar^2}{2m^2} \eta^{\sigma\alpha} R_{\lambda\tau\mu\nu,\alpha} S^{\lambda\tau} \Sigma^{\mu\nu} = 0. \tag{4.14}$$

Kinematics requires that the square of the four velocity $u^{\sigma}$ should be a constant, namely

$$u_{\sigma} u^{\sigma} = -1. \tag{4.15}$$

Therefore, we should have

$$u_{\sigma} \frac{Du^{\sigma}}{D\tau} = 0. \tag{4.16}$$

Multiplying Eq. (4.14) on the left by $u_{\sigma}$, and using the symmetry properties of the Riemann tensor leads to

$$u_{\sigma} \frac{Du^{\sigma}}{D\tau} - \frac{\hbar^2}{2m^2} u^{\alpha} R_{\lambda\tau\mu\nu,\alpha} S^{\lambda\tau} \Sigma^{\mu\nu} = 0. \tag{4.17}$$

The second term here will not vanish in general.

Thus we see that in order $\hbar^2$ our equations of motion are inconsistent. This arises from the inconsistency in discarding some $\hbar$'s and not others in our WKB approximation. A similar difficulty arises in the equivalent treatment of the "Pauli term" in the iterated Dirac equation in the presence of an electromagnetic field. As a result of this we discard the term proportional to $\hbar^2$ and restrict ourselves to first order in the spin angular momentum and the equations of motion

$$\frac{Du^{\sigma}}{D\tau} + \frac{\hbar}{m} u^{\beta} R^{\sigma}_{\beta\lambda\tau} \left( S^{\lambda\tau} + \frac{1}{2} \Sigma^{\lambda\tau} \right) = 0. \tag{4.18}$$

This is just the equation of motion which was derived previously[3] for Dirac particles [7], [8], but with the spin tensor $S^{\lambda\tau}+\frac{1}{2}\Sigma^{\lambda\tau}$ replacing $\frac{1}{2}\Sigma^{\lambda\tau}$. We are thus in the position to repeat the calculation of the earlier work in which a high-energy particle with spin is deflected as it passes near a static massive body. The body of mass $M$ is represented by its linearized Schwarzschild field centered on the origin of coordinates. The spinning particle approaches, moving in the $+z$ direction with momentum $p \approx E$. Its initial motion is in the $x$, $z$ plane along the line $x = b$. The spin of the particle is specified to be aligned with the unit vector $\hat{n}$ *in its rest frame.* As we see from Eqs (3.17) and (3.20) the values of the spin tensor $S^{\lambda\tau}+\frac{1}{2}\Sigma^{\lambda\tau}$ for such a vector-spinor have the same values as the spin tensor of a Dirac particle [7], [8] but multiplied by a factor of three. Thus the deflections of the spin $\frac{3}{2}$ particle will be just those of the spin $\frac{1}{2}$ particle with the spin contribution multiplied by three.

We thus find the deflections in the $-x$ direction to be

$$\delta\Phi_x = \frac{2GM}{b}\left(1 - \frac{3\hbar n_2}{mb}\right) \tag{4.19}$$

where $n_2$ is the $y$ component of the unit vector $\hat{n}$. The deflection in the $-y$ direction will be

$$\delta\Phi_y = +\frac{6GM\hbar}{mb^2}\, n_1. \tag{4.20}$$

Finally, we also point out that, from dimensional arguments, we can see that the contribution to the deflections which would arise from a term in the equations of motion proportional to $\hbar^2$ must contain a factor $\leqslant \dfrac{\hbar^2}{m^2b^2}$.

Since for an electron just grazing the sun $\dfrac{\hbar}{mb} \approx 10^{-21}$ we can see that (*i*) our first order corrections to the deflections are very small and (*ii*) the error caused by neglecting the $\hbar^2$ terms is a factor $10^{-21}$ smaller yet. For $n_1 = n_2 = 0$, as for massless particles, we just get geodetic motion.

## 5. *Discussion and extension of results*

The most striking feature of our results for the motion of a spin $\frac{3}{2}$ particle is the close relationship to the corresponding results for a spin $\frac{1}{2}$ particle. The generator of infinitesimal Lorentz transformations in the spin $\frac{1}{2}$ case is

$$(G^{\mu\nu}_{1/2})_{\alpha'\beta'} = \frac{1}{2}\sigma^{\mu\nu}_{\alpha'\beta'} \tag{5.1}$$

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[3] In Ref. [8] it was erroneously concluded that vector particles must follow geodesics. In this paper the calculation was restricted to linearly polarized plane wave states, which, of course, carry no spin. Also the spin term of the equation of motion in reference [8] has the wrong sign.

where the primed indices are spinor indices and the unprimed are tensor indices. The generator for the spin $\frac{3}{2}$ case is

$$(G^{\mu\nu}_{3/2})_{\alpha'\beta'\varrho\sigma} = (S^{\mu\nu})_{\varrho\sigma}\delta_{\alpha'\beta'} + \frac{1}{2}\,\sigma^{\mu\nu}_{\alpha'\beta'}\,\eta_{\varrho\sigma}\,. \tag{5.2}$$

For the spin tensors which appear in the equations of motion for the spinor and spinor-tensor fields, respectively, we have

$$\frac{1}{2}\,\Sigma^{\mu\nu} = \frac{\bar{\psi}_{\alpha'}(G^{\mu\nu}_{1/2})_{\alpha'\beta'}\psi_{\beta'}}{\bar{\psi}_{\alpha'}\psi_{\alpha'}} \tag{5.3a}$$

and

$$\frac{1}{2}\,\Sigma^{\mu\nu} + S^{\mu\nu} = \frac{\bar{\psi}^{\varrho}_{\alpha'}(G^{\mu\nu}_{3/2})_{\alpha'\beta'\varrho\sigma}\psi^{\sigma}_{\beta'}}{\bar{\psi}^{\lambda}_{\alpha'}\psi_{\alpha'\lambda}}\,. \tag{5.3b}$$

This suggests that one could use the following prescription for finding the equation of motion resulting from a field $\Phi$ with adjoint $\bar{\Phi}$ representing spin $\frac{n}{2}$ where $n$ is an integer: Take the Eq. of motion to be just Eq. (4.18) with the expression $S^{\lambda\tau}+\frac{1}{2}\Sigma^{\lambda\tau}$ replaced by

$$\mathcal{S}^{\lambda\tau} \equiv \frac{\bar{\Phi}G^{\lambda\tau}_{n/2}\Phi}{\bar{\Phi}\Phi} \tag{5.4}$$

where $G^{\lambda\tau}_{n/2}$ is the generator of infinitesimal Lorentz transformations for the spin $\frac{n}{2}$ field $\Phi$. For the vector meson (spin 1, $n = 2$) case this can be straightforwardly shown to be so. The treatment is just the same as our Rarita-Schwinger case, but leaving out the effects of the spinor index. The spin tensor is just

$$\mathcal{S}^{\lambda\tau} = S^{\lambda\tau}. \tag{5.5}$$

It also follows that the anomalous deflections of the high energy mesons grazing the sun are just twice those for spin $\frac{1}{2}$ particles (ignoring the effect of different masses). This form of equation of motion holds for higher spins too if, ignoring the question of supplementary conditions, one assumes that the dynamical field equations are just the usual covariant extension of the flat space Klein-Gordon equation, *i.e.*

$$g^{\varrho\sigma}\nabla_{\varrho}\nabla_{\sigma}\Phi_{\mu\nu\ldots} = m^2\Phi_{\mu\nu\ldots} \tag{5.6}$$

If the field represents integer spin $s$, then there are $s$ tensor indices $\mu$, $\nu$, ... If it represents spin $s+\frac{1}{2}$ there will also be a spinor index. The form of the covariant derivatives is appropriate to the number of tensor indices and the presence or absence of a spinor index. From this assumption one can show that the equations of motion for the particle are just Eq. (4.18) with the spin tensor (5.4) instead of (5.3b). Our assumption (5.6) can be shown to be consistent with the auxilliary conditions for the fields with spin $\leqslant\frac{3}{2}$, but for spin $>\frac{3}{2}$ its validity is not generally clear. It may be that in order to obtain consistent equations it is necessary to use a more complicated generalization of the flat space field equations. For a discussion of the spin 2 case see Cohen [16].

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