HERMITIAN SYMMETRY IN EINSTEIN'S UNIFIED FIELD THEORY

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The result of hermitian symmetry in Einstein's Unified Field Theory is examined in terms of pseudo-tensor $U^{\gamma}_{\mu\nu}$ for which the Ricci tensor is hermitian symmetric. Einstein's expression for $U^{\lambda}_{\mu\nu}$ is shown not to be unique. It is also shown that there is no linear expression for the affine connection in terms of such a pseudo-tensor which would lead to invariant symmetry conditions.

1. *Introduction*

The requirement that the field equations should satisfy the condition of a new kind of symmetry, called "hermitian", plays a crucial part in Einstein's Unified Field Theory (UFT) (Refs [1], [2], [3] and [4]). In general, the symmetry is defined in the following way. Given a (tensor) expression

$$A_{\mu\nu} = A_{\mu\nu}(g_{\alpha\beta}, \Gamma^{\lambda}_{\varrho\sigma}, \ldots)$$

where $g_{\alpha\beta}$ is a metric tensor and $\Gamma^{\lambda}_{\varrho\sigma}$, the affine connection, and Greek indices go from 1 to n (or 1 to 4 if we restrict ourselves to a four-dimensional space time), its conjugate is given by

$$\tilde{A}_{\mu\nu} = A_{\nu\mu}(g_{\beta\alpha}, \Gamma^{\lambda}_{\sigma\varrho}, \ldots).$$

If $A = \tilde{A}$, the tensor is said to be hermitian symmetric, or just hermitian, and if $A = -\tilde{A}$, hermitian skew symmetric or antihermitian. Any tensor can be invariantly separated into a hermitian and an antihermitian part:

$$A = \tfrac{1}{2}(A+\tilde{A})+\tfrac{1}{2}(A-\tilde{A}). \tag{1}$$

It is clearly not necessary to restrict the definition to tensors of rank 2, except that only two indices (both covariant or contravariant) take part in any given hermitian operation. The reason for the name of the symmetry is simple. If $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ are allowed to be complex

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(as in Refs [1], [2]) the above symmetry coincides with ordinary hermitian conjugation. Indeed if

$$a_{\mu\nu} = b_{\mu\nu} + ic_{\mu\nu}$$

with b and c real.

$$\tilde{a}_{\mu\nu} = b_{\nu\mu} - ic_{\nu\mu},$$

and $a_{\mu\nu}$ is hermitian if, and only if, $b_{\mu\nu}$ is symmetric and $c_{\mu\nu}$ skew symmetric. This is what Einstein and Straus did in taking

$$g_{\mu\nu} = g_{\underline{\mu\nu}} + ig_{\underset{\vee}{\mu\nu}},\ \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\underline{\mu\nu}} + i\Gamma^{\lambda}_{\underset{\vee}{\mu\nu}},$$

(a line under two indices denotes symmetry and a caret, skew symmetry). The importance of the concept lies in that it restricts the manifold of possible solutions and expresses (Ref. [4]) the invariance of the field laws with respect to the sign of electricity. It represents therefore an anchorage of a hypothetical (and complicated) mathematical structure of the theory in the physical reality. It is responsible for the selection of equation

$$g_{\underset{+-}{\mu\nu;\lambda}} \equiv g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0, \tag{2}$$

as a definition of the generalized (nonsymmetric) affine connection. Equation (2) alone, and not, for example,

$$g_{\underset{++}{\mu\nu;\lambda}} \equiv g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\lambda}g_{\mu\sigma} = 0, \tag{3}$$

is hermitian symmetric.

It is of some importance to the present investigation to recall that Hlavaty (Ref. [5]) proves that equation (3) has a solution if and only if Det $g_{\mu\nu}$/Det $g_{\underline{\mu\nu}}$ is a constant. All this is well known. It is equally well known that the Ricci tensor of the theory

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho}, \tag{4}$$

is not hermitian, the condition for it to be so, reading

$$\Gamma_{\mu,\nu} + \Gamma_{\nu,\mu} - 2\Gamma^{\sigma}_{\mu\nu}\Gamma_{\sigma} = 0. \tag{5}$$

In Ref. [4], Einstein and Kaufman introduce a new quantity $U^{\lambda}_{\mu\nu}$ defined by

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - \tfrac{1}{3}\, U^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu}, \tag{6}$$

so that

$$\Gamma^{\sigma}_{\mu\sigma} = -\tfrac{1}{3}\, U^{\sigma}_{\mu\sigma} \text{ and } U^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu}.$$

In terms of U, the Ricci tensor becomes

$$R_{\mu\nu} = -U^{\sigma}_{\mu\nu,\sigma} + U^{\sigma}_{\mu\varrho}U^{\varrho}_{\sigma\nu} - \tfrac{1}{3}\, U^{\sigma}_{\mu\sigma}U^{\varrho}_{\varrho\nu}, \tag{7}$$

and is clearly hermitian with respect to $U^{\lambda}_{\mu\nu}$.

On the other hand, the equation (2) becomes

$$g_{\mu\nu,\lambda} - U^{\sigma}_{\mu\lambda}g_{\sigma\nu} - U^{\sigma}_{\lambda\nu}g_{\mu\sigma} + \tfrac{1}{3}(U^{\varrho}_{\mu\varrho}g_{\lambda\nu} + U^{\varrho}_{\lambda\varrho}g_{\mu\nu}) = 0 \tag{8}$$

and is nonhermitian as we can easily convince ourselves.

Moreover, the condition for (8) to become hermitian is

$$U^{\varrho}_{\mu\varrho} g_{\lambda\nu} + U^{\varrho}_{\lambda\varrho} g_{\mu\nu} - U^{\varrho}_{\varrho\nu} g_{\mu\lambda} - U^{\varrho}_{\varrho\lambda} g_{\mu\nu} = 0,$$

or

$$U^{\varrho}_{\underline{\mu\varrho}} g_{\lambda\nu} - U^{\varrho}_{\underline{\nu\varrho}} g_{\mu\lambda} + U_{\mu} g_{\lambda\nu} + U_{\nu} g_{\mu\lambda} + 2 U_{\lambda} g_{\mu\nu} = 0, \tag{9}$$

where

$$U_{\lambda} = \tfrac{1}{2}(U^{\varrho}_{\lambda\varrho} - U^{\varrho}_{\varrho\lambda}).$$

It should be observed that unlike

$$\Gamma_{\lambda} = \tfrac{1}{2}(\Gamma^{\varrho}_{\lambda\varrho} - \Gamma^{\varrho}_{\varrho\lambda}).$$

U_{λ} is not a vector. In terms of the affine connection

$$U_{\lambda} = -\tfrac{1}{2}(3\Gamma^{\sigma}_{\underline{\mu\sigma}} + \Gamma_{\mu}).$$

and

$$U^{\varrho}_{\underline{\mu\varrho}} = -\tfrac{1}{2}(3\Gamma^{\sigma}_{\underline{\mu\sigma}} + 5\Gamma_{\mu}).$$

Hence, the condition (9) reads

$$\Gamma^{\varrho}_{\underline{\mu\varrho}} g_{\lambda\nu} + \Gamma^{\varrho}_{\underline{\lambda\varrho}} g_{\mu\nu} + \tfrac{1}{3}(3\Gamma_{\mu} g_{\lambda\nu} - 2\Gamma_{\nu} g_{\mu\lambda} + \Gamma_{\lambda} g_{\mu\nu}) = 0. \tag{10}$$

This is clearly not invariant or tensor equation since the quantity in brackets is a tensor. If we impose Einstein's condition

$$\Gamma_{\mu} = 0, \tag{11}$$

which is identified in the UFT with the second set of Maxwell's equations (it can be proved directly from equation (2), by the method familiar from General Relativity and used to obtain Christoffel brackets, that

$$(\sqrt{-g}\, g^{\mu\sigma})_{,\sigma} = \sqrt{-g}\, g^{\underline{\sigma\mu}}\, \Gamma_{\sigma},$$

where $g = \mathrm{Det}\; g_{\mu\nu}$, see Appendix), equation (10) becomes

$$\Gamma^{\varrho}_{\underline{\varrho\mu}} g_{\lambda\nu} + \Gamma^{\varrho}_{\underline{\lambda\varrho}} g_{\mu\nu} = 0. \tag{12}$$

It is these consequences of switching the hermitian requirement from $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ to $g_{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$ which prompted the present investigation. Einstein and Kaufman claimed the latter to be the "natural" variables for investigating the unified field. The above difficulty shows that this is hardly the case and the negative conclusion of our results throw considerable doubt on the usefulness of the hermitian concept in the theory as hitherto formulated.

2. $\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - \frac{1}{3} U^{\varrho}_{\mu\varrho} \delta^{\lambda}_{\nu}$

In this section we investigate further the consequences of using $U^{\lambda}_{\mu\nu}$ in hermitian conjugation. We prove first that it follows from (12) that

$$g = \text{const.} \tag{13}$$

Indeed, if $g \neq 0$, there exists a tensor $g^{\mu\nu}$ such that

$$g^{\mu\alpha}g_{\nu\alpha} = g^{\alpha\mu}g_{\alpha\nu} = \delta^{\mu}_{\nu}.$$

Hence multiplying (12) by $g^{\sigma\nu}$ and contracting over σ and λ,

$$5\Gamma^{\varrho}_{\underline{\mu\varrho}} = 0 = \Gamma^{\varrho}_{\underline{\mu\varrho}}.$$

But from (2)

$$\Gamma^{\varrho}_{\underline{\mu\varrho}} = \frac{\partial}{\partial x^{\mu}} \ln \sqrt{-g},$$

and therefore (13). This a severe restriction on the space used in UFT.

The reason why U-substitution leads to a noninvariant condition for hermitian symmetry of equation (8) is that in constructing hermitian conjugates

$$U^{\varrho}_{\mu\varrho} \to U^{\varrho}_{\varrho\mu},$$

and whereas $U^{\varrho}_{\mu\varrho} = 3\Gamma^{\varrho}_{\mu\varrho}$ transforms as a contracted affine connection,

$$U^{\varrho}_{\varrho\mu} = -2\Gamma_{\nu}$$

is a vector.

We can show further that equation (9) implies

$$U^{\varrho}_{\mu\varrho} = 0 = U^{\varrho}_{\varrho\mu}, \tag{14}$$

and therefore also equation (11) (together with $g = \text{const}$). Thus, multiplying (9) by $g^{\mu\nu}$ (and summing as indicated),

$$10U_{\lambda} = 0. \tag{15}$$

Hence (9) reduces to

$$U^{\varrho}_{\underline{\mu\varrho}}g_{\lambda\nu} - U^{\varrho}_{\underline{\nu\varrho}}g_{\mu\lambda} = 0, \tag{16}$$

or

$$g^{\mu\lambda}U^{\varrho}_{\underline{\mu\varrho}}g_{\lambda\nu} = 4U^{\varrho}_{\underline{\lambda\varrho}}.$$

Hence

$$g^{\sigma\nu}g^{\mu\lambda}g_{\lambda\nu}U^{\varrho}_{\underline{\mu\varrho}} = 4U^{\varrho}_{\underline{\nu\varrho}}g^{\sigma\nu} = \delta^{\varrho}_{\lambda}g^{\mu\lambda}U^{\varrho}_{\underline{\mu\varrho}} = g^{\mu\sigma}U^{\varrho}_{\underline{\mu\varrho}}. \tag{17}$$

But

$$U^{\varrho}_{\underline{\mu\varrho}}g_{\lambda\nu}g^{\sigma\nu} = g^{\sigma\nu}U^{\varrho}_{\underline{\nu\varrho}}g_{\mu\lambda},$$

or

$$\delta^{\sigma}_{\lambda}U^{\varrho}_{\underline{\mu\varrho}} = g^{\sigma\nu}U^{\varrho}_{\underline{\nu\varrho}}g_{\mu\lambda}$$

from (16), so that, contracting over σ and λ,

$$U^{\varrho}_{\underline{\mu\varrho}} = \tfrac{1}{4}\, g^{\lambda\nu}U^{\varrho}_{\underline{\nu\varrho}}g_{\mu\lambda}.$$

Substituting into (17)

$$4U^{\varrho}_{\nu\varrho}g^{\sigma\nu} = \tfrac{1}{4}\, g^{\mu\sigma}g^{\lambda\nu}U^{\varrho}_{\nu\varrho}g_{\mu\lambda} = \tfrac{1}{4}\, \delta^{\sigma}_{\lambda}g^{\lambda\nu}U^{\varrho}_{\nu\varrho} = \tfrac{1}{4}\, g^{\sigma\nu}U^{\varrho}_{\nu\varrho}.$$

Hence $U^{\varrho}_{\underline{\nu\varrho}} = 0$, and in view of (15), (14) follows. (14), therefore, is another form of the requirement that (2) should be hermitian symmetric with respect to $U^{\lambda}_{\mu\nu}$. It shows, however, that $g_{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$ must not be used as variational parameters in deriving field equations contrary to the process adopted by Einstein and Kaufman.

The meaning of hermitian invariance is best shown if we revert to complex quantities and write

$$\Gamma^{\lambda}_{\mu\nu} = P^{\lambda}_{\mu\nu} + iQ^{\lambda}_{\mu\nu},\ U^{\lambda}_{\mu\nu} = V^{\lambda}_{\mu\nu} + iW^{\lambda}_{\mu\nu},\ g_{\mu\nu} = g_{\underline{\mu\nu}} + ig_{\underset{\vee}{\mu\nu}}$$

P, Q, V and W being real. Then, retaining the relation (6) of Einstein and Kaufman,

$$Q^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu} - \tfrac{1}{3}\, W^{\varrho}_{\mu\varrho}\delta^{\lambda}_{\nu}$$

must be a tensor if we restrict ourselves to real transformations of coordinates. In that case $P^{\lambda}_{\mu\nu}$ is a connection and $P^{\lambda}_{\underset{\vee}{\mu\nu}} = V^{\lambda}_{\underset{\vee}{\mu\nu}} - \tfrac{1}{6}\,(V^{\varrho}_{\mu\varrho}\delta^{\lambda}_{\nu} = V^{\varrho}_{\nu\varrho}\delta^{\lambda}_{\mu})$, also a tensor.

Hermitian symmetry requirement of equation (2) then leads to the following relations between $P^{\lambda}_{\mu\nu}$ and $Q^{\lambda}_{\mu\nu}$:

$$P^{\sigma}_{\underset{\vee}{\lambda\mu}} g_{\underline{\sigma\nu}} + P^{\sigma}_{\underset{\vee}{\nu\lambda}} g_{\underline{\mu\sigma}} = Q^{\sigma}_{\underline{\mu\lambda}} g_{\underset{\vee}{\nu\sigma}} + Q^{\sigma}_{\underline{\nu\lambda}} g_{\underset{\vee}{\sigma\mu}}, \tag{18}$$

$$P^{\sigma}_{\underset{\vee}{\lambda\mu}} g_{\underset{\vee}{\nu\sigma}} + P^{\sigma}_{\underset{\vee}{\lambda\nu}} g_{\underset{\vee}{\mu\sigma}} = -Q^{\sigma}_{\underline{\mu\lambda}} g_{\underline{\nu\sigma}} - Q^{\sigma}_{\underline{\lambda\nu}} g_{\underline{\mu\sigma}}. \tag{19}$$

If we assume, as is necessary for a meaningful application to physics, that Det $g_{\underline{\mu\nu}} \neq 0$, (18) can be solved for $P^{\lambda}_{\mu\nu}$ and (19) for $Q^{\lambda}_{\underline{\mu\nu}}$ to give

$$P^{\lambda}_{\underset{\vee}{\mu\nu}} = g^{\underline{\lambda\alpha}}(Q^{\sigma}_{\underline{\alpha\mu}} g_{\underset{\vee}{\sigma\nu}} + Q^{\sigma}_{\underline{\nu\alpha}} g_{\underset{\vee}{\mu\sigma}}),$$

and

$$Q^{\lambda}_{\underline{\mu\nu}} = g^{\underline{\lambda\alpha}}(P^{\sigma}_{\underset{\vee}{\alpha\nu}} g_{\underset{\vee}{\mu\sigma}} + P^{\sigma}_{\underset{\vee}{\mu\alpha}} g_{\underset{\vee}{\sigma\nu}}),$$

where

$$g^{\underline{\mu\alpha}} g_{\underline{\nu\alpha}} = \delta^{\mu}_{\nu}.$$

These relations are independent of (6) which we are now going to show to be non unique as far as making $R_{\mu\nu}$ hermitian symmetric is concerned.

3. *A new "natural" field variable*

Let

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - \tfrac{1}{3}\, U^{\sigma}_{\mu}\delta^{\lambda}_{\nu} - \tfrac{1}{3}\, U_{\nu}\delta^{\lambda}_{\mu}. \tag{20}$$

A straightforward substitution into (4) gives

$$R_{\mu\nu} = -U^{\sigma}_{\mu\nu,\sigma} + \tfrac{1}{3}\,(U_{\nu,\mu} - U_{\mu,\nu}) + U^{\varrho}_{\mu\sigma}U^{\sigma}_{\varrho\nu} - \tfrac{1}{3}\, U^{\sigma}_{\mu\sigma}U^{\varrho}_{\varrho\nu}. \tag{21}$$

Only the bracketed terms need to be examined for hermitian symmetry. However, undre the conjugation $U_{\nu} \to -U_{\nu}$ first and subsequent transposition of ν and μ returns the expression to the original. Hence $R_{\mu\nu}$ is hermitian symmetric with respect to $U^{\lambda}_{\mu\nu}$ and providing (20) can be solved for $U^{\lambda}_{\mu\nu}$, these variables will be just as "natural" as Einstein's.

Contracting (20) first with respect to λ and ν, and then over λ and μ, we easily find that

$$U^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \tfrac{5}{8}\,(\Gamma^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} + \Gamma^{\sigma}_{\nu\sigma}\delta^{\lambda}_{\mu}) - \tfrac{1}{4}\,(\Gamma_{\mu}\delta^{\lambda}_{\nu} - \Gamma_{\nu}\delta^{\lambda}_{\mu}). \tag{22}$$

Hence $U^{\lambda}_{\mu\nu}$ is well determined. We may note that if the last term in (20) is replaced by $+\frac{1}{3}U_{\nu}\delta^{\lambda}_{\mu}$ or $+\frac{1}{3}U_{\mu}\delta^{\lambda}_{\nu}$, such a solution cannot be derived. $-\frac{1}{3}U_{\mu}\delta^{\lambda}_{\nu}$ is possible but $R_{\mu\nu}$ then ceases to be hermitian symmetric.

It remains to see now what happens to equation (2). The condition that it should be hermitian symmetric becomes

$$U^{\varrho}_{\underline{\mu}\varrho}g_{\lambda\nu} - U^{\varrho}_{\underline{\nu}\varrho}g_{\mu\lambda} + 2(U_{\mu}g_{\lambda\nu} + U_{\nu}g_{\mu\lambda} + 2U_{\lambda}g_{\mu\nu}) = 0. \tag{23}$$

Multiplying by $g^{\mu\nu}$ and summing as indicated,

$$U_{\lambda} = 0,$$

so that (23) reduces to

$$U^{\varrho}_{\underline{\mu}\varrho}g_{\lambda\nu} - U^{\varrho}_{\underline{\nu}\varrho}g_{\mu\lambda} = 0,$$

that is (16). Hence

$$U^{\varrho}_{\underline{\mu\varrho}} = 0,$$

and therefore, from (22),

$$\Gamma_{\mu} = 0 = \Gamma^{\varrho}_{\mu\varrho}$$

as before. The last equation implies, of course,

$$g = \text{const.}$$

Hence our new variables suffer the same deficiency as Einstein's and Kaufman's.

In the next section we shall consider, therefore, whether a linear substitution of the above kind can exist in an unrestricted space time of the non-symmetric theory.

4. *A general U-substitution*

The most general linear relation between $\Gamma^{\lambda}_{\mu\nu}$ and a $U^{\lambda}_{\mu\nu}$ variable is of the form

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} + \alpha_1 U^{\varrho}_{\mu\varrho}\delta^{\lambda}_{\nu} + \alpha_2 U^{\varrho}_{\nu\varrho}\delta^{\lambda}_{\mu} + \beta_1 U^{\varrho}_{\varrho\mu}\delta^{\lambda}_{\nu} + \beta_2 U^{\varrho}_{\varrho\nu}\delta^{\lambda}_{\mu} + \gamma_1 U_{\mu}\delta^{\lambda}_{\nu} + \gamma_2 U_{\nu}\delta^{\lambda}_{\mu}, \tag{24}$$

where α, β, γ's are constants. To be of any use, (24) must be solvable for $U^{\lambda}_{\mu\nu}$ in terms of the affine connection. If we write

$$\begin{aligned} A &= 1+4\alpha_1+\alpha_2+4\beta_1+\beta_2, \\ B &= 1+4\alpha_1+\alpha_2-4\beta_1-\beta_2+4\gamma_1+\gamma_2, \\ C &= 1+\alpha_1+4\alpha_2+\beta_1+4\beta_2, \end{aligned} \tag{25}$$

and

$$D = -1+\alpha_1+4\alpha_2-\beta_1-4\beta_2+\gamma_1+4\gamma_2,$$

the condition of solvability is easily seen to be

$$BC-AD \neq 0. \tag{26}$$

This is then the first requirement we impose on (24). The aim is to transfer hermitian condition from $\Gamma^\lambda_{\mu\nu}$ to $U^\lambda_{\mu\nu}$ in such a way that $R_{\mu\nu}$ should become hermitian symmetric with respect to $U^\lambda_{\mu\nu}$ and that the condition of symmetry of $g_{\mu\nu;\lambda}$ with respect to a joint transposition of $g_{\mu\nu}(= \underline{g}_{\mu\nu}+i\check{g}_{\mu\nu})$ and $U^\lambda_{\mu\nu}$, should result in a tensor equation. Actually, since the latter is necessarily an algebraic equation, it is sufficient to ask that the part of $R_{\mu\nu}$ in which derivatives of $\Gamma^\lambda_{\mu\nu}$ appear (in other words, $-\Gamma^\sigma_{\mu\nu,\sigma}+\Gamma^\sigma_{\mu\sigma,\nu}$) should be hermitian symmetric.

We can prove now the following theorem.

Theorem:

There is no solvable substitution of the form (24) which satisfies simultaneously

a) the condition that $g_{\underline{\mu\nu};\lambda}$ should be hermitian symmetric with respect to $g_{\mu\nu}$ and $U^\lambda_{\mu\nu}$ is a tensor equation;

b) $-\Gamma^\sigma_{\mu\nu,\sigma}+\Gamma^\sigma_{\mu\sigma,\nu}$ is hermitian symmetric in $U^\lambda_{\mu\nu}$. The proof is elementary though somewhat tedious. It depends essentially on an almost self-evident

Lemma:

A linear form

$$L_{\mu\nu\lambda} \equiv a\Gamma^\varrho_{\underline{\mu\varrho}}g_\lambda+b\Gamma^\varrho_{\underline{\lambda\varrho}}g_\mu+c\Gamma^\varrho_{\underline{\nu\varrho}}g_{\mu\lambda},$$

where a, b and c are numerical, is not a tensor.

A necessary condition for $L_{\mu\nu\lambda}$ to be a tensor can be written as

$$\frac{\partial x'^\alpha}{\partial x^\sigma}\left(a\,\frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\mu}\,\delta^\varrho_\lambda+b\,\frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\lambda}\,\delta^\varrho_\mu+c\,\frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\mu}\,\delta^\varrho_\lambda\right)=0,$$

whence

$$a=b=c=0.$$

Let us write

$$\Gamma^\lambda_{\mu\nu} = U^\lambda_{\mu\nu}+A^\lambda_{\mu\nu}, \tag{27}$$

and let $\bar{A}^\lambda_{\mu\nu}$ be obtained from $A^\lambda_{\mu\nu}$ by letting

$$U^\varrho_{\mu\varrho} \to U^\varrho_{\varrho\mu},\; U_\mu \to -U_\mu.$$

Then, $g_{\underset{+-}{\mu\nu};\lambda}$ will be hermitian symmetric if

$$(A^\sigma_{\mu\lambda}-\bar{A}^\varrho_{\lambda\mu})g_{\sigma\nu}+(A^\sigma_{\lambda\nu}-A^\sigma_{\nu\lambda})g_{\mu\sigma}$$

$$\equiv M(\underline{U}^\varrho_{\mu\varrho}g_{\lambda\nu}-\underline{U}^\varrho_{\nu\varrho}g_{\mu\lambda})+N(U_\mu g_{\lambda\nu}+2U_\lambda g_{\mu\nu}+U_\nu g_{\mu\lambda})=0 \tag{28}$$

where

$$M = \alpha_1 + \beta_1 - \alpha_2 - \beta_1,$$

$$N = \alpha_1 - \beta_1 + \gamma_1 + \alpha_2 - \beta_2 + \gamma_2.$$

By our lemma, (28) will be a tensor equation only if $A^{\sigma}_{\mu\lambda} - \bar{A}^{\sigma}_{\lambda\mu}$ is independent of $\Gamma^{\sigma}_{\underline{\mu\sigma}}$ and this gives

$$M(B-D) = 0 = N(A-C), \tag{29}$$

This gives four alternatives

$$\begin{aligned} &i) && B = D,\ A = C, \\ &ii) && M = 0,\ N = 0, \\ &iii) && M = 0,\ A = C,\ (\text{or } U_\mu \propto \Gamma_\mu), \\ &iv) && N = 0,\ B = D,\ (\text{or } U^{\varrho}_{\underline{\mu\varrho}} \propto \Gamma_\mu). \end{aligned} \tag{30}$$

The solution of (22) can be wirtten in the form

$$U_\mu = R\Gamma^{\lambda}_{\underline{\mu\lambda}} + S\Gamma_\mu,$$

$$U^{\varrho}_{\underline{\mu\varrho}} = P\Gamma^{\lambda}_{\underline{\mu\lambda}} + Q\Gamma_\mu,$$

where

$$P = \frac{B-D}{CB-AD}, \quad Q = -\frac{B+D}{CB-AD}, \quad R = -\frac{A-C}{CB-AD}, \quad S = \frac{A+C}{CB-AD}.$$

Furthermore, the antihermitian part of $-\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu}$ is

$$K(U^{\varrho}_{\underline{\mu\varrho},\nu} - U^{\varrho}_{\underline{\nu\varrho},\mu}) + L(U_{\mu,\nu} + U_{\nu,\mu})$$

$$= 2LR\Gamma^{\varrho}_{\underline{\mu\varrho},\nu} + (KQ+LS)\Gamma_{\mu,\nu} + (LS-KQ)\,\Gamma_{\nu,\mu}, \tag{31}$$

where

$$K = 1 + 3\alpha_1 + 2\alpha_2 + 3\beta_1 + 2\beta_2,$$

$$L = 1 + 3\alpha_1 - 3\beta_1 + 3\gamma_1.$$

(31) is a tensor if and only if

$$LR = LS = 0,$$

so that either

$$L = 0 \quad \text{or} \quad R = S = 0. \tag{32}$$

Also (31) vanishes identically (requirement b) of the theorem) if either

$$K = 0 \quad \text{or} \quad Q = 0. \tag{33}$$

It is now a matter of elementary algebra to show that every combination of the conditions (30), (32) and (33) necessarily contradicts (26).

Hence the theorem is proved.

5. *Conclusion*

The theorem proved in the preceding section implies that it is impossible to find an expression for the affine connection $\Gamma^{\lambda}_{\mu\nu}$ in terms of a new variable $U^{\lambda}_{\mu\nu}$ to which the operation of hermitian conjugation could be meaningfully transferred. We regard here as meaningful only an invariant condition in an unrestricted space time. We have seen that "natural" variables $U^{\lambda}_{\mu\nu}$, that is these for which $R_{\mu\nu}$ is hermitian symmetric, exist in spaces in which g is a constant. It would be premature perhaps to regard these as necessary in a unified field theory although it might be alluring to do so. (Of course, all that we require is the existence of a coordinate system in which g is globally constant.) In a way, this would explain why electromagnetism is essentially a flat space theory.

On the other hand, it seems necessary to apply the requirement that $g_{\underline{\mu}\nu;\lambda}$ should be hermitian symmetric with respect to the transposition of $g_{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$ if we decide to use the latter. Otherwise, there could be no meaning given to the field equations derived in terms of them from a variational principle (Ref. [4]). $U^{\lambda}_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ are clearly not equivalent parameters. We cannot rest satisfied with having $R^{\lambda}_{\mu\nu}$ hermitian with respect to a U-transposition, and $g_{\mu\nu;\lambda}$ with respect to a transposition of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$.

There is yet another aspect of the negative theorem of section 4. It is well known that the equation

$$\Gamma_{\mu} = 0, \tag{11}$$

cannot be derived from the remaining field equations of the field theory. The best we can do is to obtain

$$\Gamma^{\varrho}_{\mu\check{\nu}}\Gamma_{\varrho} = 0,$$

or alternatively

$$g^{\underline{\mu\nu}}(\tilde{R}_{\mu\nu} - R_{\mu\nu}) = 0,$$

where $\tilde{R}_{\mu\nu}$ is the hermitian conjugate (with respect to $\Gamma^{\lambda}_{\mu\nu}$) of $R_{\mu\nu}$. There is, likewise, no straightforward way of obtaining (11) from a variational principle so that the compatibility of the field equations including (11) could be assured. If now we were able to prove that no algebraic expression for $\Gamma^{\lambda}_{\mu\nu}$ in terms of an $U^{\lambda}_{\mu\nu}$ exists which satisfies our theorem, the result could be interpreted as an argument that (11) can be adjoined to the full set of field equations. We have only shown that there is no such linear expression. This is a strong encouragement to assert the general conclusion but not its definite proof. The existence of a nonlinear substitution would enormously complicate the field equations and remove them even further from General Relativity and thereby from physics.

APPENDIX

A Direct Proof of $g^{\underline{\mu\nu}}\Gamma_\nu = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{\underset{\vee}{\mu\nu}})_{,\nu}$.

Permuting cyclically the indices μ, ν, λ in

$$g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0, \tag{2}$$

twice, we obtain easily

$$(g^{\varepsilon\sigma} g^{\beta\lambda} g_{\nu\alpha} + g^{\sigma\varepsilon} g^{\lambda\beta} g_{\alpha\nu})\, \Gamma^{\alpha}_{\sigma\lambda} = P^{\varepsilon\beta}_{\nu}, \tag{34}$$

where

$$P^{\varepsilon\beta}_{\underset{\vee}{\nu}} = g^{\varepsilon\sigma} g^{\beta\lambda} g_{\nu\lambda,\sigma} - g^{\varepsilon\sigma} g^{\lambda\beta} g_{\lambda\sigma,\nu} + g^{\sigma\varepsilon} g^{\lambda\beta} g_{\sigma\nu,\lambda}. \tag{35}$$

Forming $P^{\varepsilon\beta}_{\underset{\vee}{\nu}}$ and contracting over β and ν.

$$(g^{\varepsilon\sigma} + g^{\sigma\varepsilon})\, \Gamma_\sigma = P^{\varepsilon\beta}_{\beta}.$$

But, from (35) we find that

$$P^{\varepsilon\beta}_{\beta} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{\underset{\vee}{\varepsilon\sigma}})_{,\sigma}.$$

Hence

$$g^{\underline{\mu\nu}}\Gamma_\mu = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{\underset{\vee}{\mu\nu}})_{,\nu}.$$

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