IRROTATIONAL SHEARFREE MOTION OF PERFECT FLUID IN GENERAL RELATIVITY

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The conditions of isotropy of pressure in a perfect fluid undergoing irrotationa shearfree motion lead to differential equations, which may be used to obtain exact solutions for inhomogeneous distributions.

1. Introduction

The purpose of the present paper is to explore the possibilities of exact solutions corresponding to perfect fluid in general relativity under certain restrictions such as $\omega_i = \sigma_{ab} = 0$, but $\Theta \neq 0$ where ω_i, σ_{ab} and Θ stand for vorticity vector, shear tensor, and expansion scalar respectively of the fluid motion (Ehlers 1961). The line element is then of the form

$$ds^2 = e^v dt^2 - e^u h_{ij} dx^i dx^j \quad (i, j = 1, 2, 3), \tag{1}$$

where $\dot{h}_{ij} = 0$. Here dot represents differentiation with respect to time and v, u are functions of space as well as time co-ordinates. If the motion is also geodetic, we get the isotropic homogeneous cosmological solution.

We consider here that the motion is nongeodetic ($V^a{}_{;b}V^b \neq 0$, V^a being the four velocity) due to the existence of pressure-gradient forces and further h_{ij} is simply the 3-flat space. There are a few such inhomogeneous cosmological solutions in the literature (Shepley and Taub 1967, Thompson and Whitrow 1967, Faulkes 1969, Banerjee and Banerji 1976) with high symmetry. We do not make any a priori assumption regarding the symmetry and differential equations are obtained from the conditions of isotropy of

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the fluid pressure, which might be considered as a generalisation of the equation obtained by Faulkes for spheres of perfect fluid.

These equations may be used to obtain exact solutions, some of which are new.

2. *Field equations and the condition of isotropy of the fluid pressure*

We start with the line element (1)

$$ds^2 = e^v dt^2 - e^u h_{ij} dx^i dx^j,$$

with $h_{ij} = (-1, -1, -1)$. The energy momentum tensor for a perfect fluid in co-moving reference frame

$$T^1{}_1 = T^2{}_2 = T^3{}_3 = -p, \quad T^4{}_4 = \varrho, \tag{2}$$

$$T^i{}_j = T^i{}_4 = 0 \qquad i \neq j, \tag{3}$$

where p is the fluid pressure and ϱ is the matter density. Now if $G^a{}_b$ represents Einstein tensor, we obtain from field equations

$$G^1{}_1 = G^2{}_2 = G^3{}_3, \tag{4}$$

$$G^i{}_4 = 0, \tag{5}$$

$$G^i{}_j = 0 \quad i \neq j. \tag{6}$$

From (5) $2u_{i4} = u_4 v_i$ and so we can write

$$u_4 = C(t)e^{v/2}, \tag{7}$$

$C(t)$ being an arbitrary function of time. The subscripts indicate differentiation with respect to corresponding coordinates. Now, from (4) and (6)

$$E = 2(u_{ii} + v_{ii}) + (v_i{}^2 - u_i{}^2 - 2u_i v_i) \quad \text{for } i = 1, 2, 3 \tag{8}$$

and

$$0 = 2(u_{ij} + v_{ij}) + (v_i v_j - u_i u_j - u_i v_j - u_j v_i). \tag{9}$$

Equations (8) and (9) may be written, in view of (7) in the form

$$(U_{11}/U^2)_{,4} = (U_{22}/U^2)_{,4} = (U_{33}/U^2)_{,4} \tag{10}$$

and

$$(U_{ij}/U^2)_{,4} = 0 \quad \text{for } i \neq j, \tag{11}$$

where $U = e^{-u/2}$. Equations (10) and (11), in turn, may be expressed as

$$U_{22} - U_{33} = AU^2, \quad U_{33} - U_{11} = BU^2, \quad U_{11} - U_{22} = CU^2 \tag{12}$$

and

$$U_{23} = DU^2, \quad U_{31} = EU^2, \quad U_{12} = FU^2, \tag{13}$$

where A, B, C, D, E, F are functions of space coordinates and $A+B+C = 0$.

3. *Solutions*

Relations, such as $U_{,i} = F_i(x, y, z)\, U$, where F_i's are any three functions of space variables, are clearly inconsistent with (13). In view of this restriction, we may show that the vanishing of any one of the functions D, E, F automatically implies vanishing of the others along with A, B, C. We may, therefore, obtain two classes of solutions according as

Case I: $D = E = F = 0$. It follows that $A = B = C = 0$; we immediately obtain

$$U_{11} = U_{22} = U_{33} \quad \text{and} \quad U_{ij} = 0 \quad (i \neq j),$$

and the solution is given by

$$U = a(t)(x^2+y^2+z^2)+b_1(t)x+b_2(t)y+b_3(t)z+c_1(t). \tag{14}$$

The matter density in this case is a function of time alone which means the matter density is uniform although pressure p is a function of space as well as time co-ordinate. In this sense the matter distribution is inhomogeneous although the motion is isotropic. The solution (14) apparently cannot be reduced to a spherically symmetric one by co-ordinate transformation and is thus different from that obtained by Thomson and Whitrow and others for a sphere of uniform density where $b_1(t) = b_2(t) = b_3(t) = 0$.

Case II: $D, E, F \neq 0$. Calculating $U_{123}, U_{312}, U_{231}$ from (13) and equating them one can obtain

$$2DU_1-2EU_2 = (E_2-D_1)U, \quad 2EU_2-2FU_3 = (F_3-E_2)U. \tag{15}$$

Again differentiating (12) suitably with respect to x, y, z and using (13) one further gets

$$\begin{aligned} 2AU_1-2FU_2+2EU_3 &= -(A_1+E_3-F_2)U, \\ 2FU_1+2BU_2-2DU_3 &= -(F_1+B_2-C_3)U, \\ -2EU_1+2DU_2+2CU_3 &= (E_1-D_2-C_3)U. \end{aligned} \tag{16}$$

The right-hand side of the relations (15) and (16) must be zero, for otherwise $U_i = F_i(x, y, z)U$ which is again inconsistent with the relations (12) and (13). It follows from equations (15) that

$$DU_1 = EU_2 = FU_3 = K(x, y, z, t) \tag{17}$$

and

$$D_1 = E_2 = F_3. \tag{18}$$

Equations (17) and (18) may be used to show that

$$(E/F)_{,1} = (F/D)_{,2} = (D/E)_{,3} = 0. \tag{19}$$

Vanishing of the right-hand sides in (16), on the other hand, leads to the following relations

$$A_1+E_3-F_2 = 0, \quad F_1+B_2-D_3 = 0, \quad E_1-D_2-C_3 = 0, \tag{20}$$

and

$$A = D(F^2-E^2)/EF, \quad B = E(D^2-F^2)/FD, \quad C = F(E^2-D^2)/DE. \tag{21}$$

In view of (19) we may write

$$D = L(x)g(y)h(z)X(x, y, z), \quad E = f(x)M(y)h(z)X(x, y, z), \quad F = f(x)g(y)N(z)X(x, y, z). \tag{22}$$

Equation (18) may now be written as

$$(XL),_1/f = (XM),_2/g = (XN),_3/h. \tag{23}$$

Using equations (20), (21) and (23) it may be shown that $(f/L)_1 = (g/M)_2 = (h/N)_3 = 2a$ (a is an arbitrary constant). It therefore follows that

$$D = Xfgh/(2ax+b_1), \quad E = Xfgh/(2ay+b_2), \quad F = Xfgh/(2az+b_3). \tag{24}$$

Equations (17) and (24) may be used to show that

$$U_1 dx + U_2 dy + U_3 dz = \frac{K}{Xfgh}\, dq, \tag{25}$$

where

$$q = a(x^2+y^2+z^2)+b_1x+b_2y+b_3z+c, \tag{26}$$

a, b_1, b_2, b_3, c being arbitrary constants. We conclude, therefore, that $U = U(q, t)$ or in other words $u = u(q, t)$ and also in view of (7) $v = v(q, t)$. Considering U as a function of q and t, the equations (12) and (13) lead to a single equation

$$U_{qq} = Q(q)U^2, \tag{27}$$

where $Q(q) = Xfgh/(q_1 q_2 q_3)$ and $q_i = q,_i$. This differential equation is analogous to the corresponding equation obtained by Faulkes for a sphere of perfect fluid where the variable q is equal to the square of the radial co-ordinate. Equation (27) can now be solved following the technique given in a previous paper by Chakravarty et al. (1976). The variable q in (26) can have either spherically symmetric or plane symmetric form according as $a \neq 0$ or $a = 0$. In this context one may take note of a very simple homogeneous isotropic solution which is apparently plane symmetric when we put $Q(q) = 0$ and $q = b_1x+b_2y+b_3z$. Such a solution is

$$U = e^{-u/2} = \frac{1+kq}{R(t)}, \tag{28}$$

k being a constant. The matter density and fluid pressure are

$$8\pi\varrho = 3(-\dot{R}^2+b^2)/R^2 \quad \text{and} \quad 8\pi p = -(2R\ddot{R}+\dot{R}^2-b^2)/R^2, \tag{29}$$

where dot denotes differentiation with respect to time and $b^2 = k^2(b_1{}^2+b_2{}^2+b_3{}^2)$.

The solution given above may be interpreted as plane symmetric and is found to have maximally symmetric three dimensional subspace in the sense that it satisfies the relation (Eisenhart 1924)

$$P_{ijkl} = A(g_{ik}g_{jl}-g_{il}g_{jk}) \text{ with } A = b^2,$$

where P_{ijkl} stands for the curvature tensor for the three dimensional subspace with the metric g_{ij} and A is a constant. The solution is thus basically equivalent to open cosmological model of Robertson-Walker with the space curvature constant equal to $-b^2$. This may be justified by considering the plane symmetry in this case to be the limiting situation for spherical symmetry.

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