ANALYTIC REGULARIZATION IN AN ARBITRARY NUMBER OF DIMENSIONS*

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The procedure of Speer's analytic regularization is repeated for d-dimensional space-time (d is arbitrary). It is shown that the generalized propagators $(m^2-p^2-i0)^{-\lambda}$ (λ is an arbitrary complex number) as well as products of such expressions can be analytically continued to all values of $\lambda-d/2$. For nonrational values of d the location of poles in the generalized amplitudes is shifted away from the physical value $\lambda = 1$. The magnitude of this dislocation remains finite for the perturbation order tending to infinity.

1. Introduction

In writing this paper we are motivated by a recent idea of Symanzik concerning the possibility of presenting certain renormalized systematic expansion for Green's functions of massless scalar field theory with self-interaction of the φ^4 type considered in a space-time of dimension greater than four (specifically $\varphi^3_{4+\varepsilon}$, where $0 < \varepsilon < 3$) [1,2]. Symanzik's programme has also been tested on the examples of massless $\varphi^6_{3+\varepsilon}(0 < \varepsilon < 2)$ and massless spinor $(\bar{\psi}\psi)^2_{2+\varepsilon}(0 < \varepsilon < 3)$ models [3]. In our opinion it is interesting to analyse even such obviously unphysical models like the φ^4_6 or the φ^6_4 as they exhibit the intriguing mathematical features of nonrenormalizable interactions.

Symanzik's idea is based on the hope that one could perhaps define the renormalized functions in the models like these quoted above as limits of traditionally renormalized functions of corresponding theories with an appropriate ultra-violet cutoff when the cutoff is removed. To this end Symanzik has proposed to use the cutoff of Pais and Uhlenbeck [4] in models which, without this cutoff, are renormalizable in a certain number of dimensions — Symanzik has proposed first to renormalize them in a space-time of the dimensionality which is of interest for us, i. e. in that of a dimension higher than this number. The dimensions we are interested in cannot of course be too high as the "improved" renormalizability (we mean the renormalizability following by power counting with the presence of the cutoff taken into account) should still take place.

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Let us stress that for essential technical reasons on all intermediate stages of his procedure Symanzik was forced to keep the dimension of the space-time generic (i. e. the ε nonrational). For example, this author does not even pass to physically interesting dimensions (like $\varepsilon = 1$ for the $\varphi^6_{3+\varepsilon}$ model) before the removal of the cutoff whereby, within the discussed framework, such dimensions can be only treated as special cases of ε rational. One can of course expect that the removal of the cutoff, which drives us back to the nonrenormalizable case, will be impossible unless some additional conditions are satisfied. Symanzik has succeeded to formulate these conditions in terms of existence conditions for a finite number of improper one-dimensional integrals. Unfortunately, the fact that it was actually possible constitutes the only virtue of the whole method thus far. First of all any further progress is tamed by the fact that within present techniques one does not know how to compute the integrals which were mentioned above. Moreover, one can also raise some doubts concerning the consistency of the whole Symanzik's method, at least when applied to massless models [5].

For our limited purposes it is important to stress once again that at the stage of the perturbative renormalization both the cutoff and the ε of assumed nonrational values are treated as if they are some realistic parameters which, in particular, have nothing to do with the renormalization procedure. Let us also recall that the cutoff of the Pais–Uhlenbeck type must be introduced together with a dimensionfull parameter and this feature excludes any direct application of Symanzik's method (in which simple dimensional analysis is used throughout) to massive models. On the other hand, it is just the masslessness which seems to lead to inconsistencies in higher orders of the Symanzik's approach [3, 5].

In view of what has been said above it is tempting to investigate at the beginning rather massive models of nonrenormalizable interactions by first introducing two kinds of regulators as in the Symanzik method, but to choose both of them to be dimensionless. We hope that then it will also be possible to retain Symanzik's idea of manipulating two regulators, i. e. to initially remove singularities of Green's functions in one of them (by an appropriate renormalization procedure, e. g. the BPHZ method) and to investigate whether it will be possible to make the second regulator tend to its physical value. For the first regulator we can choose the generic space-time dimension [6, 7], exactly as in the Symanzik's approach. For the second one we are going to choose the parameters of the analytic regularization [8, 9]. The idea of analytic regularization itself has been successfully applied to a renormalization of the S-matrix following from a renormalizable local lagrangian field theory (the analytic renormalization) [10, 11].

In the present paper we are not dealing with any renormalization programme yet, and we shall limit ourselves to the analysis of the dependence of Feynman amplitudes on the number of space-time dimensions and on the parameters of the analytic regularization. The results of this rather technical paper are based on the observation that it is very natural to continue in the number of dimensions the Gaussian integral

$$\int_{\mathscr{R}^d} dp \exp\left[i(pRp+xp)\right] = \frac{\pi^{d/2}}{\left[\det(-iR)\right]^{1/2}} \exp\left[-(i/4)xR^{-1}x\right], \tag{1.1}$$

where

$$R = A \otimes Q,\ Q = \begin{bmatrix} 1+i\eta & & & & & \\ & -1+i\eta & & & & \\ & & -1+i\eta & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix},\ \eta \geqslant 0,$$

$\operatorname{Im} \eta = 0$, $\dim Q = d$ and A is a real matrix related to the structure of Feynman graphs. Let us note that the nontrivial dependence of the r. h. s. of (1.1) on d is hidden in the determinant of the direct product. Namely, we have

$$\det A \otimes Q = (\det A)^{\dim Q} (\det Q)^{\dim A} = (\det A)^{d} (\det Q)^{\dim A} \tag{1.2}$$

and the substitution of (1.2) into (1.1) gives

$$\int d^d p \exp\left[i(pRp + xp)\right] = \pi^{d/2}(-i)^{-(d/2)\dim A}(\det A)^{-d/2}$$
$$\times (\det Q)^{-(\dim A)/2} \exp\left[-(i/4)x(A\otimes Q)^{-1}x\right]. \tag{1.3}$$

Now it is sufficient to note that in (1.3) the "metric tensor" Q never stands alone but only in the forms $\det Q$ and $xQ^{-1}x$ so that actually we shall never need to describe what we mean by Q (or by $\lim_{\eta \to 0} Q$) for noninteger d. On the other hand, the formula

$$\lim_{\eta \to 0} \det Q = (-1)^{d-1}, \tag{1.4}$$

which we shall need at the end of our procedure makes perfect sense also for noninteger values of d. A similar argument applies to the quadratic form $\lim_{\eta \to 0} xQ^{-1}x = x^2 + i0$.

In Section 2 we give an explicit form of the generalized propagator (which is the ordinary propagator to an arbitrary power) in d-dimensional configuration space. In Section 3 we derive an expression for the generalized Feynman amplitude (which is the product of generalized propagators over lines of a Feynman graph) in d-dimensional momentum space. In Section 4 we apply techniques of Speer in order to investigate, in Section 5, the structure of singularities of the generalized Feynman amplitude. In this context we shall also mention an interesting case of the perturbation order tending to infinity.

2. *The generalized propagator*

Let us begin with quoting some important properties of the generalized propagator (for scalar field)

$$\tilde{\Delta}_{F,\eta}(\lambda, p) \doteqdot -i(m^2 - i\eta - pQp)^{-\lambda} \tag{2.1}$$

in the usual integral dimensional space-time. In (2.1) λ is the parameter of the analytic regularization ($\lambda \in \mathscr{C}$) $m^2 > 0$, $\eta \geqslant 0$ (real) and $Q_{\mu\nu} = g_{\mu\nu} + i\eta\delta_{\mu\nu}$, $\mu, \nu = 0, 1, \ldots, d-1$

(at this point we understand that d is an integer number). The Q is defined so that

$$\lim_{\eta \to 0} \det Q = (-1)^{d-1}, \tag{2.2}$$

in agreement with (1.4). Note that η enters in (2.1) twice, in particular also in a manifestly nonrelativistic way and until now it is not clear whether this feature really does not affect the apparently relativistic character of various quantities which one obtains in a theory based on (2.1) in the $\eta \to 0$ limit. Let us recall that another nonrelativistic procedure has been successfully (i. e. with proofs of relativistic covariance of final results) applied by Zimmermann for an analysis of absolute covergence of Feynman amplitudes in four dimensions [12]. In Appendix A we shall merely stress a point where the nonrelativistic regularization as introduced above is needed and in this paper we shall not comment on this serious problem anymore.

The property of (2.1) of far reaching consequences is the following: for integer d (2.1) can be continued as a tempered distribution to all values of λ and this continuation is holomorfic in λ, continous in Q and continous in $m^2 - i\eta$ ($m^2 > 0$) [10, 11]. The physical propagator (in zero order of perturbation theory) corresponds to $\lambda = 1$, $\eta = 0$ and integer d, e. g. $d = 4$. Models of nonrenormalizable interactions can be realized i. e. as $\varphi^6_{d=4}$ or (obviously unphysical) $\varphi^4_{d=6}$. However, apart from the type of the interaction, it is well known that products of such physical propagators, and such products are needed for construction of perturbation expansions, are meaningless.

In order to investigate the multiplication in our general case it is advantageous to transform (2.1) to (d-dimensional) configuration space:

$$\Delta_{F,\eta}(\lambda, x) = \frac{1}{(2\pi)^{d/2}} \int d^d p e^{ipx} \tilde{\Delta}_{F,\eta}(\lambda, p). \tag{2.3}$$

In Appendix A we show that

$$\Delta_{F,\eta}(\lambda, x) = \frac{(-i)^{1-d/2} e^{\lambda\pi i/2}}{2^{d/2}\Gamma(\lambda)\sqrt{\det Q}} \int_0^\infty d\alpha \alpha^{\lambda-d/2-1} \exp\{-\alpha[i(m^2 - i\eta) - (1/4\alpha) i x Q^{-1} x\}, \tag{2.4}$$

or, after the integration over Feynman parameter α

$$\Delta_{F,\eta}(\lambda, x) = \frac{(-i)^{1-d} 2^{1-\lambda}}{\Gamma(\lambda)\sqrt{\det Q}} (\sqrt{m^2 - i\eta})^{d/2-\lambda}$$

$$\times (\sqrt{-xQ^{-1}x})^{\lambda-d/2} K_{d/2-\lambda}[(\sqrt{m^2 - i\eta})(\sqrt{-xQ^{-1}x})], \tag{2.5}$$

where $K_\nu(z)$ is the generalized Hankel function [13]. From the above mentioned continuity of (2.1) and from (2.2) we get

$$\Delta_F(\lambda, x) = \lim_{\eta\to 0} \Delta_{F,\eta}(\lambda, x) = \frac{2^{1-\lambda}}{\Gamma(\lambda)} m^{d/2-\lambda} (\sqrt{-x^2 + i0})^{\lambda - d/2} K_{d/2-\lambda}(m\sqrt{-x^2 + i0}). \tag{2.6}$$

In spite of the fact that derivation of (2.5) was based on the assumption that Re $\lambda > d/2$, the results (2.6) and (2.5) can be already viewed as analytic continuations of the generalized propagator to all values of $d/2-\lambda$. In particular, the $K_{d/2-\lambda}$ are meaningful for all (also complex) values of the parameter d. One immediately restores the familiar fact that for $\lambda = 1$ and $d = 4$ the propagator is proportional to the function $K_1(mx)$. The form with $K_{-1}(mx)$ which can also be met in the literature follows by noting that for noninteger ν we have $K_\nu(z) = K_{-\nu}(z)$ (cf. (A.4)) and from the fact that $K_\nu(z)$ are analytic in ν.

The formula (2.6) also says that light cone singularities are of the same type for the physical $\lambda = 1$, $d = 4$ case as for $\lambda = 0$, $d = 2$ (constant propagator in two dimensions) as well as for $\lambda = 2$, $d = 6$ (Pais–Uhlenbeck propagator in six dimensions). From (2.6) the reader can easily convince himself about the analogies between a usual theory in two dimensions ($\lambda = 1$, $d = 2$) and the theory of Pais–Uhlenbeck type in four dimensions ($\lambda = 2$, $d = 4$). The linear growth of potentials in the last two cases might explain the quark confinement and this hope has nowadays motivated some authors to revisit the old paper of Pais and Uhlenbeck (Ref. [4]) [14–16].

In particular, we infer from (2.5) and (2.6) that $\Delta_{F,\eta}(\lambda, x)$ and $\Delta_F(\lambda, x)$ are m-times continously differentiable functions of x only for Re $\lambda >$ Re $d/2+m$. This property is of course not connected with the specific order of $K_\nu(z)$ (we mean $\nu = d/2-\lambda$ or $\nu = \lambda-d/2$) and it is only the presence of the factor $(\sqrt{-x^2+i0})^{\lambda-d/2}$ in (2.6) which is of importance in this respect. This can be directly seen from the substitution of power series expansion

$$K_\nu(z) = z^{-\nu} \sum_{k=0}^{\infty} A_k(\nu)z^{2k} + z^{\nu} \sum_{k=0}^{\infty} B_k(\nu)z^{2k} \tag{2.7}$$

into (2.6). Namely, it gives

$$\Delta_F(\lambda, x) = \frac{2^{1-\lambda}}{\Gamma(\lambda)}\left[(-x^2)^{\lambda-d/2} \sum_{k=0}^{\infty} A_k(\nu)\,(m^2x^2)^k + (m^2)^{d/2-\lambda} \sum_{k=0}^{\infty} B_k(\nu)\,(m^2x^2)^k\right. . \tag{2.8}$$

3. *The generalized Feynman amplitude*

In Section 2 we have shown that in d dimensions the generalized Feynman amplitude

$$T_\eta(\underline{\lambda}, \underline{x}) \doteqdot \prod_{l=1}^{L} \Delta_{F,\eta}(\lambda_l, x_k e_{kl}) \tag{3.1}$$

for a graph G with $L = L(G)$ lines and $V = V(G)$ vertices is well defined for $\lambda_l > d/2$ (real parts) for each $l = 1, \ldots, L$. For the sake of generality we are following Speer and we are introducing different regulators for different lines of the graph (this is mandatory for the analytic renormalization [10, 11]). In other words, $\underline{\lambda} \doteqdot (\lambda_1, \ldots, \lambda_L) \in \mathscr{C}^L$. The e_{kl} in (3.1) is the incidence matrix for the graph:

$$e_{kl} \doteqdot \begin{cases} 1 \text{ if } k = f_l, \\ -1 \text{ if } k = i_l, \\ 0 \text{ otherwise,} \end{cases} \tag{3.2}$$

($k = 1, \ldots, V$), so that $x_k e_{kl} = x_{f_l} - x_{i_l}$, where $x_{f_l}(x_{i_l})$ denotes final (initial) point of the l-th line.

We are of course interested also in other values of $\lambda_l - d/2$ and in order to define (3.1) in such a general case it is advantageous to transform (3.1) to p-space. To this end we start from $\Delta_{\mathrm{F},\eta}(\lambda_l, x_k e_{kl})$ in the form (2.4). The (3.1) with (2.4) gives

$$T_\eta(\underline{\lambda}, \underline{x}) = \prod_{l=1}^{L} \left(\frac{(-i)^{1-d/2} e^{\lambda_l \pi i/2}}{2^{d/2} \Gamma(\lambda_l) \sqrt{\det Q}} \right) \int_0^\infty \cdots \int_0^\infty \prod d\alpha_l \alpha_l^{\lambda_l - d/2 - 1}$$

$$\times \exp \{ i[-(1/4) x_i A_{ij} Q^{-1} x_j - (m^2 - i\eta) \sum_l \alpha_l] \}, \tag{3.3}$$

where

$$A_{ij} = \sum_{l=1}^{L} \frac{e_{il}}{\alpha_l} (e^{\mathrm{T}})_{lj}. \tag{3.4}$$

It is easy to note that $T_\eta(\underline{\lambda}, \underline{x})$ actually depends only on $V-1$ points $\xi_i = x_i - x_k$, $i \neq k$ so that the exponential in (3.3) can be written as

$$\exp \{ i[-(1/4) \xi_i A'_{ij} Q^{-1} \xi_j] \},$$

where A' is obtained from A by deleting k-th row and column ($\dim A' = V-1$). For the Fourier transform we get

$$\tilde{T}_\eta(\underline{\lambda}, \underline{p}) = \frac{1}{(2\pi)^{(d/2)V}} \underbrace{\int \cdots \int}_{V} \prod_{i=1}^{V-1} d^d \xi_i d^d x_k T(\underline{\lambda}, \underline{\xi}) \exp \left\{ -i \left(\sum_{i \neq k} \xi_i p_i + x_k \sum_{i=1}^{V} p_i \right) \right\}$$

$$= \prod_{i=1}^{L} \left(\frac{(-i)^{1-d/2} e^{\lambda_l \pi i/2}}{2^{d/2} \Gamma(\lambda_l) \sqrt{\det Q}} \right) (2\pi)^{d/2} \delta \left(\sum p_i \right) \frac{1}{(2\pi)^{d/2(V-1)}} \int_0^\infty \cdots \int_0^\infty \prod$$

$$\times\, d\alpha_l \alpha_l^{\lambda_l - d/2 - 1} \underbrace{\int \cdots \int}_{V(G)-1} \prod d\xi_i \exp \{ i[-(1/4) \xi_i A'_{ij} Q^{-1} \xi_j - \xi_i p_i - (m^2 - i\eta) \sum \alpha_l] \}, \tag{3.5}$$

where

$$\delta(\sum p_i) = (2\pi)^{-d} \int d^d x \exp(-ix \sum p_i)$$

plays the role of Dirac's delta function in d-dimensions. Note that in (3.5) we have already changed the order of $\underline{\alpha}$ and $\underline{\zeta}$ integrations what is legitimate for $\lambda_l > d/2$ (real parts). In Appendix B we show that for the ζ integral in (3.5) ($\underline{\zeta}$ has $d(V-1)$ components) one obtains

$$\int d^d\xi \exp\{i[-(1/4)\xi A'\otimes Q^{-1}\xi-\xi p]\}$$

$$= \pi^{(d/2)(V-1)}(i/4)^{-(d/2)(V-1)}(\sqrt{\det Q})^{V-1}\left[A\begin{pmatrix}k\\k\end{pmatrix}\right]^{-d/2}\exp\left\{ip_i\frac{A\begin{pmatrix}k & i\\ k & j\end{pmatrix}}{A\begin{pmatrix}k\\k\end{pmatrix}}Qp_j\right\}, \quad (3.6)$$

where A (: :) denote minors of the A matrix.

For the analysis below it will be convenient to write our result in terms of structure functions for the graph:

$$d(\alpha) \doteq \left(\prod_{l=1}^{L}\alpha_l\right)A\begin{pmatrix}k\\k\end{pmatrix}, \quad (3.7)$$

$$D^k_{ij}(\alpha) \doteq \left(\prod_{l=1}^{L}\alpha_l\right)A\begin{pmatrix}k & i\\ k & j\end{pmatrix}. \quad (3.8)$$

From (3.4) it is seen that the $d(\alpha)\,(D(\alpha))$ is homogenous function of degree $L-V+1$ $(L-V+2)$. In this notation our result reads

$$\tilde{T}_\eta(\underline{\lambda}, \underline{p}) = \prod_{l=1}^{L}\left(\frac{(-i)^{1-d/2}e^{\lambda_l\pi i/2}}{2^{d/2}\Gamma(\lambda_l)\sqrt{\det Q}}\right)(2\pi)^{d/2}\delta\left(\sum p_i\right)4^{(d/4)(V-1)}(\sqrt{\det Q})^{V-1}$$

$$\times\int_0^\infty\cdots\int_0^\infty\prod d\alpha_l\alpha_l^{\lambda_l-1}[d(\alpha)]^{-d/2}\exp\left\{i\left[\sum_{i,j\neq k}p_i\frac{D^k_{ij}(\alpha)}{d(\alpha)}Qp_j-(m^2-i\eta)\sum\alpha_l\right]\right\}. \quad (3.9)$$

This formula is of course meaningful also for complex values of d. In particular, for $d = 4$ we get the familiar $d^{-2}(\alpha)$ factor under the $\underline{\alpha}$ integral.

4. *The singularities of the generalized Feynman amplitude*

Our basic observation that the generalized amplitude (3.1) can be meaningless for $\lambda_l < d/2$ (real parts) is reflected in possible divergencies of the α_l integrals in (3.9) at $\alpha_l = 0$. This can happen so because the function $d(\alpha)$ (defined in (3.7)) can vanish when some α_l's are zero. Speer has analysed the zeros of $d(\alpha)$ for arbitrary graphs (including those with overlapping loops) in four dimensions and has shown that the generalized amplitude can be continued to a meromorphic function of $\underline{\lambda}$ [10, 11]. From (3.9) we infer that this property actually does not depend on the number of dimensions and the result with the power of $d(\alpha)$ equal $-d/2$ which replaces well known power equal -2 for $d = 4$ case which has been investigated thus far can merely affect the location of poles (in α_l's) in the continued generalized amplitude. In this Section we shall find the position of poles in our general case. However, before doing so, we have also decided to repeat here some most important steps in Speer's construction of the analytic continuation because otherwise even the notation in which we are going to write our final results (cf. eg. (4.12) below) might become cumbersome without directly consulting the original papers of Speer (Refs. [10] and [11]). The first part of this construction serves for isolation of zeros of the structure function $d(\alpha)$ into powers of certain new variable. In the second part a pole part and an analytic part of these powers are extracted.

Let us recall that in Refs. [10] and [11] Speer introduces first families E of nonoverlapping subgraphs H of G, each H 2-connected or consisting of a single line, such that no union of different H's is 2-connected[1]. Assuming that the G itself is 2-connected we infer that G also belongs to E. The next step is to define for each E a region $D(E)$ in $\mathscr{C}^L$ where order of α_l's is specified. This buys us the possibility of replacing the multiple $\underline{\alpha}$ integral in (3.9) by a sum over E's of integrals over $D(E)$. Further, in each $D(E)$ one introduces scaling variables t_H $(H \in E)$ by setting

$$\alpha_l = \prod_{H;l\in H} t_H, \tag{4.1}$$

where $t_G \in \langle 0, \infty \rangle$ and $t_H \in \langle 0, 1 \rangle$ for $H \neq G$. Let us quote that under this change of variables

$$\left(\frac{\partial \underline{\alpha}}{\partial \underline{t}}\right) = \prod_H t_H^{L(H)-1}, \tag{4.2}$$

$$d(\alpha) = \prod_H t_H^{\mathscr{L}(H)} E(\underline{t}), \tag{4.3}$$

$$D_{ij}^k(\alpha) = t_G \prod_H t_H^{\mathscr{L}(H)} F_{ij}^k(\underline{t}), \tag{4.4}$$

where $\mathscr{L}(H)$ denotes number of loops in H $(\mathscr{L}(H) = L(H) - V(H) + 1)$, the function $E(\underline{t})$ does not depend on t_G and, most important, $E(\underline{t}) \neq 0$ for $t_H \geqslant 0$ (ditto the function $F_{ij}^k(\underline{t})$). For more details and, in particular, for proofs of (4.2)–(4.4) the Refs. [10] and [11] should be consulted. It is the formula (4.3) which provides us with the anticipated isolation of zeros of the structure function $d(\alpha)$. Note also that from (4.1) we can write

$$\prod_l \alpha_l^{\lambda_l - 1} = \prod_H t_H^{\Lambda(H)}, \tag{4.5}$$

where

$$\Lambda(H) = \sum_{l;l\in H} (\lambda_l - 1). \tag{4.6}$$

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[1] The graph is called 2-connected if it cannot be made disconnected by removing any vertex. From the point of view of renormalization it is satisfactory to consider only 2-connected graphs because amplitudes for graphs which are not 2-connected are just products of their 2-connected pieces.

After the substitution of (4.2)–(4.5) the expression (3.9) for the generalized amplitude in d dimensions can be written as

$$\tilde{T}_\eta(\underline{\lambda}, \underline{p}) \sim \sum_F \int_0^\infty dt_G \int_0^1 \cdots \int_0^1 \prod_{H \neq G} dt_H \prod_{H;H \in F} t_H^{A(H)-(1/2)\mu(H,d)-1} [E(\underline{t})]^{-d/2}$$

$$\exp\left\{it_G\left[p_i \frac{F_{ij}^k(\underline{t})}{E(\underline{t})} Q p_j - (m^2 - i\eta) \sum_l \beta_l\right]\right\}, \tag{4.7}$$

where

$$\mu(H, d) = d\mathscr{L}(H) - 2L(H) \tag{4.8}$$

are superficial degrees of divergence of subgraphs H in d dimensions and $B_l = t_G^{-1}\alpha_l$. As expected, for the "physical" point $\lambda_l = 1$ for all λ_l's the t_H integrals in (4.7) are convergent only for these H's for which $\mu(H, d) < 0$. For arbitrary λ_l's the t_H integrals in (4.7) are of course convergent only for $\mathrm{Re}\,[A(H) - (1/2)\,\mu(H, d)] > 0$.

The $t_H(H \neq G)$ integrals can be easily continued to regions $\mathrm{Re}[A(H) - (1/2\,\mu(H, d)] > -(k+1)$, $k \geqslant 0$, by writing

$$\int_0^1 dt_H t_H^{A(H)-1/2\mu(H,d)-1} f(\underline{t})$$

$$= \sum_{l=0}^k \frac{(\partial^l/\partial t_H^l) f(\underline{t})}{A(H) - (1/2)\mu(H, d) + l}\Bigg|_{t_H=0} + \int_0^1 dt_H t_H^{A(H)-(1/2)\mu(H,d)-1} f_k(\underline{t}), \tag{4.9}$$

where the second term on the r. h. s. of (4.9) is already analytic in the above regions ($f_k(t_H) = f(t_H) - \sum_{l=0}^{k} (1/l!)\,(\partial^l/\partial t_H^l) f(t_H)|_{t_H=0} \sim t_H^{k+1}$ at $t_H \to 0$). It is in fact possible to proceed so as in (4.9) because our

$$f(t) = [E(\underline{t})]^{-d/2} \exp\left\{it_G\left[p_i \frac{F(\underline{t})}{E(\underline{t})} Q p_j - (m^2 - i\eta) \sum_l \beta_l\right]\right\} \tag{4.10}$$

is infinitely differentiable function of t_H's. For the brevity of notation we shall below keep on writing the $\int_0^1 dt_H \ldots$ integrals so as these stand in (4.7) but we shall understand that in such places the analytic continuations given by (4.9) are already substituted.

Now we are left only with the $\int_0^\infty dt_G \ldots$ integral in (4.7). This integral can be performed with the help of formula analogous to (A.1). This formula can be still applied as we are always keeping $\eta > 0$. In such a way we finally get

$$\tilde{T}_\eta(\underline{\lambda}, \underline{p}) = \prod_{l=1}^{L} \left(\frac{(-i)^{1-d/2}}{2^{d/2}\Gamma(\lambda_l)\sqrt{\det Q}} \right) \exp\{(\pi i/2)[L(G)+(1/2)\mu(G,d)]\}$$

$$\times 4^{(d/4)(V-1)}(\sqrt{\det Q})^{V-1}\Gamma[A(G)-(1/2)\mu(G,d)](2\pi)^{d/2}\delta(\sum p_i)\sum_{E}\int_0^1 \cdots \int_0^1 \prod_{H\neq G}$$

$$\times dt_H t_H^{A(H)-1/2\mu(H,d)-1}[E(\underline{t})]^{-d/2}\left[(m^2-i\eta)\sum_l \beta_l - \sum_{i,j\neq k} p_i \frac{F_{ij}^k(\underline{t})}{F(\underline{t})} Q p_j\right]^{\mu(G,d)-A(G)}. \quad (4.11)$$

It is first on the level of this result when we may let $\eta \to 0$. In accordance with continuity of the generalized propagators (2.1) in Q and in $m^2 - i\eta$ and from (2.2) we can write

$$\tilde{T}(\underline{\lambda}, \underline{p}) = \prod_{l=1}^{L} \left(\frac{(-i)^{2-(3/2)d}}{2^{d/2}\Gamma(\lambda_l)} \right) \exp\{(\pi i/2)[L(G)-(1/2)\mu(G,d)]\}(4^{d/4}i^{1-d})^{V-1}$$

$$\times (2\pi)^{d/2}\delta(\sum p_i)\Gamma[A(G)-(1/2)\mu(G,d)]\sum_E \int_0^1 \cdots \int_0^1 \prod_{H\neq G} dt_H t_H^{A(H)-(1/2)\mu(H,d)-1}[E(\underline{t})]^{-d/2}$$

$$\times \left[(m^2-i\eta)\sum \beta_l - \sum_{i,j\neq k} p_i \frac{F_{ij}^k(\underline{t})}{E(\underline{t})} p_j\right]^{\mu(G,d)-A(G)}. \quad (4.12)$$

Let us recall that for the $\int_0^1 dt_H \ldots$ integrals in (4.12) the expressions (4.9) should be substituted.

5. *The discussion of the singularities*

From (4.9) we conclude that (4.12) constitutes the desired continuation of the generalized Feynman amplitude to a function with simple poles on the varieties in $\mathscr{C}^{L(H)}$ which are sets of points satisfying

$$A(H)-(1/2\,\mu(H,d) = 0, -1, -2, \ldots, \quad (5.1)$$

where $A(H)$ and $\mu(H,d)$ are given by (4.6) and (4.8), respectively. For $\lambda_l > d/2$ (real parts, all l) the amplitude is of course singularity free, as it can be now seen from

$$A(H)-(1/2)\mu(H,d) = \sum_{l=1}^{L(H)} (\lambda_l - d/2) + (d/2)[V(H)-1] \quad (5.2)$$

(from (4.6) and (4.8)).

The net result (4.12) can be also viewed as analytic continuation of the generalized amplitude (at fixed values of λ_l's) to all (also complex) values of the parameter d. In particular, for

$$d = 4+\varepsilon \tag{5.3}$$

the poles will be located for values of ε satisfying

$$\Lambda(H)-(1/2)\mu(H, d) = (\varepsilon/2)\mathscr{L}(H)-n, \tag{5.4}$$

where $n = 0, 1, 2, \ldots$ Note that for arbitrary values of λ_l's these are not necessarily rational values of ε. However, at the "physical" point $\underline{\lambda} = 1$ and for nonrational ε the amplitude is not singular, as expected. At this point the singularities take place for values of ε satisfying $(1/2)\mu(H, 4)+(\varepsilon/2)\mathscr{L}(H) = n$ $(n = 0, 1, 2, \ldots)$, i.e. for

$$\varepsilon = \frac{2n-\mu(H, 4)}{\mathscr{L}(H)}, \tag{5.5}$$

in agreement with well known results in dimensional regularization [6], [7].

Let us fix the ε again and let us write (5.4) in the form

$$\sum_{l=1}^{L(H)} \{\lambda_l-1-[\varepsilon/2L(H)]\mathscr{L}(H)-(1/2)\mu(H, 4)/L(H)\} = -n. \tag{5.4'}$$

Herefrom it is seen that in the $\mathscr{C}^{L(H)}$ spaces singularities take place for

$$\lambda_l = 1+\frac{\varepsilon\mathscr{L}(H)}{2L(H)}+\frac{(1/2)\mu(H, 4)-n}{L(H)}. \tag{5.6}$$

Hence, for $\varepsilon \neq [2n-\mu(H, 4)]/\mathscr{L}(H)$ (i.e. ε nonrational) the point $\underline{\lambda} = 1$ never happens to be a singular point in the generalized Feynman amplitude.

This variety in $\mathscr{C}^{L(H)}$ which is characterized by $n = 0$ corresponds to the set of points $\lambda_l = 1+[\varepsilon\mathscr{L}(H)-\mu(H, 4)]/2L(H)$. It is interesting to note how does the magnitude of this dislocation of the singularity from $\lambda_l = 1$ behave for perturbation orders tending to infinity. To see this let us take as an example a scalar $\varphi^k_{4+\varepsilon}$ (k even) theory. The "topological" relation for graphs in such a model reads

$$2L(H) = kV(H)-2E(H), \tag{5.7}$$

where $L(H)$ denotes, as before, the number of internal lines of the (sub)graph H, $V(H)$ is the number of its k-legged vertices and $E(H)$ is the number of its external lines. Upon the use of (5.7) we get

$$\lim_{V(H)\to\infty} \frac{\varepsilon\mathscr{L}(H)-\mu(H, 4)}{2L(H)} = \varepsilon\,\frac{k/2-1}{k}, \tag{5.8}$$

what means that in this limit the above dislocation is finite and depends only on the assumed dynamics.

APPENDIX A

In order to perform the Fourier transform (2.3) we represent the generalized propagator (2.1) as integral over the Feynman parameter:

$$\tilde{\Delta}_{F,\eta}(\lambda, p) = \frac{-ie^{\lambda\pi i/2}}{\Gamma(\lambda)} \int_0^\infty d\alpha\alpha^{\lambda-1} \exp\left[i\alpha(pQp - m^2 + i\eta)\right], \tag{A.1}$$

This formula is valid for $\mathrm{Im}\,(pQp-m^2+i\eta) > 0$ what is just our case as we are keeping $\eta > 0$, real. As the next step we write

$$\Delta_{F,\eta}(\lambda, x) = \frac{-ie^{\lambda\pi i/2}}{\Gamma(\lambda)} \int_0^\infty d\alpha\alpha^{\lambda-1} \int d^d p \exp\left\{i[p\alpha Qp + px - \alpha(m^2+i\eta)]\right\}. \tag{A.2}$$

The change of order of the integrations which has been done in (A.2) is legitimate only if

$$\int d^d p \int_0^\infty d\alpha\alpha^{\lambda-1} |\exp\left\{i\alpha[pQp + px/\alpha - m^2 + i\eta]\right\}|$$

$$= \int d^d p \int_0^\infty d\alpha\alpha^{\lambda-1} \exp\left[-\alpha\eta(p_\mu\delta_{\mu\nu}p_\nu + 1)\right] = \Gamma(\lambda)\eta^{-\lambda} \int d^d p (p_\mu\delta_{\mu\nu}p_\nu + 1)^{-\lambda} < \infty,$$

i.e. only for $\mathrm{Re}\,\lambda > d/2$. Note that without the nonrelativistic term $\eta p_\mu \delta_{\mu\nu} p_\nu$ the above estimate would be impossible. When keeping λ in this region we can perform first the d-dimensional integral. For doing this we make use of the formula (1.3) with the substitution $R = \alpha Q$. This gives

$$\Delta_{F,\eta}(\lambda, x) = \frac{(-i)^{1-d/2} e^{\lambda\pi i/2}}{2^{d/2}\Gamma(\lambda)\sqrt{\det Q}} \int_0^\infty d\alpha\alpha^{\lambda-d/2-1} \exp\left\{-\alpha[i(m^2 - i\eta)] - (1/4\alpha) ixQ^{-1}x\right\}$$

$$= \frac{(-i)^{1-d}}{2^{d/2}\Gamma(\lambda)\sqrt{\det Q}} (m^2 - i\eta)^{d/2-\lambda} \int_0^\infty dt t^{-(d/2-\lambda)-1} \exp\left\{-t - (1/4t)(m^2 - i\eta)(-xQ^{-1}x)\right\}. \tag{A.3}$$

The form in (A.3) directly leads to the generalized Hankel functions in (2.5) [13]. Let us also quote here, also for terminology reasons, that the relation between the generalized Hankel functions $K_\nu(z)$ and Bessel functions of imaginary argument $I_\nu(z)$ for noninteger ν reads

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}. \tag{A.4}$$

APPENDIX B

For the evaluation of the

$$\int d^d\xi \exp\{i[-(1/4)\xi A'\otimes Q^{-1}\xi - \xi p]\}$$

integral which has appeared in (3.5) we apply the formula (1.1) with $R = -(1/4)A' \otimes Q^{-1}$, where dim $A' = V-1$. For the determinant we get

$$\det(-iR) = \{\det[(i/4)A']\}^d(\det Q^{-1})^{-V+1} = (i/4)^{d(V-1)}(\det A')^d(\det Q)^{V-1} \quad \text{(B.1)}$$

and for the inverse

$$R_{ij}^{-1} = -4(A'^{-1})_{ij}Q \quad \text{(B.2)}$$

The substitution of (B.1) and (B.2) into (1.1) gives that our integral

$$\int d^d\xi \exp\{i[-(1/4)\xi A'\otimes Q^{-1}\xi - \xi p]\}$$
$$= \pi^{(d/2)(V-1)}(i/4)^{-(d/2)(V-1)}(\sqrt{\det A'})^{-d}\exp[i(pA'^{-1}\otimes Qp)]. \quad \text{(B.3)}$$

The formula (3.6) is written in a notation where $\det A' = A\begin{pmatrix}k\\k\end{pmatrix}$($A$ (:) denotes the minor of the A matrix with the factor V included in its definition) and $(A'^{-1})_{ij} = A\begin{pmatrix}k & i\\k & j\end{pmatrix}\Big/A\begin{pmatrix}k\\k\end{pmatrix}$.

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