ON SPIN AND ISOSPIN STABILITY OF DENSE NEUTRON AND NUCLEAR MATTER WITH HARD CORE INTERACTION

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Spin and isospin stability of dense neutron and nuclear matter with hard core interaction of radius c is studied using expansion in powers of gaseousness parameter, $x = k_{FC}$, and variational approach. Variational calculations are performed using Jastrow method, including the contribution from three-body clusters. The correlation function has been obtained from Euler's equation resulting from constrained minimization of two-body cluster contribution with subsidiary healing condition. The optimum value of healing integral has been subsequently determined by minimizing the sum of the two- and three-body cluster contributions. In the case of neutron matter, both methods lead to conclusion that in the region of their validity the hard core model is spin stable. Numerical results obtained for hard core model of nuclear matter suggest a spin and isospin instability at $x \cong 1-1.3$, where, however, applicability of our methods is doubtful.

1. Introduction

In the present paper, we try to answer the question whether neutron matter, in its ground state, has spin zero (i.e., is spin stable) or is spin polarized (i.e., spin unstable or ferromagnetic), and the question whether nuclear matter, in its ground state, has spin and isospin zero or is spin and isospin polarized. Obviously, most interesting is the behaviour of both systems at high densities (neutron star matter, dense nuclear matter which might be formed in high energy central collisions of heavy ions). Answer to the question stated above is essential in the problem of a possible ferromagnetism of neutron star matter, and is important in the problem of possible existence of pion condensate in dense nuclear matter.

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In the present work, we make the following simplifying assumptions:

(*i*) We use the model of pure hard core (h.c.) two-body interaction (with radius c) between nucleons.

(*ii*) We employ nonrelativistic quantum mechanics.

(*iii*) We assume that the system is homogeneous i.e., it is in the liquid phase.

We shall discuss the applicability of our simplifying assumptions later. Now, we want to stress that at sufficiently high densities, where the short range repulsion is the decisive part of nuclear forces, our model of pure hard core interaction should approximately describe real nuclear and neutron matter. Furthermore, the effect of two-body attraction on spin and isospin stability is known: the attraction increases the stability.

Needless to say that our results are relevant to other fermion systems, e.g., electrons in metals, and liquid ^{3}He.

The paper is organized as follows. In the next Section, conditions for spin and isospin instability are formulated. In Section 3, we apply to the instability conditions the expansion in powers of the gaseousness parameter $x = k_F c$ (k_F = Fermi momentum). In Section 4, we apply the Jastrow method (with the Iwamoto-Yamada [1, 2] cluster expansion, and with the healing condition imposed on the correlation function) to the instability conditions. Discussion of our results is given in Section 5, which also contains a critical review of earlier work on spin stability of hard core fermion systems. Expressions for three-body terms in the cluster expansion for total energy and for the energy of an impurity are given in Appendix A. In Appendix B the condition for spin instability is derived from the Landau theory of Fermi liquids.

Some of our results, based on conditions (2.2), (2.10) were reported in [3–5]. Some of the other results of the present paper were presented in [6].

2. *Instability conditions*

Throughout this paper, we use the short notation NM for nuclear matter, and $\widetilde{\text{NM}}$ for neutron matter. We distinguish all guantities related to $\widetilde{\text{NM}}$ from those related to NM by a tilda. For instane, E is the energy of NM, an $\tilde{E}$ is the energy of $\widetilde{\text{NM}}$.

(a) Neutron matter ($\widetilde{\text{NM}}$)

We assume that we have N neutrons in the periodicity box of volume Ω. Among the N neutrons, there are $N_\uparrow$ neutrons with spin up and $N_\downarrow$ neutrons with spin down ($N = N_\uparrow + N_\downarrow$). For a fixed density $\varrho = N/\Omega$ the ground state energy $\tilde{E}(N_\uparrow, N_\downarrow)$ depends on the relative numbers of neutrons with spin up and spin down, i.e., on the spin excess parameter $\alpha_\sigma = (N_\uparrow - N_\downarrow)/N$. This means, we have $\tilde{E}(N_\uparrow, N_\downarrow) = \tilde{E}(\alpha_\sigma)$. For spin unpolarized (i.e., normal) $\widetilde{\text{NM}}$ $\alpha_\sigma = 0$, and for totally spin polarized $\widetilde{\text{NM}}$ $\alpha_\sigma = 1$.

Let us denote the difference between the energy per neutron in totally polarized and unpolarized $\widetilde{\text{NM}}$ by

$$\tilde{\Delta} = \tilde{E}(\alpha_\sigma = 1)/N - \tilde{E}(\alpha_\sigma = 0)/N. \tag{2.1}$$

If

$$\tilde{\Delta} < 0 \tag{2.2}$$

then the spin unpolarized state of $\widetilde{\mathrm{NM}}$ is unstable. Inequality (2.2) is our first condition for the occurrence of spin instability (i.e., of ferromagnetism) of $\widetilde{\mathrm{NM}}$.

Instead of asking the question whether the totally spin polarized state has lower energy than the normal state, condition (2.2), we may ask a slightly different question: has the energy $\tilde{E}(\alpha_\sigma)$ a maximum at $\alpha_\sigma = 0$? The answer is yes, if

$$\tilde{\varepsilon}_\sigma = \tfrac{1}{2}\,[\partial^2\{\tilde{E}(\alpha_\sigma)/\mathrm{N}\}/\partial\alpha_\sigma^2]_{\alpha_\sigma=0} < 0, \tag{2.3}$$

where $\tilde{\varepsilon}_\sigma$ is the spin symmetry energy of neutron matter. Inequality (2.3) is our second condition for the occurrence of spin instability.

We may ask still another question: does the energy of $\widetilde{\mathrm{NM}}$ reach a minimum when $\widetilde{\mathrm{NM}}$ becomes totally spin polarized? The answer is yes, if the energy $\tilde{E}(\alpha_\sigma)/N$ is decreasing when α_σ approaches 1, i.e., if

$$\tilde{D} = 2[\partial\{\tilde{E}(\alpha_\sigma)/\mathrm{N}\}/\partial\alpha_\sigma]_{\alpha_\sigma=1} < 0. \tag{2.4}$$

Inequality (2.4) is our third condition for the occurrence of ferromagnetism. It may be put into a different form. With the notation $\tilde{E}(N_\uparrow, N_\downarrow)$, we have $\tilde{E}(N, 0)$ for the energy of the totally spin polarized state. If we change the spin direction of one neutron, we obtain a state with the energy $\tilde{E}(N-1, 1)$. If $\tilde{E}(N, 0)$ is a minimum, the condition

$$\tilde{E}(N, 0) - \tilde{E}(N-1, 1) < 0 \tag{2.5}$$

should be satisfied. This condition for the occurrence of complete ferromagnetism is, of course, equivalent to condition (2.4). Indeed, we have $\tilde{E}(N, 0) = \tilde{E}(\alpha_\sigma = 1)$, $\tilde{E}(N-1, 1) = \tilde{E}(\alpha_\sigma = 1-2/N)$, and consequently

$$\tilde{E}(N, 0) - \tilde{E}(N-1, 1) = \frac{2}{N}\,[\partial\tilde{E}(\alpha_\sigma)/\partial\alpha_\sigma]_{\alpha_\sigma=1} = \tilde{D}. \tag{2.6}$$

Notice that we may write the left-hand side of the last equation in the form:

$$\begin{aligned}\tilde{E}(N, 0) - \tilde{E}(N-1, 0) - [\tilde{E}(N-1, 1) - \tilde{E}(N-1, 0)] \\ = \partial\tilde{E}(N, 0)/\partial N - [\tilde{E}(N, 1) - \tilde{E}(N, 0)],\end{aligned} \tag{2.7}$$

where terms $\sim 1/N$ have been neglected. Consequently, we have

$$\tilde{D} = \frac{\partial\tilde{E}(N, 0)}{\partial N} - \tilde{e} = \frac{\tilde{E}(N, 0)}{N} + \varrho\,\frac{\partial}{\partial\varrho}\,\frac{\tilde{E}(N, 0)}{N} - \tilde{e}, \tag{2.8}$$

where

$$\tilde{e} = \tilde{E}(N, 1) - \tilde{E}(N, 0) \tag{2.9}$$

Consequently, if we know $\tilde{E}(N, 0)/N$ (the energy per neutron in spin polarized neutron matter as function of ϱ) and $-\tilde{e}$ (the binding energy of one spin down neutron in the spin up polarized neutron matter) we may determine $\tilde{D}$ from Eq. (2.8).

The mutual relation between the three conditions (2.2), (2.3), (2.4) depends on the form of the interaction. We denote by $\varrho(\tilde{A})$, $\varrho(\tilde{\varepsilon})$, $\varrho(\tilde{D})$ these densities of $\widetilde{\mathrm{NM}}$ at which

respectively $\tilde{\Delta} = 0, \tilde{\varepsilon}_\sigma = 0, \tilde{D} = 0$. Several situations may occur. For instance, if there is a range of densities for which $\tilde{D} < 0$ and $\tilde{\Delta} > 0$, then within this range of densities, the completely ferromagnetic state is a metastable state. We shall see, however, that with our model of hard core interaction, we should expect two situations: either no spin instability

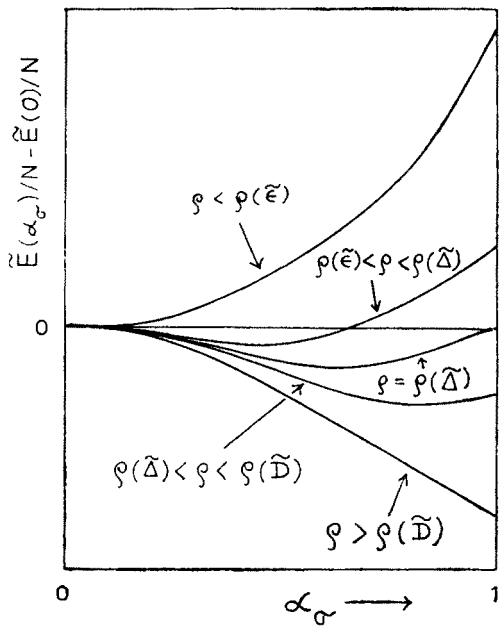

Fig. 1. Expected dependence of neutron matter energy on spin excess at different densities for Serber hard core interaction

occurs (for hard core acting in all states), or there is an instability (for hard core acting in even states only, i.e., for Serber hard core interaction) with $\varrho(\tilde{\varepsilon}) < \varrho(\tilde{\Delta}) < \varrho(\tilde{D})$. In the last case, the system is normal for $\varrho < \varrho(\tilde{\varepsilon})$ and is completely ferromagnetic for $\varrho > \varrho(\tilde{D})$. In the intermediate range of densities, $\varrho(\tilde{\varepsilon}) < \varrho < \varrho(\tilde{D})$ the ground state is partially spin polarized (partial ferromagnetism), and $\varrho(\tilde{\Delta})$ belongs to this interval. This situation is shown in Fig. 1.

(b) Nuclear matter (NM)

Among A nucleons there are $N_\uparrow(N_\downarrow)$ neutrons with spin up (down), and $Z_\uparrow(Z_\downarrow)$ protons with spin up (down), $A = N_\uparrow + N_\downarrow + Z_\uparrow + Z_\downarrow$, and the energy $E = E(N_\uparrow, N_\downarrow, Z_\uparrow, Z_\downarrow)$. Now, we have to consider three types of polarization: spin (σ) polarization, isospin (τ) polarization, and spin-isospin ($\sigma\tau$) polarization.

In case of σ polarization, we have $N_\uparrow = Z_\uparrow = A_\uparrow/2$, $N_\downarrow = Z_\downarrow = A_\downarrow/2$. For a fixed density $\varrho = A/\Omega$, the energy $E(A_\uparrow/2, A_\downarrow/2, A_\uparrow/2, A_\downarrow/2) = E_\sigma(A_\uparrow, A_\downarrow)$ depends only on the spin excess parameter $\alpha_\sigma = (A_\uparrow - A_\downarrow)/A$, $E_\sigma(A_\uparrow, A_\downarrow) = E_\sigma(\alpha_\sigma)$.

In NM with τ polarization, i.e., NM with neutron excess, characterized by $\alpha_\tau = (N-Z)/A$, we have $Z_\uparrow = Z_\downarrow = Z/2$, $N_\uparrow = N_\downarrow = N/2$. We use the notation: $E(N/2, N/2,$

$Z/2, Z/2) = E_\tau(N, Z) = E_\tau(\alpha_\tau)$. What is called here a completely τ polarized NM, is obviously $\widetilde{\text{NM}}$.

In $\sigma\tau$ polarized NM, we have an excess of spin up neutrons and spin down protons compared to spin down neutrons and spin up protons. This excess is characterized by $\alpha_{\sigma\tau} = (A_+ - A_-)/A$, where $A_+/2 = N_\uparrow = Z_\downarrow$, and $A_-/2 = N_\downarrow = Z_\uparrow$. For fixed density, the energy $E(A_+/2,\ A_-/2,\ A_-/2,\ A_+/2) = E_{\sigma\tau}(A_+, A_-)$ depends only on $\alpha_{\sigma\tau}$, $E_{\sigma\tau}(A_+, A_-) = E_{\sigma\tau}(\alpha_{\sigma\tau})$.

The discussion in (a) of the polarization of $\widetilde{\text{NM}}$ may be repeated for the three types of polarization of NM. With the notation of κ polarization, where $\kappa = \sigma, \tau, \sigma\tau$ we obtain the following three conditions for the occurrence of κ instability of NM:

$$\Delta_\kappa = E_\kappa(\alpha_\kappa = 1)/A - E_\kappa(\alpha_\kappa = 0)/A < 0, \tag{2.10}$$

$$\varepsilon_\kappa = \tfrac{1}{2}\left[\partial^2\{E_\kappa(\alpha_\kappa)/A\}/\partial\alpha_\kappa^2\right]_{\alpha_\kappa = 0} < 0, \tag{2.11}$$

$$D_\kappa = 2\left[\partial\{E_\kappa(\alpha_\kappa)/A\}/\partial\alpha_\kappa\right]_{\alpha_\kappa = 1} < 0. \tag{2.12}$$

An alternative expression for D_κ is:

$$D_\kappa = \partial E_\kappa(A, 0)/\partial A - e_\kappa = E_\kappa(A, 0)/A + \varrho\,\frac{\partial}{\partial\varrho}\,\frac{E_\kappa(A, 0)}{A} - e_\kappa, \tag{2.13}$$

where

$$e_\kappa = \tfrac{1}{2}\left[E_\kappa(A, 2) - E_\kappa(A, 0)\right]. \tag{2.14}$$

The slightly different definition of e_κ compared to $\tilde{e}$, Eq. (2.9), reflects the fact that, e.g., for σ polarization of NM, we consider the spin direction of two nucleons (neutron and proton).

3. Expansion in powers of $k_F c$

The expansion of the energy of an infinite system of fermions interacting with a hard core potential, in powers of the gaseousness parameter $x = k_F c$ (c = hard core radius, k_F = Fermi momentum in units of $\hbar$) has been investigated since a long time [7–23]. In general, a single particle state of a given momentum $\boldsymbol{k}$ may be occupied by ν particles, where ν is the number of spin and isospin degrees of freedom per particle ($\nu = 1$ for spin polarized $\widetilde{\text{NM}}$; $\nu = 2$ for unpolarized $\widetilde{\text{NM}}$, and for σ, τ, or $\sigma\tau$ polarized NM; $\nu = 4$ for symmetric ($N = Z$) and spin unpolarized NM). Usually, the expansion of the energy is presented for the case of a general value of ν. Well established is the x^3-approximation of this general expansion (i e., including terms $\sim x^3$). It appears that beyond the x^3-approximation, problems arise with logarithmic terms ($\sim x^4 \ln x$) [14, 15, 18–20].

The x^3-approximation for the ground state energy $E_\nu(\mathcal{N})$ of a hard core fermion system with a given ν is (see, e. g., Efimov and Amusya [13]):

$$E_\nu(\mathcal{N})/\mathcal{N} = \varepsilon_{F\nu}\left[\tfrac{3}{5} + (\nu - 1)\left\{\frac{2}{3\pi}\,x_\nu + \frac{4}{35\pi^2}\,(11 - 2\ln 2)x_\nu^2 + \left(\frac{1}{15\pi} + [0.0755 + 0.059(\nu - 3)]\right)x_\nu^3\right\}_S + (\nu + 1)\left(\frac{1}{5\pi}\,x_\nu^3\right)_P\right], \tag{3.1}$$

where $\mathcal{N}$ is the number of particles, $x_\nu = k_{F\nu}c$, and $k_{F\nu}$ is the Fermi momentum, connected with the density $\varrho = \mathcal{N}/\Omega$ by

$$k_{F\nu}^3 = 6\pi^2\varrho/\nu, \tag{3.2}$$

and

$$\varepsilon_{F\nu} = \hbar^2 k_{F\nu}^2/2M$$

is the Fermi energy.

In (3.1), and throughout this paper, the subscripts S and P denote terms which arise from the interaction in the S and P state respectively.

Another quantity of interest for us is the removal energy of a different particle (impurity) form a system of $\mathcal{N}$ hard core fermions with a given ν:

$$e_{I\nu} = E_\nu(\mathcal{N}+1_I) - E_\nu(\mathcal{N}), \tag{3.3}$$

where $E_\nu(\mathcal{N}+1_I)$ is the ground state energy of the system of $\mathcal{N}$ hard core fermions plus one impurity particle. The quantity $e_{I\nu}$ was calculated by Bishop [21] in case when the mass of the impurity particle is the same as the mass of the $\mathcal{N}$ fermions, and when the interaction between the impurity and each of the $\mathcal{N}$ fermions is the same as the interaction between the $\mathcal{N}$ fermions (i. e., hard core interaction with radius c). In this case, we have in the x^3-approximation:

$$e_{I\nu} = \varepsilon_{F\nu}\nu\left[\left\{\frac{4}{3\pi}x_\nu + \frac{2}{\pi^2}x_\nu^2 + \left[\frac{8}{3\pi^3} - \frac{1}{9\pi}\right.\right.\right.$$

$$\left.\left.\left. + (\nu-1)\tfrac{16}{3}\left(\frac{1}{8\pi} + \frac{2}{15\pi^3}\ln 2 - \frac{7}{30\pi^3}\right)\right]x_\nu^3\right\}_S + \left(\frac{1}{5\pi}x_\nu^3\right)_P\right]. \tag{3.4}$$

Now, let us apply Eqs (3.1) and (3.4) in the analysis of the stability of $\widetilde{\text{NM}}$ and NM.

(a) Neutron matter ($\widetilde{\text{NM}}$)

Expressions for the energy of $\widetilde{\text{NM}}$ are obtained from Eq. (3.1) by putting $\mathcal{N} = N$, $\nu = 2$ and $\nu = 1$:

$$\tilde{E}(0)/\text{N} = E_2(\text{N})/\text{N} = \varepsilon_{F2}\left[\tfrac{3}{5} + \left\{\frac{2}{3\pi}x_2 + \frac{4}{35\pi^2}\right.\right.$$

$$\left.\left.\times(11-2\ln 2)x_2^2\right\}_S + \left(\left\{\frac{1}{15\pi} + 0.0165\right\}_S + \left\{\frac{3}{5\pi}\right\}_P\right)x_2^3\right], \tag{3.5}$$

$$\tilde{E}(1)/\text{N} = E_1(\text{N})/\text{N} = 2^{2/3}\varepsilon_{F2}\left[\tfrac{3}{5} + \left\{\frac{4}{5\pi}\right\}_P x_2^3\right], \tag{3.6}$$

where we have used the relation $k_{F1} = 2^{1/3}k_{F2}$ and $x_1 = 2^{1/3}x_2$.

The factor $2^{2/3}$ in (3.6) represents the increase in kinetic energy when passing from normal ($\alpha_\sigma = 0$) to ferromagnetic ($\alpha_\sigma = 1$) state. Terms that arise from the interaction in the S state (terms linear and quadratic in x_ν, and part of the cubic term in Eq. (3.1)), do not appear in $\tilde{E}(1)$, due to the Pauli principle. This is in accordance with the usual argument

in favour of ferromagnetism: although in the ferromagnetic transition the kinetic energy of the system increases, at high density this is more than balanced by the disappearance of the repulsive interaction. What has been overlooked in this argument, is the interaction in the P state, which contributes to the cubic term in x_v. Now, by the same argument, the P state interaction acts in the opposite direction and, in fact, makes the ferromagnetic state less favourable.

From Eqs (3.5), (3.6), we get for $\tilde{\Delta}$, Eq. (2.1):

$$\tilde{\Delta}/\varepsilon_{F2} = (2^{2/3}-1)\tfrac{3}{5} - \left\{\frac{2}{3\pi}x_2 - \frac{4}{35\pi^2}(11-2\ln 2)x_2^2\right\}_S + \left(-\left\{\frac{1}{15\pi}+0.0165\right\}_S + \left\{(2^{8/3}-3)\frac{1}{5\pi}\right\}_P\right)x_2^3. \tag{3.7}$$

The function $\tilde{\Delta}/\varepsilon_{F2}$, calculated with the help of (3.7), is shown in Fig. 2. The curves denoted as L, Q, and C have been obtained by keeping in (3.7) terms linear, quadratic,

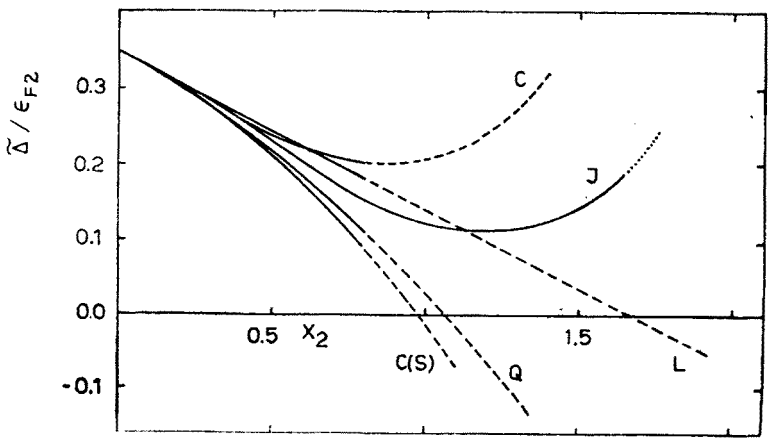

Fig. 2. The function $\tilde{\Delta}/\varepsilon_{F2}$ calculated in the linear (L), quadratic (Q), cubic (C) approximations in x_2, and according to the Jastrow method (J). Broken (dotted) parts of the curves lie outside of the region of applicability of the x-expansion (Jastrow) method

and cubic in x_2. The curve C(S) has been obtained by keeping in (3.7) all terms, except for the contribution of P state interaction.

To calculate $\tilde{D}$, Eq. (2.8), we must know $\tilde{e}$, the removal energy of a spin down neutron from the ground state of the system: spin polarized $\widetilde{\mathrm{NM}}$ + one spin down neutron. Since the hard core interaction does not affect neutron spins, we may look at the one spin down neutron as at an impurity, i. e., a particle which is different from the spin up neutrons. Consequently, we have

$$\tilde{e} = e_{\mathrm{Iv}=1}, \tag{3.8}$$

and may use (3.4) to determine $\tilde{e}$. Since $\tilde{E}(N, 0) = E_1(N)$ we may use Eq. (3.1) to calculate the remaining terms in (2.8). In this way, we find (in the x^3-approximation):

$$\tilde{D}/\varepsilon_{F1} = 1 - \left\{\frac{4}{3\pi} x_1 + \frac{2}{\pi^2} x_1^2 + \left(\frac{8}{3\pi^3} - \frac{1}{9\pi}\right) x_1^3\right\}_S + \left\{\frac{13}{15\pi} x_1^3\right\}_P . \tag{3.9}$$

We may express the right-hand side of (3.9) in terms of $x_2 = 2^{-1/3} x_1$ and use the relation $\varepsilon_{F1} = 2^{2/3} \varepsilon_{F2}$ to obtain the function $\tilde{D}/\varepsilon_{F2}$ shown in Fig. 3.

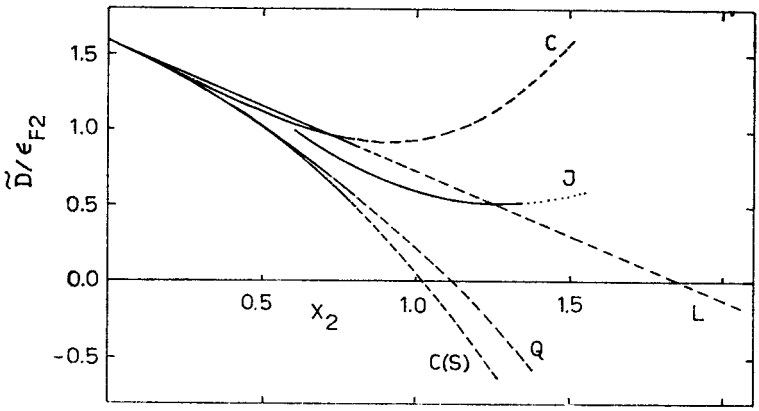

Fig. 3. The function $\tilde{D}/\varepsilon_{F2}$ (notation as in Fig. 2)

The spin symmetry energy of $\widetilde{\mathrm{NM}}$, $\tilde{\varepsilon}_\sigma$, was calculated in [17], however, in the x^2-approximation only:

$$\tilde{\varepsilon}_\sigma = \tfrac{2}{3} \varepsilon_{F2} \left[1 - \frac{2}{\pi} x_2 - \tfrac{16}{15} (2 + \ln 2) \frac{1}{\pi^2} x_2^2 + \cdots\right]. \tag{3.10}$$

A calculation of the next, cubic terms in (3.10) would require very tedious computations. In this situation, it is important to get at least an approximate estimate of this term. To get such an estimate, we proceed in the following way. We assume that we may approximate $\tilde{E}(\alpha_\sigma)/N$ by the equation

$$\tilde{E}(\alpha_\sigma)/\mathrm{N} \cong \tilde{E}(0)/\mathrm{N} + \tfrac{1}{2} \tilde{\varepsilon}_\sigma \alpha_\sigma^2 + \tilde{\gamma} \alpha_\sigma^4, \tag{3.11}$$

from which the approximate relation follows:

$$\tilde{\varepsilon}_\sigma \cong 4\tilde{A} - \tfrac{1}{2} \tilde{D}. \tag{3.12}$$

By substituting in (3.12) expressions (3.7) and (3.9) for $\tilde{A}$ and $\tilde{D}$, we get an estimate of the cubic term in the expansion of $\tilde{\varepsilon}_\sigma$. In this way, we obtain:

$$\tilde{\varepsilon}_\sigma \cong \tfrac{2}{3} \varepsilon_{F2} \left[1 - \left\{\frac{2}{\pi} x_2 + \tfrac{16}{15} (2 + \ln 2) \frac{1}{\pi^2} x_2^2 + 0.106 x_2^3\right\}_S + \{0.624 x_2^3\}_P\right]. \tag{3.13}$$

Let us mention that our approximate procedure, applied to the linear term, leads to the exact coefficient $-2/\pi$ and applied to the quadratic term leads to the coefficient -0.285, whereas the exact result is $-(16/15\,\pi^2)\,(2+\ln 2) = -0.291$.

Let us denote by $x_2(\tilde{\Delta})$, $x_2(\tilde{D})$, and $x_2(\tilde{\varepsilon}_\sigma)$ these values of x_2 for which $\tilde{\Delta} = 0$, $\tilde{D} = 0$, and $\tilde{\varepsilon}_\sigma = 0$, respectively. These values, determined with the help of expressions (3.7), (3.9) and (3.13) respectively, are collected in Table I.

TABLE I

Minimal values of x_v at which unpolarized $\widetilde{\text{NM}}$ and NM becomes unstable, calculated in the L, Q, C(S) and C approximations and with the Jastrow method (J)

		L	Q	C(S)	C	J
$\widetilde{\text{NM}}$	$x_2(\tilde{\varepsilon}_\sigma)$	1.57	1.05	0.98	a	
	$x_2(\tilde{\Delta})$	1.66	1.065	0.99	a	a
	$x_2(\tilde{D})$	1.87	1.12	1.03	a	a
NM	$x_4(\varepsilon_\kappa)$	1.57	1.38	0.74	0.6	
	$x_4(\Delta_\kappa)$	1.66	1.26	0.77	1.05	~ 1.3
	$x_4(D_\kappa)$	1.87	1.22	0.82	1.25	b

[a] stability at all values of x_2; [b] zero of D_κ outside the range of validity of J method.

If we restricted ourselves to terms quadratic (linear) in x_2 (the Q(L) approximation) we would conclude that a transition to a partially ferromagnetic state would start at $x_2 = 1.05$ (1.57), and a complete spin polarization would start at $x_2 = 1.12$ (1.87) (see Fig. 1). This, however, would not be correct, because at all these values of x_2 one must not neglect the P state interaction. When the P state interaction is taken into account, as it is the case in the cubic (C) approximation, none of the instability conditions can be satisfied, and $\widetilde{\text{NM}}$ turns out to be spin stable.

Whereas the L and C approximations are incorrect approximations (at $x_2 \sim 1$) when applied to the hard core interaction model, the C(S) approximation has a physical meaning. It is a cubic approximation for hard core interaction acting only in the S state, or simply an approximation for a Serber exchange hard core interaction (acting only in even angular momentum states). Since Serber interaction does not act at all in spin polarized $\widetilde{\text{NM}}$ (because of Pauli principle), we certainly expect here a ferromagnetic transition to occur. And indeed, as shown in Table I, we predict here a transition to a partially ferromagnetic state at $x_2 \geqq 0.98$, and a complete spin polarization at $x_2 \geqq 1.03$.

(b) Nuclear matter (NM)

The hard core interaction does not depend on spin and isospin, and the energy of σ polarized NM is the same as the energy of the τ and $\sigma\tau$ polarized NM. All the following equations will be written for κ polarization of NM, and all the conclusions will be the same for each of the three types of polarization, $\kappa = \sigma$, τ, and $\sigma\tau$.

Expressions for the energy of NM are obtained from Eq. (3.1) by putting $\mathcal{N} = A$, $\nu = 4$ and $\nu = 2$:

$$E_\kappa(0)/A = E_4(A)/A = \varepsilon_{F4}\left[\tfrac{3}{5} + \left\{\frac{2}{\pi}x_4 + \frac{12}{35\pi^2}(11-2\ln 2)x_4^2\right\}_S + \left(\left\{\frac{1}{5\pi}+0.4035\right\}_S + \left\{\frac{1}{\pi}\right\}_P\right)x_4^3\right], \tag{3.14}$$

$$E_\kappa(1)/A = E_2(A)/A = 2^{2/3}\varepsilon_{F4}\left[\tfrac{3}{5} + \left\{\frac{2}{3\pi}2^{1/3}x_4 + \frac{4}{35\pi^2}(11-2\ln 2)2^{2/3}x_4^2\right\}_S + \left(\left\{\frac{2}{15\pi}+0.033\right\}_S + \left\{\frac{6}{5\pi}\right\}_P\right)x_4^3\right], \tag{3.15}$$

where we have used the relation $k_{F2} = 2^{1/3}k_{F4}$, and $x_2 = 2^{1/3}x_4$.

For Δ_κ, Eq. (2.10), we get:

$$\Delta_\kappa/\varepsilon_{F4} = \left[(2^{2/3}-1)\tfrac{3}{5} - \left\{\frac{2}{3\pi}x_4 + \frac{4(3-2^{4/3})}{35\pi^2}\times(11-2\ln 2)x_4^2\right\}_S - \left(\left\{\frac{1}{5\pi}(1-2^{5/3}/3)+0.3511\right\}_S - \{(6\times 2^{2/3}-5)/5\pi\}_P\right)x_4^3\right]. \tag{3.16}$$

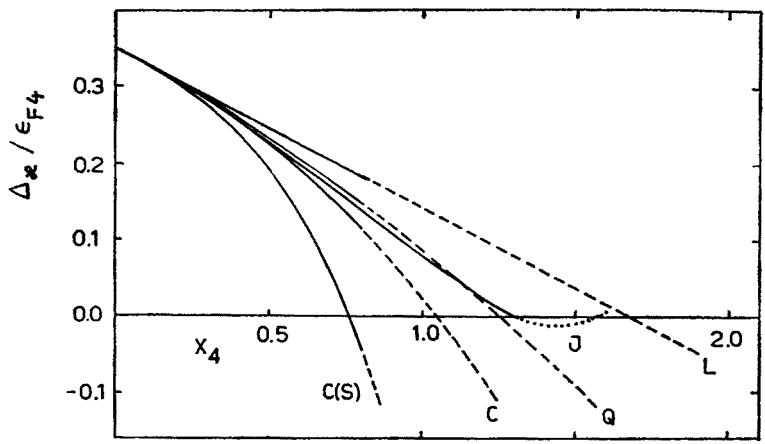

Fig. 4. The function $\Delta_\kappa/\varepsilon_{F4}$ (notation as in Fig. 2)

The function $\Delta_\kappa/\varepsilon_{F4}$ is shown in Fig. 4.

To calculate D_κ, we apply Eq. (2.14), with $E_\kappa(A, 0) = E_2(A)$, with

$$e_\kappa = e_{I,\nu=2}, \tag{3.17}$$

and with the help of (3.4) we obtain:

$$D_\kappa/\varepsilon_{F2} = 1 - \left\{\frac{4}{3\pi}x_2 + \frac{4}{\pi^2}\left[1-\tfrac{1}{15}(11-2\ln 2)\right]x_2^2 + \left[\frac{14}{15\pi} + \frac{64}{45\pi^3}(2+\ln 2) - 0.0440\right]x_2^3\right\}_S + \left\{\frac{6}{5\pi}\right\}_P x_2^3. \tag{3.18}$$

If we express the right-hand side of (3.18) in terms of $x_4 = 2^{-1/3}x_2$ and use the relation $\varepsilon_{F2} = 2^{2/3}\varepsilon_{F4}$ we obtained the function $D_\kappa/\varepsilon_{F4}$ shown in Fig. 5.

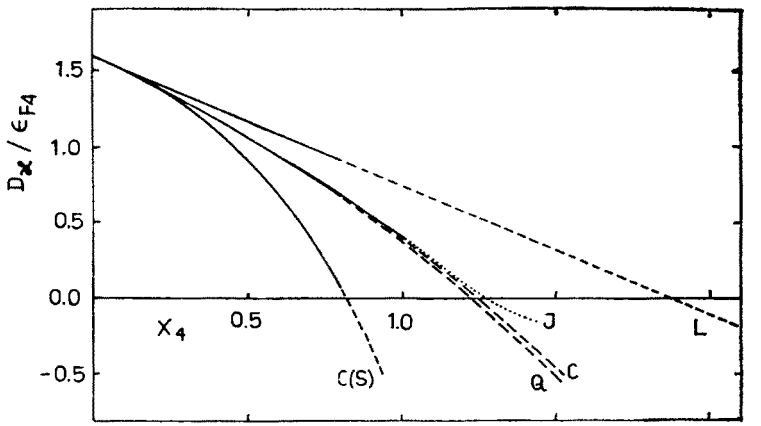

Fig. 5. The function $D_\kappa/\varepsilon_{F4}$ (notation as in Fig. 2)

The symmetry energy of NM, ε_κ, was calculated in [16, 17] only in the x^2-approximation:

$$\varepsilon_\kappa = \tfrac{2}{3}\varepsilon_{F4}\left[1 - \frac{2}{\pi}x_4 - \tfrac{8}{5}(2\ln 2 - 1)\frac{1}{\pi^2}x_4^2 + \cdots\right]. \tag{3.19}$$

To get an estimate of the next cubic term, we proceed in the same way, as in case of $\widetilde{\mathrm{NM}}$, and obtain the approximate relation

$$\varepsilon_\kappa \cong 4\Delta_\kappa - \tfrac{1}{2}D_\kappa. \tag{3.20}$$

Expressions (3.16) and (3.18) allow us to determine the cubic term in ε_κ from Eq. (3.20). In this way, we obtain:

$$\varepsilon_\kappa \cong \tfrac{2}{3}\varepsilon_{F4}\left[1 - \left\{\frac{2}{\pi}x_4 + \tfrac{8}{5}(2\ln 2 - 1)\frac{1}{\pi^2}x_4^2 + 0.5583x_4^3\right\}_S + \{0.8187x_4^3\}_P\right]. \tag{3.21}$$

Let us mention, that relation (3.20), applied to the linear term, leads to the exact coefficient $-2/\pi$, and applied to the quadratic term leads to the coefficient -0.047, whereas the exact coefficient is $-(8/5)(2\ln 2 - 1)/\pi^2 = -0.063$.

We denote by $x_4(\Delta_\kappa)$, $x_4(D_\kappa)$, and $x_4(\varepsilon_\kappa)$ these values of x_4 for which $\Delta_\kappa = 0$, $D_\kappa = 0$, and $\varepsilon_\kappa = 0$ respectively. These values, determined with the help of expressions (3.16), (3.18), and (3.21), are collected in Table I. Notice that in the Q approximation $x_4(D_\kappa) < x_4(\Delta_\kappa) < x_4(\varepsilon_\kappa)$, which corresponds to a situation which differs from that shown in Fig. 1 in case of $\widetilde{\text{NM}}$. We do not discuss this situation because the Q approximation is certainly incorrect for big values of x_4.

The x^3-approximation, applied to NM, leads to the following conclusions. NM with hard core interaction acting in all states is expected to become κ-unstable at $x_4 \cong 1$ (partial κ-polarization of NM groundstate), and to be completely κ-polarized for $x_4 \gtrsim 1.24$ (here, however, the x^3-approximation is not expected to be reliable). In case of Serber hard core, NM is expected to become κ-unstable at $x_4 \cong 0.7$, and to be completely κ-polarized for $x_4 \gtrsim 0.8$.

Now, we would like to answer the question for how big values of x is the x^3-approximation reliable (in describing the liquid phase). We do not know the answer to this question, and restrict ourselves to the following comments. In case of $v = 1$, an expansion of E_v in the x_1^8-approximation is known [16, 17]. The absolute value of the subsequent coefficients of this expansion is only slowly decreasing. Some of the coefficients are positive, and some of them negative. This means that for $x_1 > 1$, the magnitude of E_1 is dominated by the last term, kept in the expansion of E_1 in powers of x_1, and it does not make sense to apply the expansion with a finite number of terms for $x_1 > 1$. We expect a similar situation for $v = 2, 4$, and conclude that the expansion of E_v in powers of x_v is not reliable for $x_v > 1$.

On the other hand, we expect that for $x_v < 1$, the x^3-approximation should be reasonable. This may be demonstrated for $v = 1$, by applying subsequent terms, calculated in [16, 17]. In general, there is the following physical argument in favour of the x_v^3-approximation for $x_v < 1$. The maximum value of the relative momentum k in Fermi gas (with a given v) is k_{Fv} and its average value is $\sqrt{0.3}\, k_{Fv}$. By applying the classical argument with the impact parameter, we expect that the P state interaction should become important at $k_{Fv}c = x_v \cong 1$. Furthermore, we expect that P wave should be most effective around $x_v \cong 1/\sqrt{0.3} = 1.8$. The same argument shows that the D state interaction would be of no importance for $x_v \lesssim 2$. Now, the x_v^3-approximation contains already the leading contribution of the P state interaction, and consequently should be reliable for values of x_v up to $x_v \cong 1$.

Assuming the validity of the arguments presented above, we hope that in deriving our expressions for $\tilde{\Delta}$, $\tilde{D}$, and $\tilde{\varepsilon}_\sigma$ in Section 3(a), we could rely on the x^3-approximation for E_1 for $x_1 \lesssim 1$ (and for E_2, for $x_2 \lesssim 1$, which, however, is a weaker restriction). Since $x_1 = 2^{1/3} x_2$, the condition $x_1 \lesssim 1$ implies that $x_2 \lesssim 0.8$. Consequently, our expressions for $\tilde{\Delta}$, $\tilde{D}$, and $\tilde{\varepsilon}_\sigma$ should be applicable only for $x_2 \lesssim 0.8$. Similarly, the expressions for Δ_κ, D_κ, and ε_κ, derived in Section 3(b), should be applicable only for $x_4 \lesssim 0.8$. For this reason, only the solid part of the curves in Figs 2–5 is expected to be reliable. Also, among our x^3-approximation results collected in Table I (C, and C(S)), only those with $x_2 \lesssim 0.8$ and $x_4 \lesssim 0.8$ seem to be reliable. This means, however, that the most interesting possi-

bility of the κ-instability of NM is predicted to occur at values of x which appear to lie outside the region of validity of the x-expansion method.

In an attempt to clarify the situation, in the next section, we analyze the hard-core fermion system with a completely different method which should work also for bigger values of x.

4. The Jastrow method

(a) Cluster expansion of the energy

In the Jastrow method [24], we start with the following Ansatz for the ground state wave function of $\mathcal{N}$ particles with spin-isospin degeneracy v:

$$\Psi(1 \cdots \mathcal{N}) = \prod_{i<j} f(r_{ij}) \Phi(1 \cdots \mathcal{N}) \tag{4.1}$$

where the arguments of the functions denoted by numbers indicate the full set of space, and spin-isospin coordinates of the corresponding particles. The Slater determinant function Φ is an antisymmetrized product of single particle wave functions (spin-isospin functions times plane wave functions). Actually, we might use a more precise notation, Ψ_v, Φ_v, f_v, for the wave functions and correlation functions for different values of v. However, we drop the subscript v in all our equations, whose meaning concerning the value of v is obvious.

To calculate the expectation value of the hamiltonian, we apply the Iwamoto-Yamada (IY) cluster expansion [1–2]:

$$\langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle = \mathscr{E} = \tfrac{3}{5} \varepsilon_{\mathrm{F}v} \mathcal{N} + \mathscr{E}^{(2)} + \mathscr{E}^{(3)} + \cdots . \tag{4.2}$$

Similarly, as in [2] and in the extensive work by Clark and his collaborators (see, e. g., [25]), we shall restrict ourselves in expansion (4.2) to $\mathscr{E}^{(2)}$, and $\mathscr{E}^{(3)}$, i. e., we shall use the approximation

$$\mathscr{E}/\varepsilon_{\mathrm{F}v} = \hat{\mathscr{E}} = \tfrac{3}{5} \mathcal{N} + \hat{\mathscr{E}}^{(2)} + \hat{\mathscr{E}}^{(3)}, \tag{4.3}$$

where

$$\hat{\mathscr{E}}^{(2,3)} = \mathscr{E}^{(2,3)}/\varepsilon_{\mathrm{F}v}. \tag{4.4}$$

Notice that all the $\mathscr{E}$'s are functionals of f, i. e., $\hat{\mathscr{E}}^{(2,3)} = \hat{\mathscr{E}}^{(2,3)}[f]$.

Explicit expressions for $\mathscr{E}^{(2)}$ and $\mathscr{E}^{(3)}$ were derived in [2]. Here, we shall restrict ourselves to writing down final expressions for $\mathscr{E}^{(2)}$ and $\mathscr{E}^{(3)}$ in case of pure hard core interaction. In this case H in Eq. (4.2) reduces to pure kinetic energy.

$$\langle \Psi | H | \Psi \rangle = -\frac{\hbar^2}{2M} \sum_i \langle \Psi | \Delta_i | \Psi \rangle, \tag{4.5}$$

and the hard core repulsion enters as boundary condition imposed on the correlation function, $f = 0$ at $r = c$.

For $\mathscr{E}^{(2)}$, we have

$$\frac{1}{\mathcal{N}}\mathscr{E}^{(2)} = \tfrac{1}{2}\varrho\int dr\,\frac{\hbar^2}{M}\left(\frac{df}{dr}\right)^2 g_{\mathrm{F}\nu}(k_{\mathrm{F}\nu}r), \tag{4.6}$$

where the radial distribution function for noninteracting system

$$g_{\mathrm{F}\nu}(y) = 1-\frac{1}{\nu}\,l^2(y), \tag{4.7}$$

where $l(y) = 3j_1(y)/y$.

It is most convenient to use $k_{\mathrm{F}\nu}^{-1}$ as our unit of length, and to introduce the notation:

$$\xi = k_{\mathrm{F}\nu}r. \tag{4.8}$$

With the help of (3.2), we get from (4.6):

$$\hat{\mathscr{E}}^{(2)}[f]/\mathcal{N} = \frac{\nu}{6\pi^2}\int d\xi f'(\xi)^2 g_{\mathrm{F}\nu}(\xi), \tag{4.9}$$

where $f'(\xi) = df(\xi)/d\xi$.

In deriving the expression for $\mathscr{E}^{(3)}$, we may simply use Eq. (4.5). This basic prescription (BP) for the kinetic energy has been used by Iwamoto, Yamada [1, 2], and in other early papers on the Jastrow method. Another, equivalent form of the kinetic energy,

$$\langle\Psi|H|\Psi\rangle = \frac{\hbar^2}{2M}\sum_i\int d\tau(\nabla_i\Psi^*)\,(\nabla_i\Psi) \tag{4.10}$$

may be obtained from the BP form by an integration *per partes*. This CW form was first used by Clark and Westhaus [26] and leads to an expression for $\mathscr{E}^{(3)}$ which differs from that which follows from the BP form (the two expressions for $\mathscr{E}^{(3)}$ may be transformed into each other by partial integration, and are equivalent). Both BP and CW expressions for $\hat{\mathscr{E}}^{(3)}[f]$ are given in Appendix A. They involve functions $f(\xi)$, $l(\xi)$ and their derivatives, as well as the function

$$h(\xi) = f(\xi)^2 - 1. \tag{4.11}$$

In applying the Jastrow method to the instability conditions of Section 2, the following difficulties arise.

For any trial function Ψ_ν, the expectation value $\mathscr{E}_\nu$, Eq. (4.2), satisfies the variational principle:

$$\mathscr{E}_\nu \geqslant E_\nu, \tag{4.12}$$

where E_ν is the exact ground state energy. We want to use $\mathscr{E}_\nu$ in our instability conditions, e. g., for estimating $\tilde{\Delta}$, Eq. (2.1), by making the approximation:

$$\tilde{\Delta} \cong \tilde{\Delta}_{\mathrm{J}} = \frac{\mathscr{E}_{\nu=1}}{\mathcal{N}} - \frac{\mathscr{E}_{\nu=2}}{\mathcal{N}}. \tag{4.13}$$

However, the variational principle is valid only separately for $\mathscr{E}_{\nu=1}$ and $\mathscr{E}_{\nu=2}$, and does not allow us to say whether $\tilde{\Delta}_J$ is bigger or smaller than $\tilde{\Delta}$, for a necessarily approximate form of $\Psi_{\nu=1}$ and $\Psi_{\nu=2}$.

Thus, even if we calculated $\mathscr{E}_\nu$ exactly with a given Ψ_ν, we would not be able to make a precise statement about the error in the approximation (4.13). In reality, however, we calculate $\mathscr{E}_\nu$ approximately (we cut off the cluster expansion at $\mathscr{E}^{(3)}$), and consequently, even the variational principle, Eq. (4.12), does not apply to our calculated $\mathscr{E}_\nu$.

If nevertheless we apply the Jastrow method to our instability conditions, we must try to choose such optimal forms of the correlation functions f_ν, that $\mathscr{E}_\nu$, which we calculate in approximation (4.3), is very close to E_ν,

$$\mathscr{E}_\nu \cong E_\nu. \tag{4.14}$$

The need for a high accuracy of calculating $\mathscr{E}_\nu$ is enhanced by the fact that, e. g., $\tilde{\Delta}_J$ in (4.13) is a small difference between two big numbers, $\mathscr{E}_{\nu=1}$ and $\mathscr{E}_{\nu=2}$.

Fortunately, there are possibilities of testing the accuracy of calculating $\mathscr{E}_\nu$, and the degree to which (4.14) is satisfied. Some of them are: comparison with the results of the expansion in powers of x, calculation of a proper smallness parameter of the cluster expansion, and calculation of the ratio $\mathscr{E}^{(3)}/\mathscr{E}^{(2)}$. We shall use all these tests in estimating the range of densities for which our Jastrow method may be applied.

(b) Determination of the correlation function

For hard core interaction, the correlation function $f = f(\xi)$ vanishes for $r \leqslant c$, i. e., for $\xi \leqslant x = k_F c$. In particular,

$$f(x) = 0. \tag{4.15}$$

Furthermore, $f(\xi)$ should show healing

$$\lim_{\xi \to \infty} f(\xi) = 1, \tag{4.16}$$

and the healing should be sufficiently rapid, to secure the convergence of the cluster expansion.

To the best of our knowledge, an unrestricted minimalization of $\mathscr{E}^{(2)} + \mathscr{E}^{(3)}$ does not lead to a correlation function which would exhibit a sufficiently fast healing. One possible way of overcomig this healing difficulty is to assume an analytical form of f, which assures healing, and to fix free parameters of this assumed analytical form by minimizing $\mathscr{E}^{(2)} + \mathscr{E}^{(3)}$. This is the way followed by Clark and his collaborators (see, e. g., [25]). Here, we follow another way, suggested already by Jastrow [27], and impose on f the healing condition,

$$\kappa_J[f] = \varrho \int dr[1-f]^2 g_{F\nu} = \frac{\nu}{6\pi^2} \int d\xi[1-f(\xi)]^2 g_{F\nu}(\xi) = \text{const.}, \tag{4.17}$$

which is apparently the simplest form of subsidiary condition giving the desired asymptotic behaviour of f, Eq. (4.16).

Our procedure, for any fixed value of x_ν (i. e., fixed density or $k_{F\nu}$), consists of two steps. First, we find f which minimizes $\mathscr{E}^{(2)}$ with subsidiary condition (4.17), i. e., we solve Euler's equation of the variational problem

$$\delta\{\hat{\mathscr{E}}^{(2)}[f]+\mathscr{N}\beta^2\kappa_J[f]\} = 0. \tag{4.18}$$

Second, we determine the Lagrange multiplier β^2 by minimizing $\hat{\mathscr{E}}^{(2)}+\hat{\mathscr{E}}^{(3)}$. This minimum value of $\hat{\mathscr{E}}^{(2)}+\hat{\mathscr{E}}^{(3)}$, inserted into Eq. (4.3), gives the desired value of $\hat{\mathscr{E}}$.

This procedure was applied recently by Ali, Grypeos, and Kargas [28] in a calculation of the binding energy of a Λ particle in NM.

The Euler equation, which follows from (4.18), is

$$f''+[2/\xi+g'_{F\nu}/g_{F\nu}]f'-\beta^2(f-1) = 0. \tag{4.19}$$

The asymptotic behaviour of the solution is

$$f(\xi) \to 1+(Ae^{-\beta\xi}+Be^{\beta\xi})/\xi \quad \text{for } \xi \to \infty \tag{4.20}$$

i. e., for $B = 0$ healing is achieved. (This is the reason for our choice, β^2, of the Lagrange multiplier. With a negative multiplier healing would not be possible).

For a given value of x_ν, the solution of Eq. (4.19) with $B = 0$, and which vanish at $\xi = x_\nu$, were determined numerically for a few values of β. With these f functions, $\hat{\mathscr{E}}^{(2)}$ and $\hat{\mathscr{E}}^{(3)}$ were calculated by numerical integration, and the values of β was determined, for which $\hat{\mathscr{E}}^{(2)}+\hat{\mathscr{E}}^{(3)}$ attains its minimum. (The minimum is attained because of opposite behaviour of $\hat{\mathscr{E}}^{(2)}$ and $\hat{\mathscr{E}}^{(3)}$. With increasing range of f, i. e., with decreasing β, the curvature of f is decreasing, and $\hat{\mathscr{E}}^{(2)}$ is decreasing. On the other hand, the three body cluster term, $\hat{\mathscr{E}}^{(3)}$, obviously increases with increasing range of the correlation function).

(c) Cluster expansion of D

We want to apply the Jastrow method to calculate $\tilde{D}$ and D_κ, Eq. (2.8) and (2.13). The Jastrow method of calculating $\tilde{E}(N, 0) = E_{\nu=1}(N)$ and $E_\kappa(A, 0) = E_{\nu=2}(A)$ has been explained already in points (a), (b) of this Section. The only quantities remaining unknown in (2.8) and (2.13) are $\tilde{e} = e_{1\nu=1}$ and $e_\kappa = e_{1\nu=2}$, Eqs (3.3) and (3.17). Here, we shall outline the Jastrow method of calculating $e_{1\nu}$, the removal energy of an impurity from a system of $\mathscr{N}$ hard core fermions with a given ν. In our case, the impurity has the same mass M as the particles of the background of $\mathscr{N}$ fermions, and its interaction with the fermions of the background is the same hard core repulsion as that between the fermions of the background. This simplifies our expressions for $e_{1\nu}$.

We start with the equation

$$e_{1\nu} \cong \mathscr{E}_\nu(\mathscr{N}+1_1)-\mathscr{E}_\nu(\mathscr{N}), \tag{4.21}$$

where $\mathscr{E}_\nu(\mathscr{N})$ is the expectation value of Eq. (4.2), and

$$\mathscr{E}_\nu(\mathscr{N}+1_1) = \langle\Psi_\nu(0; 1 \cdots \mathscr{N})|H(0; 1 \cdots \mathscr{N})\,|\Psi_\nu(0; 1 \cdots \mathscr{N})\rangle$$

$$/\langle\Psi_\nu(0; 1 \cdots \mathscr{N})\,|\Psi(0; 1 \cdots \mathscr{N})\rangle,$$

where

$$\Psi_\nu(0; 1 \cdots \mathcal{N}) = \prod_{i=1}^{\mathcal{N}} \bar{f}(r_{0i}) \bar{\varphi}(0) \tilde{\Psi}_\nu(1 \cdots \mathcal{N}).$$

Here $\bar{f}(r_{0i})$ is the correlation function between the impurity (particle number zero) and the i-th particle of the background, $\bar{\varphi}(0)$ is the single particle wave function of the impurity (spin-isospin function times a plane wave function with zero momentum), and $\tilde{\Psi}(1 \ldots \mathcal{N})$ is defined in (4.1) with f replaced by $\tilde{f}$. A priori, we cannot assume that $f = \tilde{f}$: the correlation function f in the pure background might be changed slightly ($\sim 1/\mathcal{N}$) by introducing the impurity.

Similarly, as in the case of Eq. (4.13) for $\tilde{\Delta}$, (4.21) is an approximate equation, because of the assumed simple forms of $\Psi_\nu(1 \ldots \mathcal{N})$, $\tilde{\Psi}_\nu(\ldots \mathcal{N})$, and $\Psi_\nu(0; 1 \ldots \mathcal{N})$, and because of the approximate way, in which we calculate $\mathscr{E}_\nu(1 \ldots \mathcal{N})$ and $\mathscr{E}_\nu(\mathcal{N} + 1_\mathrm{I})$. We shall test the accuracy of our procedure here in the same way, as in case of Δ. One may expect the accuracy of calculating $e_{1\nu}$, and $\tilde{D}$, D_κ to be better than the accuracy of calculating $\tilde{\Delta}$, Δ_κ. Here e. g., in case of $\tilde{D}$, we avoid subtracting two big numbers, $\mathscr{E}_1(\mathcal{N})$ and $\mathscr{E}_2(\mathcal{N})$. Instead, we calculate directly the removal energy of a spin down neutron (the impurity) from the background of spin up neutrons. While doing it, we describe correlations between the spin down neutron and the spin up neutrons with a correlation function $\bar{f}$, which differs from f, which describes correlations between the spin up neutrons.

Now, we apply for both $\mathscr{E}_\nu(\mathcal{N})$ and $\mathscr{E}_\nu(\mathcal{N} + 1_\mathrm{I})$ the IY cluster expansion. For $\mathscr{E}_\nu(\mathcal{N})$, we use the approximation (4.3). Similarly, in the IY cluster expansion for $\mathscr{E}_\nu(\mathcal{N} + 1_\mathrm{I})$, we keep only $\mathscr{E}_\nu^{(2,3)}(\mathcal{N} + 1_\mathrm{I})$. The derivation for the two first cluster terms, $\mathscr{E}^{(2,3)}(\mathcal{N} + 1_\mathrm{I})$ proceeds analogically to the original derivation of the corresponding two terms in IY cluster expansion of $\mathscr{E}_\nu(\mathcal{N})$ [1–2]. Here, we restrict ourselves to writing down the final expression:

$$\mathscr{E}_\nu(\mathcal{N} + 1_\mathrm{I})/\varepsilon_{\mathrm{F}\nu} = \tfrac{3}{5}\mathcal{N} + \hat{\mathscr{E}}_\nu^{(2)}[\tilde{f}] + \hat{\mathscr{E}}_\nu^{(3)}[\tilde{f}] + \hat{e}_{1\nu}^{(2)}[\bar{f}] + \hat{e}_{1\nu}^{(3)}[\tilde{f}, \bar{f}], \tag{4.22}$$

where

$$\hat{e}_{1\nu}^{(2)}[\bar{f}] = 2\left(\frac{\nu}{6\pi^2}\right)\int d\xi \bar{f}'(\xi)^2. \tag{4.23}$$

Expressions for $\tilde{e}_{1\nu}^{(3)}$ are given in Appendix A for two forms assumed for the kinetic energy (BP and CW).

To find optimal correlation functions, $\tilde{f}$ and $\bar{f}$, we proceed similarly as in part (b) of this Section. We impose condition (4.17) on $\tilde{f}$, and a similar condition on $\bar{f}$:

$$\bar{\kappa}_\mathrm{J}[\bar{f}] = \varrho \int d\boldsymbol{r}[1 - \bar{f}]^2 = \frac{\nu}{6\pi^2}\int d\xi[1 - \bar{f}(\xi)]^2 = \text{const.} \tag{4.24}$$

Both these conditions insure the proper asymptotic behaviour of $\tilde{f}$ and $\bar{f}$ ($\tilde{f} \to 1, \bar{f} \to 1$ for $\xi \to \infty$), necessary for the convergence of the cluster expansion of $\mathscr{E}_\nu(\mathcal{N} + 1_\mathrm{I})$.

We determine $\tilde{f}$ and $\bar{f}$ in two steps. First, we minimize the two-body cluster part of $\mathscr{E}_\nu(\mathcal{N}+1_I)$, Eq. (4.22), with subsidiary conditions (4.17), (4.24), by solving Euler's equations of the variational problem

$$\delta\{\hat{\mathscr{E}}_\nu^{(2)}[\tilde{f}]+\mathcal{N}\tilde{\beta}^2\kappa_J[\tilde{f}]+\hat{e}_{I\nu}^{(2)}+2\bar{\beta}^2\bar{\kappa}_J[\bar{f}]\} = 0. \tag{4.25}$$

Second, we determine the Lagrange multipliers $\tilde{\beta}^2$ and $\bar{\beta}^2$ by minimixing $\mathscr{E}_\nu(\mathcal{N}+1_I)$, given in (4.22).

The Euler equation for $\tilde{f}$ is obviously identical with Eq. (4.19), and that for $\bar{f}$ is:

$$\bar{f}''+(2/\xi)f'-\bar{\beta}^2(\bar{f}-1) = 0. \tag{4.26}$$

It is easy to see that our present procedure leads to $\tilde{\beta}$ which differs by a quantity $\Delta\beta \sim 1/\mathcal{N}$ from β, determined in case of pure background according to the procedure described in part (b) of this Section. According to this procedure $\hat{\mathscr{E}}_\nu^{(2)}[\tilde{f}]+\hat{\mathscr{E}}_\nu^{(3)}[\tilde{f}]$ in (4.22) attains a minimum for $\tilde{f} = f$, i. e., for $\tilde{\beta} = \beta$. Hence, by replacing $\tilde{f}$ by f in (4.22), we introduce a change $\sim (\Delta\beta)^2 \sim (1/\mathcal{N})^2$. Even though $\hat{\mathscr{E}}_\nu^{(2)}$ and $\hat{\mathscr{E}}_\nu^{(3)}$ are proportional to $\mathcal{N}$, the whole change is still only of order $1/\mathcal{N}$ and may be neglected. Consequently, Eq. (4.22), takes the form:

$$\mathscr{E}_\nu(\mathcal{N}+1_I) = \mathscr{E}_\nu(\mathcal{N})+\varepsilon_{F\nu}\{\hat{e}_{I\nu}^{(2)}[\bar{f}]+\hat{e}_{I\nu}^{(3)}[f,\bar{f}]\}. \tag{4.27}$$

It should be stressed that this equation, in which the difference between β and $\tilde{\beta}$ is neglected, is possible *only* because our procedure of determining β is a minimalization procedure.

If we insert expression (4.27) into (4.21), we obtain our final formula for $e_{I\nu}$:

$$e_{I\nu}/\varepsilon_{F\nu} = \hat{e}_{I\nu} = \hat{e}_{I\nu}^{(2)}[\bar{f}]+\hat{e}_{I\nu}^{(3)}[f,\bar{f}]. \tag{4.28}$$

Practical determination of $e_{I\nu}$ is simplified by the fact that the desired solution of Eq. (4.26) has the simple analytical form:

$$\bar{f}(\xi) = 1-e^{-\bar{\beta}(\xi-x)}/(\xi/x). \tag{4.29}$$

From Eq. (4.23), we obtain easily the result

$$\hat{e}_{I\nu}^{(2)} = \frac{4\nu}{3\pi}[x_\nu+\tfrac{1}{2}\bar{\beta}^2x_\nu^2]. \tag{4.30}$$

(Notice that the first term coincides with the linear term in expansion (3.4). A similar coincidence of the quadratic terms would require that $\bar{\beta} = 3/\pi$). What remains, is to calculate numerically $\hat{e}_{I\nu}^{(3)}[f,\bar{f}]$ for a few values of $\bar{\beta}$, and to find the value of $\bar{\beta}$ for which $\hat{e}_{I\nu}$, Eq. (4.28), attains its minimum.

To calculate $\tilde{D}$, Eq. (2.8), we substitute e_{I1} for $\tilde{e}$, Eq. (3.8), $\mathscr{E}_1$ for $\tilde{E}(N, 0)$, Eq. (4.11), and using (4.3), (4.28) we obtain

$$\tilde{D}/\varepsilon_{F1} = 1+\tfrac{5}{3}[\hat{\mathscr{E}}_1^{(2)}+\hat{\mathscr{E}}_1^{(3)}]/N+\tfrac{1}{3}x_1\frac{d}{dx_1}[\hat{\mathscr{E}}_1^{(2)}+\hat{\mathscr{E}}_1^{(3)}]/N-\hat{e}_{I1}. \tag{4.31}$$

With $[\hat{\mathscr{E}}_1^{(2)}+\hat{\mathscr{E}}_1^{(3)}]/N$ being calculated as function of x_1, we compute numerically the derivative (d/dx_1) [ ]$/N$. Expressing ε_{F1} by ε_{F2}, and x_1 by x_2, we finally obtain $\tilde{D}/\varepsilon_{F2}$ as function of x_2. Similarly, we have

$$D_\kappa/\varepsilon_{F2} = 1+\tfrac{5}{3}\,[\hat{\mathscr{E}}_2^{(2)}+\hat{\mathscr{E}}_2^{(3)}]/A+\tfrac{1}{3}\,x_2\,\frac{d}{dx_2}\,[\hat{\mathscr{E}}_2^{(2)}+\hat{\mathscr{E}}_2^{(3)}]/A-\hat{e}_{12}, \tag{4.32}$$

and may calculate $D_\kappa/\varepsilon_{F4}$ as function of x_4.

(d) Results

Results obtained for the energy, $\hat{\mathscr{E}}_\nu/\mathscr{N}$, calculated according to the procedure described in points (a), and (b), are shown in Table II for $\nu = 1$, 2, and 4.

Most remarkable is the agreement between $\hat{\mathscr{E}}/\mathscr{N}$ for $\nu = 1$ and the x^3-approximation to $(E_1/\mathscr{N})/\varepsilon_{F1}$, Eq. (3.1), for $x_1 \lesssim 1$. The fact that the "variational" values of $\hat{\mathscr{E}}/\mathscr{N}$ are slightly smaller than the "$\sim x^3$" values indicates that the x^3-approximation is not sufficiently accurate. Actually, our values of $\hat{\mathscr{E}}/\mathscr{N}$, $\nu = 1$, agree for $x_1 \leqslant 1$ completely (within the

TABLE II

Jastrow method results for total energy (with the BP form of the kinetic energy). Values of $(E_\nu/\mathscr{N})/\varepsilon_{F\nu}$, calculated in the x^3-approximation, Eq. (3.1), are denoted by "$\sim x^3$"

ν	x	β	$\hat{\mathscr{E}}^{(2)}/\mathscr{N}$	$\hat{\mathscr{E}}^{(3)}/\mathscr{N}$	$\hat{\mathscr{E}}/\mathscr{N}$	$\sim x^3$	$\mathscr{E}^{(3)}/\mathscr{E}^{(2)}$	κ_J	ζ
1	0.6	0.6	0.02	0.00	0.63	0.63	0.03	0.01	0.14
	0.8	0.7	0.06	0.00	0.66	0.67	0.07	0.02	0.24
	0.95	0.7	0.09	0.01	0.70	0.71	0.10	0.04	0.48
	1.27	0.8	0.20	0.05	0.84	0.86	0.23	0.10	0.75
	1.59	0.8	0.35	0.14	1.09	1.11	0.41	0.21	1.29
	1.75	0.8	0.44	0.24	1.28	1.28	0.54	0.30	1.69
	1.91	0.8	0.54	0.36	1.50	1.49	0.67	0.43	2.02
	2.22	0.9	0.74	0.71	2.11	1.97	0.96	0.58	2.49
2	0.76	0.8	0.27	0.09	0.96	0.93	0.33	0.11	1.05
	1.01	0.8	0.40	0.21	1.23	1.16	0.53	0.23	1.57
	1.26	0.8	0.62	0.42	1.64	1.50	0.68	0.43	2.36
	1.39	0.9	0.76	0.55	1.91	1.72	0.72	0.52	2.52
	1.51	0.9	0.88	0.76	2.24	1.96	0.86	0.66	2.83
	1.76	1.0	1.20	1.31	3.11	2.57	1.09	1.00	3.40
	2.02	1.0	1 54	2.21	4.35	3.38	1.44	1.48	3.82
4	0.4	1.0	0.32	0.06	0.98	0.96	0.19	0.06	0.75
	0.6	1.0	0.53	0.18	1.32	1.27	0.34	0.16	1.23
	0.8	1.0	0.81	0.39	1.81	1.72	0.48	0.37	1.90
	1.0	1.1	1.12	0.80	2.52	2.30	0.71	0.52	2.24
	1.1	1.1	1.29	1.09	2.98	2.75	0.84	0.66	2.73
	1.2	1.2	1.52	1.41	3.53	3.20	0.93	0.80	2.90
	1.4	1.2	1.94	2.40	4.94	4.30	1.24	1.20	3.94
	1.6	1.2	2.42	3.88	6.89	5.69	1.60	1.72	5.07

accuracy of Table II) with values of $(E_1/\mathcal{N})\varepsilon_{F1}$ calculated in the x^8-approximation of [22, 23]. The particular accuracy of our Jastrow method for $v = 1$ should be attributed to the fact that only odd orbital momentum states appear in the expectation value, $\hat{\mathscr{E}}$. To achieve a comparable accuracy for $v = 2, 4$, one should introduce different correlation functions for even and odd values of l. But even with our state-independent correlation functions our results for $\hat{\mathscr{E}}/\mathcal{N}$ for $v = 2, 4$, are very close to the "$\sim x^3$" results for $x_v \leqslant 1$ (in this respect, our results for $v = 4$ are not worse than the FHNC results reported by Zabolitzky [29]).

We have tested our numerical accuracy by calculating $\hat{\mathscr{E}}$, $v = 1, 2, 4$, with the BP and CW form of $\hat{\mathscr{E}}^{(3)}$. The results obtained with the two equivalent forms of $\hat{\mathscr{E}}$ agree within the numerical accuracy of Table II.

Now we would like to answer the question, for which range of x's is our method of calculating $\hat{\mathscr{E}}$ reliable. A comparison with the x^3-approximation is useless for big values of x for which this approximation is not expected to work.

The IY cluster expansion, applied in our method of calculating $\mathscr{E}$, is essentially an expansion in the number of h factors. As a smallness parameter ζ of the expansion, Iwamoto and Yamada [1] have considered $\omega/(\Omega/\mathcal{N}) = \varrho\omega$, where ω is the volume within which the correlation of two particles is strong. A reasonable measure of ω is $\int drh$ and by incorporating statistical correlation, one is led to the definition

$$\xi = \varrho \int drhg_{Fv} = \frac{v}{6\pi^2} \int d\xi h(\xi) g_{Fv}(\xi). \tag{4.33}$$

The hope, however, that ζ is a proper measure of the ratio of successive terms in the IY cluster expansion is, at least in our prodecure, not fulfilled. Values of ζ shown in Table I are much bigger than the ratio $\mathscr{E}^{(3)}/\mathscr{E}^{(2)}$.

Another reasonable measure of ω is $\int dr(1-f)^2$ which suggests to consider the Jastrow wound integral κ_J, Eq. (4.17), as the smallness parameter. Values of κ_J in Table I, and their comparison with values of $\mathscr{E}^{(3)}/\mathscr{E}^{(2)}$ suggest that κ_J is a much better indicator of the convergence of our cluster expansion, than ζ. In particular, $\kappa_J \cong 1$ in the same range of values of x, in which $\mathscr{E}^{(3)}/\mathscr{E}^{(2)} \cong 1$. (At very high values of x both ζ and κ_J are of the same order of magnitude, determined by integration in (4.17) and (4.33) over $\xi < x(r < c)$ whose contribution to both ζ and κ_J is the same, and does not depend on the shape of f).

We settle the question of the range of reliability of our results in the following way. We shall consider our results as completely unreliable in the range $x > \bar{x}$ in which $\mathscr{E}^{(3)}/\mathscr{E}^{(2)} > 1$ (this is about the same range in which $\kappa_J > 1$). Consequently, we shall restrict our discussion to the range $x < \bar{x}$. Obviously, for values of x approaching $\bar{x}$ our results are becoming less reliable. According to the results for $\mathscr{E}^{(3)}/\mathscr{E}^{(2)}$ in Table II, we have $\bar{x}_1 = 2, 3$, $\bar{x}_2 = 1.7$, $\bar{x}_4 = 1.25$ as the upper limits for the applicability of our method of calculating $\mathscr{E}_v$ for $v = 1$, 2, and 4 respectively.

With the results obtained for $\hat{\mathscr{E}}_1$ and $\hat{\mathscr{E}}_2$ we may calculate $\tilde{\Delta}$. Eq. (4.13). Expressing ε_{F1} and x_1 by ε_{F2} and x_2, we obtain $\tilde{\Delta}/\varepsilon_{F2}$ as function of x_2 which is shown as curve J in

Fig. 2. Since at fixed density, $\bar{x}_1 = 2.3$ corresponds to the value of $2.3/2^{1/3} = 1.8$ of x_2, we see that our upper limit for the applicability of our method of calculating $\tilde{\Delta}/\varepsilon_{F2}$ is determined by $\bar{x}_2 = 1.7$. The dotted part of curve J in Fig. 2 lies in the range of values of x_2 beyond this upper limit.

Curve J should be compared with curve C in Fig. 2 (with our state independent correlation function, we are not able to treat the Serber hard core, curve C(S)). The fact that curve J lies below curve C should be attributed to the exceptional accuracy of our "variational" Jastrow results for $\mathscr{E}_1$. A similar accuracy in case of $\nu = 2$ should lower the value of $\mathscr{E}_2$, and shift up curve J. Consequently, an improvement in calculating $\mathscr{E}_2$ should only strengthen the inequality $\tilde{\Delta} > 0$.

In an analogical way, we obtain $\Delta_\kappa/\varepsilon_{F4}$, shown as curve J in Fig. 4. Here, our upper limit for the reliability of our results is $\bar{x}_4 = 1.28$. (There is a difference at high values of x_4 between curve J, obtained with a corrected computer code, and the preliminary results represented as crosses in Fig. 2 of [5] and in Fig. 3 of [6]). Our Jastrow method results indicate that $\tilde{\Delta} > 0$, and $\Delta_\kappa = 0$ at $x_4 \cong 1.2$–1.3.

Results obtained for the removal energy $\hat{e}_I$, calculated in the way described in point (c), are shown in Table III.

TABLE III

Jastrow method results for $\hat{e}_I$ (with the BP form of the kinetic energy). Values of $\hat{e}_I$, calculated in the x^3-approximation, Eq. (3.4), are denoted by "$\sim x^3$"

ν	x	$\bar{\beta}$	$\hat{e}_I$	$\sim x^3$	$\hat{e}_I^{(3)}/\hat{e}_I^{(2)}$	$\bar{\kappa}$
1	0.60	0.5	0.36	0.35	0.23	0.09
	0.80	0.5	0.55	0.53	0.35	0.17
	0.95	0.6	0.72	0.68	0.45	0.25
	1.27	0.6	1.25	1.16	0.65	0.41
	1.59	0.7	2.02	1.67	0.91	0.65
	1.91	0.8	3.15	2.32	1.26	1.0
	2.22	0.8	4.80	3.17	1.6	1.4
2	0.76	0.8	1.20	1.16	0.46	0.22
	1.01	0.8	2.07	1.89	0.69	0.40
	1.26	0.9	3.33	2.94	1.0	0.67
	1.39	0.9	4.13	3.62	1.1	0.82
	1.51	0.9	5.13	4.24	1.4	1.0
	1.76	1.0	7.73	6.10	1.7	1.4
	2.02	1.0	11.35	8.45	2.3	2.0

Similarly, as in case of $\mathscr{E}$, we consider as the upper limit for applicability of our method of calculating $\hat{e}_{Iv}$ the value $\bar{x}_v$ beyond which $\hat{e}_I^{(3)}/\hat{e}_I^{(2)} > 1$. We have $\bar{x}_1 = 1.7$ (the corresponding value of x_2 is 1.35), and $\bar{x}_2 = 1.3$ (the corresponding value of x_4 is 1.0). To calculate $\tilde{D}$ and D_κ, we apply Eqs (4.31) and (4.32), in which we use the previously calculated values of $\mathscr{E}_1$ and $\mathscr{E}_2$. Since our upper limits for the validity of our results for $\hat{\mathscr{E}}_1$ and $\hat{\mathscr{E}}_2$ are bigger than $\bar{x}_1 = 1.7$ and $\bar{x}_2 = 1.3$, the latter ones determine the range of validity of our results obtained for $\tilde{D}$ and D_κ.

Our results for $\tilde{D}/\varepsilon_{F2}$, and $D_\kappa/\varepsilon_{F4}$ are shown as curves in Fig. 3 and Fig. 5. We see that $\tilde{D} > 0$. In case of NM, Fig. 5, we have $D_\kappa = 0$ at $x_4 \cong 1.2$–1.3, however this value of x_4 is far beyond our limit, $\bar{x}_4 = 1.0$, of applicability of our method of calculating D_κ.

5. *Discussion*

(a) Neutron matter ($\widetilde{\text{NM}}$)

All our results, shown in Fig. 2 and Fig. 3, as well as the discussion in Section 3(a) of $\tilde{\varepsilon}_\sigma$, reveal spin stability of dense $\widetilde{\text{NM}}$ treated as a nonrelativistic hard core gas. Even at x_2 values beyond the limit of applicability of our Jastrow approach we do not see any tendency towards ferromagnetism. Obviously, our results apply to any hard core fermion system, e. g., the hard core interaction model has been applied since a long time in the theory of magnetism of metals (for a review, see [30]).

Our results contradict previous analyses [31–34] (see also [35]) which have suggested the possibility that hard core $\widetilde{\text{NM}}$ becomes ferromagnetic at a density, comparable to neutron star densities. On the other hand, existing calculations (see, e. g., [36]) of magnetic susceptibility of $\widetilde{\text{NM}}$, performed with realistic forces, do not reveal any tendency towards ferromagnetism (see, e. g., the review article by Baym and Pethick [37]). Consequently, the opinion has been accepted that pure hard core interaction would lead to ferromagnetism, and only the presence of attractive forces makes $\widetilde{\text{NM}}$ spin stable. Unfortunately, this opinion relies entirely on calculations with realistic n–n forces, and the approximation methods applied in these calculations are not expected to work at high densities. Furthermore, this opinion rises serious doubts, because the model of pure hard core interaction should approximately describe real $\widetilde{\text{NM}}$, especially at high densities.

These difficulties are resolved by our results which show spin stability of $\widetilde{\text{NM}}$ (at least in a wide range of densities of the liquid phase). It should be stressed that to demonstrate spin stability of $\widetilde{\text{NM}}$, it is sufficient to consider pure hard core interaction. Namely, the attraction between neutrons increases spin stability (see, e. g., [35]).

Now let us discuss the previous analyses. Morita et al. [34] restricted themselves to the linear, i. e., x-approximation and predicted a transition to a partially and totally ferromagnetic state at $x_2 = 1.57$ and 1.87, in agreement with the L values of Table I. Because of the neglect of terms higher than linear, their result is of no significance.

Rice [31] applied the condition for a ferromagnetic transition in the Landau theory of normal Fermi liquids [38]. He restricts himself to terms quadratic in x, and concludes that $\widetilde{\text{NM}}$ undergoes a ferromagnetic transition for $x_2 \geqslant 0.98$. If not for an error in his Eq. (4) he would predict ferromagnetism for $x_2 \geqslant x_f$ with $x_f = x_2(\tilde{\varepsilon}_\sigma) = 1.05$ in the quadratic approximation (see Table I). The correct derivation of the expression for x_f from the Landau theory is given in the Appendix B. However, even the corrected result of Rice is of no significance because of his neglect of the cubic terms in x whose importance has been stressed in Section 3(a).

Other estimates of x_f are more sophisticated and by their complexity are better protected from demonstration of their inadequacy. In his early work Ehrman [33] applied a simplified version of Brueckner theory. However, he neglected the exclusion principle

in his reaction matrix equation, the only source of the quadratic term in (3.10). In consequence, in Ehrman's expression for energy the term quadratic in x_2 is missing. Also, the coefficient 76/45 in his Eq. (20) should be replaced by $-1/9$; the positive sign of this coefficient would correspond to an attractive D-state contribution produced by hard core. Consequently, one cannot attach any significance to his result, $x_f = 1.674$.

Brownell and Callaway [32] have applied Brueckner theory with the reference spectrum approximation. However, the reference spectrum approximation is known to be inadequate for dense neutron matter. Namely, even for realistic potentials with attractive component the self-consistent potential energy of the hole states near the Fermi surface is not as attractive as in nuclear matter, so that the healing parameter γ becomes imaginary [40]. An expansion of the condition for ferromagnetism, considered by Brownell and Callaway, in powers of x_2 may be compared with our formulae, and shows the sensitivity of their result to the value of healing parameter γ. The value of γ taken by Brownell and Callaway *ad hoc* from nuclear matter calculations with realistic forces, leads to a cubic term in their expression for $\tilde{\varepsilon}_\sigma$, which is 30 times smaller than the cubic term in our expression (3.13) (and 20 times smaller than the analogical cubic term in our exact expression (3.7) for $\tilde{\Delta}$). Consequently, the meaning of their result, $x_f = 0.86$, is obscure.

The paper of Silverstein [35] is sometimes interpreted [31], [41] as indicating that hard core neutron matter would become ferromagnetic at sufficiently high density, even with attractive Wigner force added. In fact, Silverstein applied first order perturbation theory to an effective interaction consisting of a purely S state repulsion of zero range, $I\delta(\boldsymbol{r})$, and of an attractive Wigner force, $V_1(r)$, and shows that by adjusting I one may always get a ferromagnetic transition. Since the contribution of the effective repulsion $I\delta$ to the spin symmetry energy is proportional to $-I$, one may always get a negative value of total $\tilde{\varepsilon}_\sigma$ for any V_1 by taking a sufficiently big value of I. However, an effective S state repulsion is not equivalent to the hard core interaction, and the otherwise correct result of Silverstein is not relevant in the dense hard core Fermi gas problem. (The proper form of an effective interaction in this problem is the pseudopotential of Huang and Yang [8] and its application leads to our results.)

In all our considerations, we have treated $\widetilde{\mathrm{NM}}$ as nonrelativistic, homogeneous gas of hard spheres. We have shown that within this model, $\widetilde{\mathrm{NM}}$ is spin stable for $x_2 \lesssim \bar{x}_2$, and we have argued that $\bar{x}_2 \cong 1.7$. Actually, the precise value of $\bar{x}_2$ is probably irrelevant for the behaviour of real $\widetilde{\mathrm{NM}}$. Let us assume that $c = 0.4$ fm. Then, for $\bar{x}_2 = 1.7$, we have $\varrho = 2.6\ \mathrm{fm}^{-3}$, and $M\varrho = 4.3 \times 10^{15}\ \mathrm{g/cm}^3$. At such high density one certainly expects important relativistic effects, the appearance of other barions and pions, a possible solidification (see, e. g., [42]), not to mention such exotic possibilities as formation of an abnormal phase or of quark matter (for a review see, e. g., [37, 43]). Notice that already at $x_2 = 1$ (i. e., at $\varrho = 0.5\ \mathrm{fm}^{-3}$, $M\varrho = 0.9 \times 10^{15}\ \mathrm{g/cm}^3$) the average kinetic energy of a neutron (i. e., $\mathscr{E}_2/N$) is equal about 156 MeV (see Table II), i. e. about 17% of its rest mass.

In conclusion, our method of establishing spin stability of $\widetilde{\mathrm{NM}}$, especially the Jastrow method, is reliable at all densities at which the assumed model of $\widetilde{\mathrm{NM}}$ is applicable (probably at $M\varrho \lesssim 10^{15}\ \mathrm{g/cm}^3$).

(b) Nuclear matter (NM)

The results shown in Fig. 4, and the discussion of ε_κ in Section 3(b), suggest the occurrence of κ-instability of NM at $x_4 \cong 1 - 1.3$. Also, the results shown in Fig. 5 support this suggestion. There are, however, the following difficulties in drawing any definite conclusions. The x^3-approximation suggests the onset of partial κ-polarization at $x_4 = x_4(\varepsilon_\kappa) = 0.96$, i. e., for $c = 0.4$ fm, at about six times the equilibrium density of NM. But the discussion in Section 3(b) led us to the conclusion that we should not rely on the x^3-approximation for $x_4 \gtrsim 0.8$. In the Jastrow method we did not calculate $x_4(\varepsilon_\kappa)$ but only $x_4(\Delta_\kappa)$ and $x_4(D_\kappa)$, and we expect that $x_4(\varepsilon_\sigma) < x_4(\Delta_\kappa) < \kappa_4(D_\kappa)$ (see Fig. 1). And indeed, the x^3-approximation gives $x_4(\Delta_\kappa) = 1.05$, and $x_4(D_\kappa) = 1.24$. Now, our Jastrow method gives $x_4(\Delta_\kappa) \cong 1.3$, but at $x_4 \cong 1.3$ we have already $\mathscr{E}_4^{(3)}/\mathscr{E}_4^{(2)} \cong 1$ and the question arises wheather our cluster expansion of Δ_κ converges at this high value of x_4. A similar convergence problem of our cluster expansion of D_κ arises already at $x_4 \cong 1$.

As we have mentioned in the preceding point of the present Section, the attractive part of nuclear forces is expected to increase the κ-stability of NM (at equilibrium density of NM, nuclear forces give a positive contribution to ε_κ [44]). Consequently, a possible κ-instability in case of pure hard core interaction may be removed by attractive part of nuclear forces, or shifted to higher densities where all the new phenomena, mentioned in the preceding point should be considered.

We are now working on improving our method on investigating the κ-stability of dense NM. It appears particularly important to clarify the possibility of the $\sigma\tau$ instability. If it turns out that NM is $\sigma\tau$ unstable for $\varrho > \varrho'$, then one could interpret it in Landau theory as instability with respect to excitation of the spin-isospin wave [45]. The unstable state, developing from normal ground state, is locally spin and isospin polarized and is spatially inhomogeneous. Thus in this unstable state all the symmetries of the normal ground state are broken. However, due to this symmetry breaking the expectation value of the pion field does not vanish and one may consider (after taking account of pion-nucleon interaction in NM) a stable "abnormal" state of nuclear matter in which spatially inhomogeneous, locally spin and isospin polarized NM coexists with a pion condensate. This strongly oversimplified discussion shows the relevance of the possible $\sigma\tau$ instability of NM for the very much discussed problem of possible existence of pion condensate in dense NM [46–48].

APPENDIX A

1. Expressions for $\hat{\mathscr{E}}^{(3)}$

We apply Clark's notation (see, e. g., [26]), and write

$$\hat{\mathscr{E}}^{(3)}[f] = \hat{\mathscr{E}}_h + \hat{\mathscr{E}}_{hh} + \hat{\mathscr{E}}_t + \hat{\mathscr{E}}_{\mathrm{IV}}, \tag{A.1}$$

where the four-body cluster term linear in h is approximated, as suggested in [2], by

$$\hat{\mathscr{E}}_{\mathrm{IV}} \cong -(17/105)\hat{\mathscr{E}}_h. \tag{A.2}$$

In the case of the BP kinetic energy, we have:

$$\frac{1}{\mathcal{N}}\hat{\mathscr{E}}_t(\mathrm{BP}) = -\left(\frac{v}{6\pi^2}\right)^2 \int d\xi_{12}d\xi_{13}f_{23}^2 f_{12}f'_{12}f_{13}f'_{13}\hat{\xi}_{12}\hat{\xi}_{13}$$

$$\times\left[1+\frac{2}{v^2}l_{12}l_{13}l_{23}-\frac{1}{v}l_{23}^2-\frac{2}{v}l_{12}^2\right], \tag{A.3}$$

$$\frac{1}{\mathcal{N}}\hat{\mathscr{E}}_x(\mathrm{BP}) = \hat{\mathscr{E}}_x(\mathrm{BP},0)+\hat{\mathscr{E}}_x(\mathrm{BP},1) \quad \text{for} \quad x = h, hh, \tag{A.4}$$

where

$$\frac{1}{\mathcal{N}}\hat{\mathscr{E}}_h(\mathrm{BP},0) = \left(\frac{v}{6\pi^2}\right)^2 \int d\xi_{12}d\xi_{13}\mathscr{V}_{12}h_{13}\left[\frac{2}{v^2}l_{12}l_{13}l_{23}-\frac{2}{v}l_{23}^2\right], \tag{A.5}$$

$$\frac{1}{\mathcal{N}}\mathscr{E}_h(\mathrm{BP},1) = -\left(\frac{v}{6\pi^2}\right)^2 \int d\xi_{12}d\xi_{13}f_{12}f'_{12}h_{13}\left\{\frac{2}{v^2}[l_{13}l_{23}l'_{12}\right.$$

$$\left.+l_{23}l_{12}l'_{13}\hat{\xi}_{12}\hat{\xi}_{13}]-\frac{2}{v}l_{23}l'_{23}\hat{\xi}_{12}\hat{\xi}_{32}\right\}, \tag{A.6}$$

$$\frac{1}{\mathcal{N}}\hat{\mathscr{E}}_{hh}(\mathrm{BP},0) = \left(\frac{v}{6\pi^2}\right)^2 \int d\hat{\xi}_{12}d\hat{\xi}_{13}\mathscr{V}_{12}h_{13}h_{23}$$

$$\times\left[1+\frac{2}{v}l_{12}l_{13}l_{23}-\frac{1}{v}l_{12}^2-\frac{2}{v}l_{23}^2\right], \tag{A.7}$$

$$\frac{1}{\mathcal{N}}\hat{\mathscr{E}}_{hh}(\mathrm{BP},1) = -\left(\frac{v}{6\pi^2}\right)^2 \int d\hat{\xi}_{12}d\hat{\xi}_{13}f_{12}f'_{12}h_{13}h_{23}$$

$$\times\left\{\frac{2}{v^2}[l_{13}l_{23}l'_{12}+l_{12}l_{23}l'_{13}\hat{\xi}_{12}\hat{\xi}_{13}]-\frac{2}{v}l_{12}l'_{12}-\frac{2}{v}l_{13}l'_{13}\hat{\xi}_{12}\hat{\xi}_{13}\right\}, \tag{A.8}$$

where

$$\mathscr{V}(\xi) = -f(\xi)[f''(\xi)+2f'(\xi)/\xi]. \tag{A.9}$$

Here, we use the general notation: primes and double primes denote first and second derivatives with respect to the argument, subscript *ij* at a function indicates that the argument of the function is $\xi_{ij} = k_{\mathrm{F}v}r_{ij}$, and $\hat{\xi}_{ij} = \xi_{ij}/\xi_{ij}$.

In the case of the CW kinetic energy, we have

$$\hat{\mathscr{E}}_t(\mathrm{CW}) = -\hat{\mathscr{E}}_t(\mathrm{BP}), \tag{A.10}$$

and for $x = h, hh$:

$$\hat{\mathscr{E}}_x(\mathrm{CW}) = \hat{\mathscr{E}}_x(\mathrm{BP},0) \quad \text{with} \quad \mathscr{V}(\xi) \to f'(\xi)^2. \tag{A.11}$$

The scalar products $\hat{\xi}_{ij}\hat{\xi}_{ik}$ may be easily expressed by ξ_{ij}, ξ_{ik}, and ξ_{jk}, e. g., we have:

$$\hat{\xi}_{12}\hat{\xi}_{13} = (\xi_{12}^2+\xi_{13}^2-\xi_{23}^2)/2\xi_{12}\xi_{13}. \tag{A.12}$$

The integrals over ξ_{12} and ξ_{13} may be expressed as triple integrals over ξ_{12}, ξ_{13}, and ξ_{23}:

$$\int d\xi_{12}d\xi_{13} = 8\pi^2 \int_0^\infty d\xi_{12}\xi_{12} \int_0^\infty d\xi_{13}\xi_{13} \int_{|\xi_{12}-\xi_{13}|}^{\xi_{12}+\xi_{13}} d\xi_{23}\xi_{23}. \tag{A.13}$$

2. *Expressions for* $\hat{e}_{\mathrm{I}}^{(3)}$

We write

$$\hat{e}_{\mathrm{I}\nu}^{(3)}[f,\bar{f}] = \hat{e}_{\mathrm{I}}^{(3)} = e_h + e_{hh\mathrm{I}} + e_{hhN} + e_t, \tag{A.14}$$

where we have suppressed the subscript ν, to simplify notation.

In the case of the BP kinetic energy, we have

$$\hat{e}_t(\mathrm{BP}) = -\left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\{2\bar{f}_{02}^2\bar{f}_{01}\bar{f}'_{01}f_{12}f'_{12}\hat{\xi}_{01}\hat{\xi}_{21}$$

$$+f_{12}^2\bar{f}_{01}\bar{f}'_{01}\bar{f}_{02}\bar{f}'_{02}\hat{\xi}_{01}\hat{\xi}_{02}\}g_{\mathrm{F}\nu}(\xi_{12}), \tag{A.15}$$

$$\hat{e}_x(\mathrm{BP}) = \hat{e}_x(\mathrm{BP},0)+\hat{e}_x(\mathrm{BP},1) \quad \text{for} \quad x = h, hh\mathrm{I}, hh\mathrm{N}, \tag{A.16}$$

where

$$\hat{e}_h(\mathrm{BP},0) = -2\left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\overline{\mathscr{V}}_{01}\bar{h}_{02}\frac{1}{\nu}l_{12}^2, \tag{A.17}$$

$$\hat{e}_h(\mathrm{BP},1) = 2\left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\bar{f}_{01}\bar{f}'_{01}\bar{h}_{02}\frac{1}{\nu}l_{12}l'_{12}\hat{\xi}_{01}\hat{\xi}_{21}, \tag{A.18}$$

$$\hat{e}_{hh\mathrm{I}}(\mathrm{BP},0) = 2\left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\overline{\mathscr{V}}_{01}\bar{h}_{02}h_{12}g_{\mathrm{F}\nu}(\xi_{12}), \tag{A.19}$$

$$\hat{e}_{hh\mathrm{I}}(\mathrm{BP},1) = 2\left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\bar{f}_{01}\bar{f}'_{01}\bar{h}_{02}h_{12}\frac{1}{\nu}l_{12}l'_{12}\hat{\xi}_{01}\hat{\xi}_{21}, \tag{A.20}$$

$$\hat{e}_{hhN}(\mathrm{BP},0) = \left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}\mathscr{V}_{12}\bar{h}_{01}\bar{h}_{02}g_{\mathrm{F}\nu}(\xi_{12}), \tag{A.21}$$

$$\hat{e}_{hhN}(\mathrm{BP},1) = \left(\frac{\nu}{6\pi^2}\right)^2 \int d\xi_{01}d\xi_{02}f_{12}f'_{12}\bar{h}_{01}\bar{h}_{02}\frac{2}{\nu}l_{12}l'_{12}, \tag{A.22}$$

where

$$\overline{\mathscr{V}}(\xi) = -\bar{f}(\xi)\,[\bar{f}''(\xi)+2\bar{f}'(\xi)/\xi], \tag{A.23}$$

$$\bar{h}(\xi) = \bar{f}(\xi)^2-1. \tag{A.24}$$

Otherwise the notation is the same as in first part of this Appendix.

APPENDIX B

Spin instability condition in Landau theory

The condition for a ferromagnetic transition (which is equivalent to condition (2.3)) may be formulated in a simple way in Landau theory of normal Fermi liquids [38]. The central quantity in Landau theory is the quasi-particle interaction,

$$f(\cos\theta, \boldsymbol{\sigma}, \boldsymbol{\sigma}') = \frac{\hbar^2\pi^2}{M^* k_{F2}\Omega}[F(\cos\theta) + G(\cos\theta)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'], \tag{B.1}$$

where θ is the angle between the quasi-particle momenta, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are Pauli matrices, and M^* is a quasi-particle effective mass at the Fermi surface. The effective mass M^* of a quasi-particle is related to dimensionless parameter F_1,

$$F_1 = \tfrac{1}{3}\int_{-1}^{1} \frac{d\cos\theta}{2} F(\cos\theta)\cos\theta \tag{B.2}$$

by

$$M^*/M = 1 + \tfrac{1}{3}F_1. \tag{B.3}$$

Magnetic susceptibility of a Fermi liquid is determined by dimensionless parameter

$$G_0 = \int_{-1}^{1} \frac{d\cos\theta}{2} G(\cos\theta), \tag{B.4}$$

so that the spin symmetry energy of a Fermi liquid

$$\tilde{\varepsilon}_\sigma = \frac{\hbar^2 k_{F2}^2}{3M^*}(1+G_0). \tag{B.5}$$

Thus, condition (2.3) for spin instability of unpolarized ground state reads

$$G_0 < -1. \tag{B.6}$$

The quasi-particle interaction in a hard core Fermi gas has been calculated by Abrikosov and Khalatnikov [39], who used the quasi-potential method of Huang and Yang [8] and restricted themselves to quadratic approximation in x_2. They obtained

$$f(\cos\theta, \boldsymbol{\sigma}, \boldsymbol{\sigma}') = \frac{2\pi x_2\hbar^2}{\Omega M k_{F2}}\left[1 + \frac{2x_2}{\pi}\left(2 + \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\ln\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)\right]$$

$$- \frac{2\pi x_2\hbar^2}{\Omega M k_{F2}}\left[1 + \frac{2x_2}{\pi}\left(1 - \frac{\sin\frac{\theta}{2}}{2}\ln\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)\right]\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'. \tag{B.7}$$

From equations (B.1–B.3) and (B.7) one obtains the following formula for quasi-particle effective mass at the Fermi surface,

$$M/M^* = 1 - \frac{8}{15\pi^2}(7\ln 2 - 1)x_2^2. \tag{B.8}$$

The parameter G_0 may be calculated from equations (B.1, B.4, B.7):

$$G_0 = -2\frac{M^*}{M}\frac{x_2}{\pi}\left[1 + \frac{4x_2}{3\pi}(1 - \ln 2)\right]. \tag{B.9}$$

Hence, the condition for the spin instability of unpolarized ground state reads, in quadratic approximation in x_2,

$$-\frac{16x_2^2}{15\pi^2}(\ln 2 + 2) - \frac{2}{\pi}x_2 + 1 < 0. \tag{B.10}$$

Our formula for G_0 (and, in consequence, our condition (B.10)) differs from that given by Rice [31]. This difference results from an error in the calculation of G_0 in [31]. Our correct expression (B.9), inserted into (B.5) yields, within the quadratic approximation in x_2, a formula for $\tilde{\varepsilon}_\sigma$, which is identical with formula (3.10).

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