THEOREMS ON VECTOR FIELDS IN THE GENERAL RELATIVISTIC μ-SPACE

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The relativistic μ-space (state-space of a free particle in the gravitational field) is taken as a tangent bundle (fibre-space) over the space-time V_4 and investigated as a special Riemannian space V_8 with the methods of Ricci-calculus. Anholonomic coordinates are used to make clear the symmetries of V_8, to define tensors in V_8, to relate them to tensors of V_4, and to prove theorems on vector fields and multivector fields in V_8. Among these theorems is the relativistic Liouville-theorem in different forms.

1. Statistics of free particles in the gravitational field

Statistical thermodynamics investigates the macroscopic properties of systems which consist of a great number of subsystems (particles). The basis of the theory is formed by the laws which describe the behaviour of the subsystems and by methods which permit the application of the theory of probability. Instead of an exact solution of the equations of motion for a system of interacting particles, one's incomplete knowledge of the whole system is applied to make assumptions about the probability with which a certain solution for a subsystem will occur [1].

Therefore, statistical thermodynamics for a system of particles is a probability theory on a manifold of solutions (trajectories) of equations of motion of single particles (subsystems), in which the most important interactions may be included. This statement[1], which possibly represents the most fundamental hypothesis of statistical thermodynamics, should be accepted for the generalization of the usual theory whenever the special equations of motion (e.g., Hamiltonian equations of classical mechanics) would be substituted by others.

Accordingly, for a system of particles, which between their collisions move freely in the (exterior) gravitational field, one should not try to construct a relativistic statistics

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[1] The physical content of this hypothesis is that the macroscopic properties of the system must be defined by the properties of the subsystems and their interactions. This predicts that the system may be divided into subsystems which can be characterized by equations of motion including interactions [2].

which uses canonical equations of motion [9], but the equations of motion[2]

$$\frac{d^2x^i}{d\sigma^2} + \begin{Bmatrix} i \\ k \quad l \end{Bmatrix} \frac{dx^k}{d\sigma}\frac{dx^l}{d\sigma} = 0 \tag{1.1}$$

should be directly applied. The manifold of solutions of (1.1) is the manifold of all geodesics $x^k = x^k(x_0^k; \sigma)$ of the space-time V_4. This set of ∞^7 trajectories may be described uniquely in the manifold M_8 with the coordinates $(x^{\hat{\kappa}}) \sim (x^k, p^k)$. The tangent vectors of the geodesics having the components $p^k \equiv dx^k/d\sigma$ figure, firstly, as vectors which mark the directions of special trajectories. Secondly, they may be taken as 4-momentum of particles, if p^k has time-like or null direction. The degree of freedom $m \equiv \sqrt{-p^k p_k}$ characterizes an interior property (rest mass) of the particle. The parameter m remains unessential for the probability theory on the ∞^7 trajectories as long as all particles of the system have the same rest mass and collisions altering m are excluded (no decay, no fusion of particles).

The eight coordinates $(x^{\hat{\kappa}}) \sim (x^k, p^k)$ of a particle for a fixed value of the parameter σ on a time-like or null geodesic may be called the "state" of the particle. This would be in accordance with the possible interpretation "state of motion + interior property + spatial position + time" of the eight coordinates $(x^{\hat{\kappa}}) \sim (x^k, p^k)$ of the particle. Thus, M_8 may be called the manifold of the possible states of a particle or "relativistic state manifold".

In [3,4] the geometry of tangent bundles (fibre bundles) has been applied for the construction of a geometry[3], including the whole manifold M_8. We accept these ideas but use the methods of Riemannian geometry and Ricci-calculus [7] which are well known.

2. *The 8-dimensional state space*

The transformation of the coordinates $x^{k'} = x^{k'}(x^k)$ in V_4 and the linear orthogonal transformations of the coordinates p^k in the local pseudo-Euclidean tangent spaces E_4 of V_4 with the coefficients $A_k^{k'} \equiv \partial x^{k'}/\partial x^k$, which follow from the first ones, define the holonomic "transformations of states"

$$x^{k'} = x^{k'}(x^k), \quad p^{k'} = A_k^{k'} p^k \Leftrightarrow x^{\hat{\kappa}'} = x^{\hat{\kappa}'}(x^{\hat{\kappa}}) \tag{2.1}$$

in M_8. The corresponding groups may be written as ${}^{(8)}G(x^{\hat{\kappa}}) = {}^{(4)}G(x^k) \oplus {}^{(4)}L(p^k; x^k)$. Tensors in M_8 are quantities which transform like the differentials of coordinates $(dx^{\hat{\kappa}}) \equiv (dx^k, dp^k)$. For instance,

$$t^{\hat{\kappa}'} = A_{\hat{\kappa}}^{\hat{\kappa}'} t^{\hat{\kappa}}, \quad \left(A_{\hat{\kappa}}^{\hat{\kappa}'} \equiv \frac{\partial x^{\hat{\kappa}'}}{\partial x^{\hat{\kappa}}}\right). \tag{2.2}$$

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[2] The indices $h, i, \ldots = 1, \ldots, 4$; $\hat{\mu}, \hat{\nu}, \ldots = \hat{1}, \ldots, \hat{8}$ number holonomic coordinates; $\mu, \nu, \ldots = 1, \ldots, 8$ anholonomic coordinates.

[3] The Finslerian geometry (see e.g. [6]), as applied in [5], relates to a submanifold of M_8, which is equivalent to the manifold of ∞^7 trajectories in V_4 with a discrete depending on the direction norm of tangent vectors.

The transformation of coordinates in V_4 lead also to the anholonomic transformation of states

$$dx^{k'} = A^{k'}_{k} dx^{k}; \quad Dp^{k'} = A^{k'}_{k} Dp^{k} \Leftrightarrow dx^{\kappa'} = A^{\kappa'}_{\kappa} dx^{\kappa}. \tag{2.3}$$

According to (2.3) the components of tensors in M_8 transform like the differentials of coordinates $(dx^{\kappa}) \equiv (dx^{k}, Dp^{k})$ in M_8, which represent the direct sum of the differentials of coordinates in V_4 and the covariant differentials of p^k on V_4.

The relation between the holonomic and the anholonomic components of tensors in M_8 is given by the anholonomic transformation of coordinates

$$dx^{m} = dx^{\hat{m}}, \quad dx^{4+m} = dx^{\widehat{4+m}} + \left\{ {m \atop l \ n} \right\} p^{l} dx^{n} \tag{2.4}$$

and may be written as $t^{\hat{\kappa}} = A^{\hat{\kappa}}_{\kappa} t^{\kappa}$ etc., using the coefficients

$$A^{\hat{\nu}}_{\mu} = \delta^{\nu}_{\mu} + \delta^{\nu}_{4+n} \delta^{m}_{\mu} \left\{ {n \atop l \ m} \right\} p^{l}, \quad A^{\mu}_{\hat{\nu}} = \delta^{\mu}_{\nu} - \delta^{\mu}_{4+m} \delta^{n}_{\nu} \left\{ {m \atop l \ n} \right\} p^{l}, \tag{2.5}$$

where $\delta^{\cdot}_{\cdot}$ are Kronecker symbols.

Tensors in M_8 may be constructed by extensions of tensors in V_4. Such extended tensors with respect to anholonomic or holonomic coordinates are represented by the "horizontal extensions"[4]

$$({}^{(H)}t_{\kappa}) = (t_k, 0), \quad ({}^{(H)}t_{\hat{\kappa}}) = (t_k, 0), \tag{2.6}$$

$$({}^{(H)}t^{\kappa}) = (t^{k}, 0), \quad ({}^{(H)}t^{\hat{\kappa}}) = \left(t^{k}, - \left\{ {k \atop l \ m} \right\} p^{l} t^{m} \right), \tag{2.7}$$

the "vertical extensions",

$$({}^{(V)}t^{\kappa}) = (0, t^{k}), \quad ({}^{(V)}t^{\hat{\kappa}}) = (0, t^{k}), \tag{2.8}$$

$$({}^{(V)}t_{\kappa}) = (0, t_k), \quad ({}^{(V)}t_{\hat{\kappa}}) = \left(\left\{ {m \atop l \ k} \right\} p^{l} t_{m}, t_k \right), \tag{2.9}$$

the "direct extensions", e.g.,

$$(t^{\kappa}) = (t^{k}, t^{k}), \qquad (t^{\hat{\kappa}}) = \left(t^{k}, t^{k} - \left\{ {k \atop l \ m} \right\} p^{l} t^{m} \right), \tag{2.10}$$

and Sasaki's extended tensors [3] having the structure[5]

$$({}^{(S)}t^{\kappa}) = (t^{k}, t^{k}_{;l} p^{l}), \quad ({}^{(S)}t^{\hat{\kappa}}) = (t^{k}, t^{k}{}_{,l} p^{l}). \tag{2.11}$$

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[4] In [4] called "lift". Sasaki ([3], 1958) uses only the term "lift of a curve". According to [7], p. 78, the "extensions" might also be called "prolongations".

[5] The covariant derivation is indicated by (;), the partial one by (,). The signature of $V_4(+++-)$. Gothic kernels, e.g. $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}^1, \ldots, \mathfrak{t}, \mathfrak{p}, {}^{(H)}\mathfrak{p}, \mathfrak{u}, \ldots$, symbolize vectors. The calculus of basis vectors is given e.g. in [10], p. 53.

The field of basis vectors $\mathfrak{e}_m$, or, accordingly, $\mathfrak{e}^m$, of V_4 with the property $g_{mn} = \mathfrak{e}_m \cdot \mathfrak{e}_n$ after direct extension induces a field

$$(\mathfrak{e}_\mu) = (\mathfrak{e}_m, \mathfrak{e}_m), \quad (\mathfrak{e}_{\hat{\mu}}) = \left(\mathfrak{e}_m + \left\{ {k \atop l \quad m} \right\} p^l \mathfrak{e}_k, \mathfrak{e}_m\right) \tag{2.12}$$

of eight basis vectors $\mathfrak{e}_\mu$, which defines a metric field

$$(g_{\mu\nu}) = \begin{pmatrix} g_{mn} & 0 \\ 0 & g_{mn} \end{pmatrix}, \quad (g_{\hat{\mu}\hat{\nu}}) = \begin{pmatrix} g_{mn} + g_{ij} \left\{ {i \atop k \quad m} \right\} \left\{ {j \atop l \quad n} \right\} p^k p^l & g_{kn} \left\{ {k \atop l \quad m} \right\} p^l \\ g_{km} \left\{ {k \atop l \quad n} \right\} p^l & g_{mn} \end{pmatrix}, \tag{2.13}$$

in M_8. So the state manifold M_8 underlies a metrization to a Riemannian space V_8, which in accordance with [8] shall be called "relativistic state space". The choice of the metric of V_8 as a tangent bundle $\bigcup_{x \in V_4} E_4(x)$ is unambiguous. Any metric field in M_8 which has a physical sense must be defined by the metric of V_4 and the Euclidean metric of the local E_4 in all admissible systems of coordinates (see (2.1)) in M_8: The metrical fundamental form

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \tag{2.14}$$

of V_8 splits into a horizontal and a vertical term of a sum if taken in non-holonomic coordinates. For that reason, it may be spoken of that locally an anholonomic horizontal subspace of V_8 is stretched up by $\mathfrak{e}_m$ and an anholonomic vertical subspace of V_8 is spanned by $\mathfrak{e}_{4+m}$.

3. *Horizontal and vertical flows*

We regard the tangent vectors

$$\bar{\mathfrak{t}} = \frac{dx^\mu}{d\bar{\sigma}} \mathfrak{e}_\mu = \frac{dx^m}{d\bar{\sigma}} \mathfrak{e}_m + \frac{Dp^m}{d\bar{\sigma}} \mathfrak{e}_{4+m} \tag{3.1}$$

of the curves $x^{\hat{\mu}} = x^{\hat{\mu}}(x_0^{\hat{\mu}}; \bar{\sigma})$ in V_8 for the purpose of the interpretation of these curves of V_8 in the basic space V_4. A curve in V_8 shall be called a "horizontal (vertical) curve", if $\bar{\mathfrak{t}}$ in each of its points lies in the anholonomic horizontal, or, respectively, vertical, subspace.

A point P in V_8 having the coordinates $(x^{\hat{\mu}}) \sim (x^m, p^m)$ corresponds in V_4 to a point $P \sim (x^m)$ and a vector $\mathfrak{p} = p^m \mathfrak{e}_m$ at that point. This means that:

A. A horizontal curve $x^{\hat{\mu}} = x^{\hat{\mu}}(x_0^{\hat{\mu}}; \bar{\sigma})$ corresponds in V_4 to a curve $x^m = x^m(x_0^m; \sigma)$ and a sequence of parallelly transported vectors $\mathfrak{p}(x^m)$ at the points of this curve.

B. A vertical curve $x^{\hat{\mu}} = x^{\hat{\mu}}(x_0^{\hat{\mu}}; \bar{\sigma})$ corresponds to a bundle of ordered vectors $\mathfrak{p}(x_0^m) = p^m \mathfrak{e}_m$ at the point $P \sim (x_0^m)$ in V_4.

Arbitrary curves in V_8 may be defined inversely to the given interpretation by sequences of points with vectors attached to them or points with attached successions of vectors in V_4. The geodesics $x^m = x^m(x_0^m; \sigma)$ in V_4, defined by

$$\frac{dx^k}{d\sigma} = p^k, \quad \frac{Dp^k}{d\sigma} = 0, \tag{3.2}$$

together with their tangent vectors, correspond to horizontal curves in V_8, which we will call "horizontal flow"[6]. The radius vectors of points on straight lines through the points of contact (origin of coordinates) of E_4, tangent to V_4, correspond to vertical curves in V_8, which we will call "vertical flow".

Tangent vector fields of the horizontal and the vertical flows are

$$({}^{(H)}p^{\kappa}) = (p^k, 0) \quad \text{and} \quad ({}^{(V)}p^{\kappa}) = (0, p^k). \tag{3.3}$$

It is possible to include the condition of time-like or null orientation in the definition of the flows.

A remark should be added on the parametrization of the curves $x^{\hat\mu} = x^{\hat\mu}(x_0^{\hat\mu}; \overset{H}{\sigma})$ of the horizontal and the curves $x^{\hat\mu} = x^{\hat\mu}(x_0^{\hat\mu}; \overset{V}{\sigma})$ of the vertical flow. The parameters σ of the basic geodesics $x^m = x^m(x_0^m; \sigma)$ in V_4 may be taken as parameters $\overset{H}{\sigma} = \sigma$. For $\overset{V}{\sigma}$ we propose the use of parameters $\overset{V}{\sigma} = \overset{V}{\sigma}(m)$, depending on the length $m = (-p^k p_k)^{1/2}$ of the straight lines $p^k = p^k(p_0^k; m)$ through the origin of coordinates in E_4 which are the preimage of the vertical flow (cp. footnote 10 to equations (4.20)).

The distinction of the definitions of the horizontal and the vertical flow reminds the distinction of the free motion of the particles and their interactions in the kinetic theory. The number of geodesics with different directions through each of the ∞^4 points of V_4 is ∞^3. Hence, the manifold of all possible free motions of particles corresponding to the solutions of (1.1) is made up of ∞^7 curves and coincides with the trajectories of the horizontal flow. The number of curves $p^k = p^k(p_0^k; \mu)$ with different directions through each

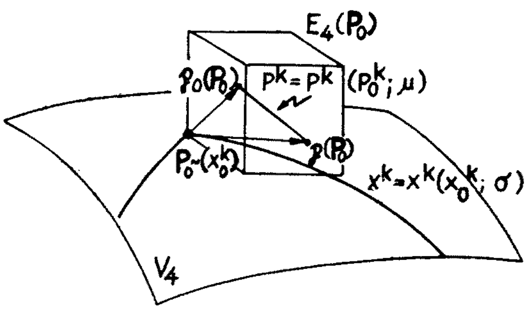

Fig. 1

of the ∞^4 points of the ∞^4 different tangent E_4 to V_4 is ∞^3 (cp. Fig. 1). So we have a manifold of ∞^{11} curves, which may be taken as characteristic lines of possible transitions of particle states by interactions which are regarded as instantaneous. For the purpose of illustration it might be supposed that the jump-like change of momentum at collision occurs along straight lines in E_4 with the directions $\mathfrak{p}(x_0^k) - \mathfrak{p}_0(x_0^k) = \tilde{\mathfrak{p}}(x_0^k; p^k, p_0^k)$. Since each E_4 is flat the vectors $\tilde{\mathfrak{p}}(x_0^k; p^k, p_0^k)$ with variable p_0^k may be taken as free and be transported to the points of contact $p^k = 0$ of the E_4 with V_4. In this way the ∞^4 bundles of

[6] In [3] — "geodesic flow". Indeed, these curves turn out to be geodesics in V_8 (cp. (4.19)).

straight lines in E_4 may be represented essentially by one bundle of straight lines fixed to the origin of coordinates of E_4. So a manifold of ∞^7 tangent straight lines to V_4 remains which coincides with the vertical flow.

4. *Theorems on vector fields in the state space*

In discussing fields of vectors or tensors in a space one has to predict that uniquely a vector or a tensor is given in each point of the space. This must be taken into account for the interconnection between tensor fields in V_4 and in V_8 since one point x^k in V_4 is connected with ∞^4 points in V_8. Above (cp. (2.6)–(2.10)) we have defined a tensor at a point $\bar{P} \sim (x^k, p^k)$ in V_8 as the extension of a tensor at a point $P \sim (x^k)$ in V_4. Taking for each point $\bar{P} \sim (x^k, p^k)$ in V_8 (p^k variable, x^k fixed) the same extension of the tensor in $P \sim (x^k)$, we may construct a tensor field in V_8 as an extension of a tensor field in V_4. The field $g_{\mu\nu}(x^{\hat{\kappa}})$ is an example of a direct extension of a field into V_8 (cp. (2.12), (2.13)). The tensor $g_{\mu\nu}(x^k, p^k)$ remains constant for variable p^k; the partial derivative $g_{\mu\nu,4+m}$ is zero.

Let us regard other extensions of fields. It is well known, that for the interpretation of the general theory of relativity in terms of classical physics, 3-frames of reference must be introduced which may be given by 3-dimensional arrangements of bodies with the world-lines $x^k = x^k(x_0^k; \tau)$ and the 4-velocities $u^k(\tau)$. Analogously, for setting up the relations between the relativistic state space and the classical phase space we must introduce congruences with the tangent vectors[7]

$$({}^{(\mathrm{H})}u^{\kappa}) = (u^k, 0), \qquad ({}^{(\mathrm{V})}u^{\kappa}) = (0, u^k). \tag{4.1}$$

The parameters $\overset{\mathrm{H}}{\tau}$ of the curves with the tangent field ${}^{(\mathrm{H})}u^{\kappa}(\overset{\mathrm{H}}{\tau})$ in V_8 may be identified with the parameters $\overset{\mathrm{H}}{\tau} = \tau$ (eigen-time) of the world lines $x^k = x^k(x_0^k; \tau)$ in V_4. The curves in V_8 with the tangent vector field ${}^{(\mathrm{V})}u^{\kappa}(\overset{\mathrm{V}}{\tau})$ correspond to congruences of parallel straight lines $p^k = p^k(p_0^k; \varepsilon)$ in the E_4 having the direction of the vectors u^k at the points of contact, and may be interpreted, according to (3.B), as successions of radius-vectors in the E_4 showing to the points of these parallel straight lines. The parameter $\overset{\mathrm{V}}{\tau}$ may be chosen as a function $\overset{\mathrm{V}}{\tau} = \overset{\mathrm{V}}{\tau}(\varepsilon)$ of the distance $\varepsilon = -p^k u_k$ between the origin of coordinates of E_4 and the 3-planes of constant relative energy which are orthogonal to the parallel straight lines.

In general, a tensor field in V_8 depends on all coordinates $x^{\hat{\kappa}}$, and it corresponds to a manifold of tensors in each point $P \sim (x^k)$ of V_4 (cp. (3.A), (3.B)). The components p^k of the tangent vectors in P may be taken as parameters for this manifold of tensors.[8] The fields of the tangent vectors of the horizontal and of the vertical flow in V_8 are not extensions of fields in V_4. At a point $P \sim (x^k)$ in V_4 such vectors correspond to ${}^{(\mathrm{H})}\mathfrak{p}$ or

[7] Normalization $u^k u_k = -1$.

[8] In the case of spherical tangent bundles ([3], 1962) or of the Finslerian geometry [5], the tensors would depend on the direction of the p^k only.

to ${}^{(V)}\mathfrak{p}$, which, after displacement to the origin of the local E_4, point to each of the points of E_4, or to the points of E_4 lying in the light-cone, if the convention after (3.3) should be applied.

The definition of the covariant derivative of geometrical objects in V_8 will be illustrated by the example of the covariant derivative of a vector $\mathfrak{V}$, respectively, to non-holonomic or to holonomic coordinates:

$$V^{\nu}{}_{;\mu} = V^{\nu}{}_{,\mu} + \Gamma^{\nu}_{\kappa\mu} V^{\kappa}, \qquad V^{\hat{\nu}}{}_{;\hat{\mu}} = V^{\hat{\nu}}{}_{,\hat{\mu}} + \begin{Bmatrix} \hat{\nu} \\ \hat{\kappa} \;\; \hat{\mu} \end{Bmatrix} V^{\hat{\kappa}}. \tag{4.2}$$

The partial derivative in non-holonomic coordinates is defined by $({}_{,\mu}) \equiv A^{\hat{\mu}}_{\mu}({}_{,\hat{\mu}})$. The relation between the coefficients of the affine connection $\Gamma^{\nu}_{\kappa\mu} = \mathfrak{e}_{\kappa,\mu} \cdot \mathfrak{e}^{\nu}$ and the Christoffel symbols $\begin{Bmatrix} \hat{\nu} \\ \hat{\kappa} \;\; \hat{\mu} \end{Bmatrix} = \mathfrak{e}_{\hat{\kappa},\hat{\mu}} \cdot \mathfrak{e}^{\hat{\nu}}$ of the metric $g_{\hat{\mu}\hat{\nu}}$ is (cp. [7,10])

$$\Gamma^{\nu}_{\kappa\mu} = A^{\nu}_{\hat{\nu}} A^{\hat{\mu}}_{\mu} A^{\hat{\kappa}}_{\kappa} \begin{Bmatrix} \hat{\nu} \\ \hat{\kappa} \;\; \hat{\mu} \end{Bmatrix} - A^{\hat{\kappa}}_{\kappa} A^{\hat{\mu}}_{\mu} A^{\nu}_{\underline{\hat{\kappa},\hat{\mu}}}. \tag{4.3}$$

The coefficients $\Gamma^{\nu}_{\kappa\mu}$ may be calculated also using the Christoffel symbols $\begin{Bmatrix} \nu \\ \kappa \;\; \mu \end{Bmatrix}$ of the anholonomic components $g_{\mu\nu}$ of the metrical tensor by

$$\Gamma^{\nu}_{\kappa\mu} = \begin{Bmatrix} \nu \\ \kappa \;\; \mu \end{Bmatrix} - [g^{\sigma\nu} g_{\kappa\varrho} \Omega_{\sigma\mu}{}^{\varrho} + g^{\sigma\nu} g_{\mu\varrho} \Omega_{\sigma\kappa}{}^{\varrho} - \Omega_{\kappa\mu}{}^{\nu}] \tag{4.4}$$

where the objects of anholonomity are

$$\Omega_{\kappa\mu}{}^{\nu} = \Gamma^{\nu}_{[\kappa\mu]} = A^{\nu}_{\hat{\nu}} A^{\hat{\nu}}_{[\kappa,\mu]}. \tag{4.5}$$

The computation of the components $\begin{Bmatrix} \hat{\nu} \\ \hat{\kappa} \;\; \hat{\mu} \end{Bmatrix}$ yields lengthy expressions, in which the curvature tensor $R^{k}{}_{lmn}$, the Christoffel symbols $\begin{Bmatrix} k \\ l \;\; m \end{Bmatrix}$ and the components p^k of the tangent vectors of V_4 appear. For the results see Sasaki ([3], 1958) or [11]. The transformation of Sasaki's formulae using (4.3) or a direct computation using (4.4) leads to:

$$\Gamma^{n}_{km} = \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix}, \quad \Gamma^{n}_{4+km} = \tfrac{1}{2} R^{n}{}_{mks} p^{s}, \quad \Gamma^{n}_{k4+m} = \tfrac{1}{2} R^{n}{}_{kms} p^{s}, \quad \Gamma_{km}{}^{4+n} = \tfrac{1}{2} R^{n}{}_{smk} p^{s},$$

$$\Gamma_{4+km}{}^{4+n} = \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix}, \quad \Gamma_{k4+m}^{4+n} = 0, \quad \Gamma_{4+k4+m}^{n} = 0, \quad \Gamma_{4+k4+m}^{4+n} = 0. \tag{4.6}$$

The only non-vanishing components of the object of anholonomity are

$$\Omega_{km}{}^{4+n} = \tfrac{1}{2} R^{n}{}_{smk} p^{s}, \quad \Omega_{4+km}{}^{4+n} = \tfrac{1}{2} \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix}, \quad \Omega_{k4+m}{}^{4+n} = -\tfrac{1}{2} \begin{Bmatrix} n \\ k \;\; m \end{Bmatrix}. \tag{4.7}$$

The advantage of the application of anholonomic coordinates in V_8 appears in the simplicity of the results (4.6), (4.7) and in the simplification of the proof of theorems on vector fields in the state space, which will be given below.

Let us regard an arbitrary differentiable tensor field $(t^\nu) = (t^n, t^{4+n})$ in V_8. The components of its covariant derivative are

$$(t^\nu{}_{;\mu}) = \left(\begin{array}{c:c} t^n{}_{,m} + \left\{ {n \atop k\ \ m} \right\} t^k + \frac{1}{2} R^n{}_{mks} p^s t^{4+k} & t^n{}_{,4+m} + \frac{1}{2} R^n{}_{kms} p^s t^k \\ \hdashline t^{4+n}{}_{,m} + \left\{ {n \atop k\ \ m} \right\} t^{4+k} + \frac{1}{2} R^n{}_{smk} p^s t^k & t^{4+n}{}_{,4+m} \end{array}\right). \tag{4.8}$$

If the horizontal and the vertical extensions $({}^{(\mathrm{H})}t^\kappa) = (t^k, 0)$ and $({}^{(\mathrm{V})}t^\kappa) = (0, t^k)$ of the field $t^k(x^l)$ in V_4 are taken, we obtain

$$({}^{(\mathrm{H})}t^\nu{}_{;\mu}) = \left(\begin{array}{c:c} t^n{}_{;m} & \frac{1}{2} R^n{}_{kms} p^s t^k \\ \hdashline \frac{1}{2} R^n{}_{smk} p^s t^k & 0 \end{array}\right), \quad ({}^{(\mathrm{V})}t^\nu{}_{;\mu}) = \left(\begin{array}{c:c} \frac{1}{2} R^n{}_{mks} p^s t^k & 0 \\ \hdashline t^n{}_{;m} & 0 \end{array}\right). \tag{4.9}$$

Here the symbol $(;m)$ has the meaning usually attributed to it in V_4.

With (4.8) it follows for the derivatives of the tangent fields of the horizontal and the vertical flow[9] that

$$({}^{(\mathrm{H})}p^\nu{}_{;\mu}) = \left(\begin{array}{c:c} 0 & \delta^n_m + \frac{1}{2} R^n{}_{kms} p^s p^k \\ \hdashline \frac{1}{2} R^n{}_{smk} p^s p^k & 0 \end{array}\right), \quad ({}^{(\mathrm{V})}p^\nu{}_{;\mu}) = \left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \delta^n_m \end{array}\right). \tag{4.10}$$

Using (4.9), (4.10) the following theorems (cp. [3], 1958, for A, C, D, E) may be shown:

A. The horizontal extension ${}^{(\mathrm{H})}\mathbf{t}$ of a field $\mathbf{t}$ in V_4 is geodesic (respectively incompressible, a gradient, normal to a hypersurface) if and only if $\mathbf{t}$ has the same property in V_4, i.e.,

$$t^m{}_{;n} t^n = 0 \Leftrightarrow {}^{(\mathrm{H})}t^\mu{}_{;\nu} {}^{(\mathrm{H})}t^\nu = 0, \tag{4.11}$$

$$t^m{}_{;m} = 0 \Leftrightarrow {}^{(\mathrm{H})}t^\mu{}_{;\mu} = 0, \tag{4.12}$$

$$t_{[m;n]} = 0 \Leftrightarrow {}^{(\mathrm{H})}t_{[\mu;\nu]} = 0, \tag{4.13}$$

$$t_{[m;n}t_{r]} = 0 \Leftrightarrow {}^{(\mathrm{H})}t_{[\mu;\nu} {}^{(\mathrm{H})}t_{\varrho]} = 0. \tag{4.14}$$

B. The vertical extension ${}^{(\mathrm{V})}\mathbf{t}$ of a field $\mathbf{t}$ in V_4 is geodesic and incompressible in V_8, i.e.,

$${}^{(\mathrm{V})}t^\nu{}_{;\mu} {}^{(\mathrm{V})}t^\mu = 0, \quad {}^{(\mathrm{V})}t^\nu{}_{;\nu} = 0; \tag{4.15}$$

it is a gradient, if the underlying field $\mathbf{t}$ in V_4 is covariantly constant, and it is hypersurface orthogonal, if the field $\mathbf{t}$ in V_4 is recurrent (with k_n arbitrary), i.e.,

$$t_{m;n} = 0 \quad \Leftrightarrow \quad {}^{(\mathrm{V})}t_{[\mu;\nu]} = 0, \tag{4.16}$$

$$t_{m;n} = t_m k_n \Leftrightarrow {}^{(\mathrm{V})}t_{[\mu;\nu} {}^{(\mathrm{V})}t_{\varrho]} = 0. \tag{4.17}$$

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[9] Here the independence of coordinates in V_8 is applied, which means that $\partial x^{\hat\kappa}/\partial x^{\hat\mu} = \delta^{\hat\kappa}_{\hat\mu}$ or $\partial x^\kappa/\partial x^\mu = \delta^\kappa_\mu$.

C. The Sasaki extensions (2.11) of an incompressible field $\mathbf{t}$ in V_4 is incompressible in V_8, i.e.

$$^{(S)}t^{\kappa}{}_{;\kappa} = 2t^{k}{}_{;k} = 0. \tag{4.18}$$

D. The tangent vector field $^{(H)}\mathfrak{p}$ of the horizontal flow is geodesic and incompressible, i.e.,

$$^{(H)}p^{\kappa}{}_{;\mu}{}^{(H)}p^{\mu} = 0, \qquad {}^{(H)}p^{\kappa}{}_{;\kappa} = 0. \tag{4.19}$$

E. The tangent vector field $^{(V)}\mathfrak{p}$ of the vertical flow is geodesic and a gradient field, i.e.

$$^{(V)}p^{\kappa}{}_{;\mu}{}^{(V)}p^{\mu} = {}^{(V)}p^{\kappa}\,{}^{10}, \qquad {}^{(V)}p_{[\kappa;\mu]} = 0\,{}^{11}. \tag{4.20}$$

F. The Lie derivatives[12] of the fields $^{(H)}\mathfrak{p}$, $^{(V)}\mathfrak{p}$, $^{(H)}\mathfrak{u}$, $^{(V)}\mathfrak{u}$ defined by (3.3), (4.1) with respect to each other have the results given in Table I. There, the following notations are applied:

$$^{(H)}\dot{u}_{\kappa} = (\dot{u}_k, 0), \qquad {}^{(V)}\dot{u}_{\kappa} = (0, \dot{u}_k) \tag{4.21}$$

TABLE I

	$\underset{(H)p}{\mathscr{L}}$	$\underset{(V)p}{\mathscr{L}}$	$\underset{(H)u}{\mathscr{L}}$	$\underset{(V)u}{\mathscr{L}}$
$^{(H)}p^{\kappa}$	0	$^{(H)}p^{\kappa}$ *	$-{}^{(H)}l^{\kappa} - {}^{(V)}r^{\kappa}$	$^{(H)}u^{\kappa} - {}^{(V)}l^{\kappa}$
$^{(H)}p_{\kappa}$	$^{(V)}p_{\kappa}$ *	$^{(H)}p_{\kappa}$ *	$-\varepsilon_{,\kappa} - {}^{(V)}u_{\kappa}$	$^{(H)}u_{\kappa}$
$^{(V)}p^{\kappa}$	$-{}^{(H)}p^{\kappa}$ *	0	0	$^{(V)}u^{\kappa}$ *
$^{(V)}p_{\kappa}$	0	$2{}^{(V)}p_{\kappa}$ *	0	$-\varepsilon_{,\kappa}$
$^{(H)}u^{\kappa}$	$^{(H)}l^{\kappa} + {}^{(V)}r^{\kappa}$	0	0	$-{}^{(V)}\dot{u}^{\kappa}$
$^{(H)}u_{\kappa}$	$^{(H)}l_{\kappa} + {}^{(V)}u_{\kappa}$	0	$^{(H)}\dot{u}_{\kappa}$	0
$^{(V)}u^{\kappa}$	$-{}^{(H)}u^{\kappa} + {}^{(V)}l^{\kappa}$	$-{}^{(V)}u^{\kappa}$ *	$^{(V)}\dot{u}^{\kappa}$	0
$^{(V)}u_{\kappa}$	$-{}^{(H)}r_{\kappa} + {}^{(V)}l_{\kappa}$	$^{(V)}u_{\kappa}$ *	$^{(H)}k_{\kappa} + {}^{(V)}\dot{u}_{\kappa}$	0

[10] Using $^{(V)}p_{\kappa}\,{}^{(V)}p^{\kappa} = -m^2$ and the directional derivative $D/d\overset{V}{\sigma} \equiv (;\mu)\ {}^{(V)}p^{\mu}$ we find the relation $\overset{V}{\sigma} = \text{const} \cdot e^{m}$ between the parameter $\overset{V}{\sigma}$ and the function $m(x^{\hat{\kappa}})$ of the particle state.

[11] With (4.10) we obtain $^{(V)}p_{\mu} = -mm_{,\mu}$ and $g_{mn} = -(m^2/2)_{;4+m;4+n}$ (cp. the Finslerian geometry [6]).

[12] The definition of the Lie derivative of multivector fields $s^{\kappa_1\kappa_2\ldots\kappa_p} = \underset{1}{s}^{[\kappa_1}\underset{2}{s}^{\kappa_2}\ldots\underset{p}{s}^{\kappa_p]}$ and $s_{\kappa_1\kappa_2\ldots\kappa_p}$ with respect to the field t^{κ} (cp. [7] p. 106) may be written in the form $\underset{t}{\mathscr{L}}s^{\kappa_1\kappa_2\ldots\kappa_p} \equiv t^{\mu}s^{\kappa_1\kappa_2\ldots\kappa_p}{}_{;\mu} - pt^{[\kappa_p|}{}_{;\mu}s^{|\kappa_1\kappa_2\ldots\kappa_{p-1}]\mu}$, $\underset{t}{\mathscr{L}}s_{\kappa_1\kappa_2\ldots\kappa_p} \equiv t^{\mu}s_{\kappa_1\kappa_2\ldots\kappa_p;\mu} + pt^{\mu}{}_{;[\kappa_p}s_{\kappa_1\kappa_2\ldots\kappa_{p-1}]\mu}$ being applicable for anholonomic components as well.

extensions of the 4-acceleration field $\dot{u}_k \equiv u_{k;m}u^m$ in V_4;

$$\varepsilon_{,\mu} = -(u_{s;m}p^s, u_m) \tag{4.22}$$

gradient in V_8 of the relative energy $\varepsilon = -{}^{(H)}p^{\kappa(H)}u_k = -{}^{(V)}p^{\kappa(V)}u_\kappa$ of a particle with p^k ($\varepsilon \equiv -p^k u_k$) in V_4;

$${}^{(H)}r_\kappa = (R_{kstn}p^s p^t u^n, 0), \qquad {}^{(V)}r_\kappa = (0, R_{kstn}p^s p^t u^n) \tag{4.23}$$

$${}^{(H)}k_\kappa = (R_{kmls}u^l u^m p^s, 0), \tag{4.24}$$

$${}^{(H)}l_\kappa = (u_{k;s}p^s, 0), \qquad {}^{(V)}l_\kappa = (0, u_{k;s}p^s) \tag{4.25}$$

fields in V_8.

Instead of the results in Table I indicated by (*), vanishing Lie derivatives may be produced after changing the norm either of the derivated field or of the field with respect to which the derivative is taken by factors depending on m, e.g.,

$$\underset{{}^{(H)}p/m}{\mathscr{L}}\, {}^{(H)}p_\kappa = 0, \quad \underset{{}^{(V)}p}{\mathscr{L}}\, ({}^{(H)}p^\kappa/m) = 0, \quad \underset{m\cdot{}^{(V)}u}{\mathscr{L}}\, {}^{(V)}p^\kappa = 0. \tag{4.26}$$

TABLE II

	$\underset{{}^{(H)}p}{\mathscr{L}}$	$\underset{{}^{(V)}p}{\mathscr{L}}$	$\underset{{}^{(H)}u}{\mathscr{L}}$	$\underset{{}^{(V)}u}{\mathscr{L}}$
$p^{\kappa\lambda}$	0	$p^{\kappa\lambda}$ *	$-({}^{(H)}l^{[\kappa}+{}^{(V)}r^{[\kappa})^{(V)}p^{\lambda]}$	$\underline{q^{\kappa\lambda}}+v^{\kappa\lambda}-{}^{(V(}l^{[\kappa(V)}p^{\lambda]}$ *
$p_{\kappa\lambda}$	0	$3p_{\kappa\lambda}$ *	$r_{\kappa\lambda}-\varepsilon_{,[\kappa}{}^{(V)}p_{\lambda]}$	$v_{\kappa\lambda}-{}^{(H)}p_{[\kappa}\varepsilon_{,\lambda]}$
$q^{\kappa\lambda}$	$w^{\kappa\lambda}+{}^{(H)}p^{[\kappa(V)}l^{\lambda]}$	0	${}^{(H)}p^{[\kappa(V)}\dot{u}^{\lambda]}-({}^{(H)}l^{[\kappa}$ $+{}^{(V)}r^{[\kappa})^{(V)}u^{\lambda]}$	$u^{\kappa\lambda}-{}^{(V)}l^{[\kappa(V)}u^{\lambda]}$
$q_{\kappa\lambda}$	$\underline{r_{\kappa\lambda}}+{}^{(H)}p_{[\kappa}({}^{(V)}l_{\lambda]}$ $\underline{-}{}^{(H)}r_{\lambda]})$ *	$2q_{\kappa\lambda}$ *	${}^{(H)}p_{[\kappa}({}^{(H)}k_{\lambda]}+{}^{(V)}\dot{u}_{\lambda]})$ $-\varepsilon_{,[\kappa}{}^{(V)}u_{\lambda]}$	$u_{\kappa\lambda}$
$r^{\kappa\lambda}$	$v^{\kappa\lambda}-q^{\kappa\lambda}+{}^{(V)}p^{[\kappa(V)}l^{\lambda]}$	$-r^{\kappa\lambda}$ *	${}^{(V)}p^{[\kappa(V)}\dot{u}^{\lambda]}$	0
$r_{\kappa\lambda}$	${}^{(V)}\overline{p}_{[\kappa}({}^{(V)}l_{\lambda]}-{}^{(H)}r_{\lambda]})$	$3r_{\kappa\lambda}$ *	${}^{(V)}p_{[\kappa}({}^{(H)}k_{\lambda]}+{}^{(V)}\dot{u}_{\lambda]})$	$-\varepsilon_{,[\kappa}{}^{(V)}u_{\lambda]}$
$u^{\kappa\lambda}$	$({}^{(H)}l^{[\kappa}+{}^{(V)}r^{[\kappa})^{(V)}u^{\lambda]}$ $+{}^{(H)}u^{[\kappa(V)}l^{\lambda]}$	$-u^{\kappa\lambda}$ *	${}^{(H)}u^{[\kappa(V)}\dot{u}^{\lambda]}$	$-{}^{(V)}\dot{u}^{[\kappa(V)}u^{\lambda]}$
$u_{\kappa\lambda}$	${}^{(H)}l_{[\kappa}{}^{(V)}u_{\lambda]}$ $+{}^{(H)}u_{[\kappa}({}^{(V)}l_{\lambda]}-{}^{(H)}r_{\lambda]})$	$u_{\kappa\lambda}$ *	${}^{(H)}\dot{u}_{[\kappa}{}^{(V)}u_{\lambda]}$ $+{}^{(H)}u_{]\kappa}({}^{(H)}k_{\lambda]}+{}^{(V)}\dot{u}_{\lambda]})$	0
$v^{\kappa\lambda}$	$({}^{(H)}l^{[\kappa}+{}^{(V)}r^{[\kappa})^{(V)}p^{\lambda]}$ $\underline{-w^{\kappa\lambda}}$ *	0	0	$\underline{u^{\kappa\lambda}}-{}^{(V)}\dot{u}^{[\kappa(V)}p^{\lambda]}$ *
$v_{\kappa\lambda}$	${}^{(H)}l_{[\kappa}{}^{(V)}p_{\lambda]}-r_{\kappa\lambda}$	$2v_{\kappa\lambda}$ *	${}^{(H)}\dot{u}_{[\kappa}{}^{(V)}p_{\lambda]}$	$-{}^{(H)}u_{[\kappa}\varepsilon_{,\lambda]}$
$w^{\kappa\lambda}$	$({}^{(H)}l^{[\kappa}+{}^{(V)}r^{[\kappa})^{(H)}p^{\lambda]}$	$w^{\kappa\lambda}$	$-{}^{(H)}u^{[\kappa}({}^{(H)}l^{\lambda]}+{}^{(V)}r^{\lambda]})$	$-{}^{(V)}\dot{u}^{[\kappa(H)}p^{\lambda]}$ $-{}^{(H)}u^{[\kappa(V)}l^{\lambda]}$
$w_{\kappa\lambda}$	$v_{\kappa\lambda}-q_{\kappa\lambda}+{}^{(H)}l_{[\kappa}{}^{(H)}p_{\lambda]}$	$w_{\kappa\lambda}$ *	${}^{(H)}\dot{u}_{[\kappa}{}^{(H)}p_{\lambda]}-{}^{(H)}u_{[\kappa}\varepsilon_{,\lambda]}$ $-u_{\kappa\lambda}$	0

This is proved easily using ${}^{(V)}p_\mu = -mm_{,\mu}$ and the definitions[11,12]. In particular, a consequence of this is

F. The fields ${}^{(H)}p^\kappa/m, \quad {}^{(V)}p^\kappa, \quad {}^{(H)}u^\kappa, \quad m{}^{(V)}u^\kappa,$

$${}^{(H)}p_\kappa/m^2, \quad {}^{(V)}p_\kappa/m^2, \quad {}^{(H)}u_\kappa, \quad {}^{(V)}u_\kappa/m$$

and the multivector fields being defined as alternating products of them (cp. (4.27), (4.28)) are absolutely invariant with respect to the vertical flow field ${}^{(V)}p^\kappa$ (i.e., have vanishing $\underset{{}^{(V)}p}{\mathscr{L}}$).

G. The Lie derivatives of the bivector fields

$$p_{\kappa\lambda} = {}^{(H)}p_{[\kappa}{}^{(V)}p_{\lambda]}, \quad q_{\kappa\lambda} = {}^{(H)}p_{[\kappa}{}^{(V)}u_{\lambda]}, \quad r_{\kappa\lambda} = {}^{(V)}p_{[\kappa}{}^{(V)}u_{\lambda]},$$
$$u_{\kappa\lambda} = {}^{(H)}u_{[\kappa}{}^{(V)}u_{\lambda]}, \quad v_{\kappa\lambda} = {}^{(H)}u_{[\kappa}{}^{(V)}p_{\lambda]}, \quad w_{\kappa\lambda} = {}^{(H)}u_{[\kappa}{}^{(H)}p_{\lambda]} \tag{4.27}$$

with respect to the fields ${}^{(H)}\mathfrak{p}$, ${}^{(V)}\mathfrak{p}$, ${}^{(H)}\mathfrak{u}$, ${}^{(V)}\mathfrak{u}$ have the results given in Table II. There, in the same way as above, instead of (*), results may be obtained which vanish (cp. F′) or in which the underlined terms cancel, e.g., $\underset{m\cdot{}^{(V)}u}{\mathscr{L}} v^{\kappa\lambda} = -m{}^{(V)}\dot{u}^{[\kappa(V)}p^{\lambda]}$.

TABLE III

	$\underset{{}^{(H)}p}{\mathscr{L}}$
$p^{\mu\lambda\kappa}$	$p^{[\kappa\lambda(V)}l^{\mu]} - v^{\kappa\lambda\mu}$
$p_{\kappa\lambda\mu}$	$p_{[\kappa\lambda}({}^{(V)}l_{\mu]} - {}^{(H)}r_{\mu]})$
$q^{\kappa\lambda\mu}$	$-w^{[\kappa\lambda(V)}l^{\mu]} - q^{[\kappa\lambda}({}^{(H)}l^{\mu]} + {}^{(V)}r^{\mu]})$
$q_{\kappa\lambda\mu}$	$u_{\kappa\lambda\mu} - q_{[\kappa\lambda}{}^{(H)}l_{\mu]} + w_{[\kappa\lambda}({}^{(H)}r_{\mu]} - {}^{(V)}l_{\mu]})$
$u^{\kappa\lambda\mu}$	$-q^{\kappa\lambda\mu} - v^{[\kappa\lambda(V)}l^{\mu]} - r^{[\kappa\lambda}({}^{(H)}l^{\mu]} + {}^{(V)}r^{\mu]})$
$u_{\kappa\lambda\mu}$	$v_{[\kappa\lambda}({}^{(H)}r_{\mu]} - {}^{(V)}l_{\mu]}) - r_{[\kappa\lambda}{}^{(H)}l_{\mu]}$
$v^{\kappa\lambda\mu}$	$p^{[\kappa\lambda}({}^{(H)}l^{\mu]} + {}^{(V)}r^{\mu]})$
$v_{\kappa\lambda\mu}$	$p_{\kappa\lambda\mu} + p_{[\kappa\lambda}{}^{(H)}l_{\mu]}$

H. The Lie derivatives of the trivector fields

$$p_{\kappa\lambda\mu} = {}^{(H)}p_{[\kappa}{}^{(V)}p_\lambda{}^{(V)}u_{\mu]}, \quad q_{\kappa\lambda\mu} = {}^{(H)}p_{[\kappa}{}^{(H)}u_\lambda{}^{(V)}u_{\mu]},$$
$$u_{\kappa\lambda\mu} = {}^{(H)}u_{[\kappa}{}^{(V)}u_\lambda{}^{(V)}p_{\mu]}, \quad v_{\kappa\lambda\mu} = {}^{(H)}u_{[\kappa}{}^{(H)}p_\lambda{}^{(V)}p_{\mu]} \tag{4.28}$$

with respect to the horizontal flow field ${}^{(H)}p^\kappa$ have the results given in Table III.[13]

[13] Other Lie derivatives of these trivector fields may be computed with the results in Tables I, II applying the rule of Leibniz for products.

I. Divergences of multivectors defined by (4.27), (4.28) are[14]

$$p^{\kappa\lambda}{}_{;\lambda} = \tfrac{5}{2}\,{}^{(H)}p^{\kappa}, \quad q^{\kappa\lambda}{}_{;\lambda} = \tfrac{1}{2}\,{}^{(H)}u^{\kappa} - \tfrac{1}{2}\,{}^{(V)}l^{\kappa}, \quad r^{\kappa\lambda}{}_{;\lambda} = -\tfrac{3}{2}\,{}^{(V)}u^{\kappa}, \tag{4.29}$$

$$u^{\kappa\lambda}{}_{;\lambda} = -\tfrac{1}{2}\,{}^{(V)}\dot{u}^{\kappa} - \tfrac{1}{2}\,u^{l}{}_{;l}{}^{(V)}u^{\kappa}, \quad v^{\kappa\lambda}{}_{;\lambda} = 2\,{}^{(H)}u^{\kappa} - \tfrac{1}{2}\,u^{l}{}_{;l}{}^{(V)}p^{\kappa}, \tag{4.30}$$

$$p^{\kappa\lambda\mu}{}_{;\mu} = \tfrac{1}{3}\,v^{\kappa\lambda} - \tfrac{4}{3}\,q^{\kappa\lambda} - \tfrac{1}{3}\,{}^{(V)}l^{[\kappa\,(V)}p^{\lambda]}, \tag{4.31}$$

$$u^{\kappa\lambda\mu}{}_{;\mu} = u^{\kappa\lambda} + \tfrac{1}{3}\,{}^{(V)}\dot{u}^{[\kappa\,(V)}p^{\lambda]} - \tfrac{1}{3}\,u^{l}{}_{;l}r^{\kappa\lambda}. \tag{4.32}$$

5. *Concluding remarks*

Among the above theorems the incompressibility (4.19) of the horizontal flow and the first equation in (4.29) according to [8] and [4] may be regarded as relativistic generalizations of the Liouville theorem of classical statistical mechanics. The results for the Lie-derivatives of fields with respect to the horizontal flow field given in Tables I, II express equivalent generalizations of this classical theorem (cp. [11]).

Here as internal property of the particles their rest mass only has been taken into account. Similar theorems as the above are of importance for the description of the motion and the statistical mechanics of particles with internal degrees of freedom. The proof of the theorems given above is a first step toward dealing with such problems.

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[14] These and other divergences can be obtained using the formulae

$$a^{\kappa\lambda}{}_{;\lambda} = \tfrac{1}{2}(\underset{s}{\mathscr{L}}r^{\kappa} + s^{\lambda}{}_{;\lambda}r^{\kappa} - r_{\lambda;\lambda}s^{\kappa}),$$

$$a^{\kappa\lambda\mu}{}_{;\mu} = \tfrac{1}{3}\,(\underset{t}{\mathscr{L}}a^{\kappa\lambda} + t^{\mu}{}_{;\mu}a^{\kappa\lambda} + 2a^{\mu[\kappa|}{}_{;\mu}t^{|\lambda]}$$

for $a^{\kappa\lambda} = r^{[\kappa}s^{\lambda]}$, $a^{\kappa\lambda\mu} = a^{[\kappa\lambda}t^{\mu]}$ and application of the theorems (A—G).