Vol. B10 (1979) ACTA PHYSICA POLONICA No 10

FURTHER REMARKS ON THE GLOBAL SUPERENERGETIC QUANTITIES OF A CLOSED SYSTEM AND ON THE PHYSICAL MEANING OF THE SUPERENERGY

By J. Garecki

Department of Physics, Pedagogical University, Szczecin*

(Received March 30, 1979)

In this paper we discuss uniqueness of the global superenergetic quantities of a closed system in GRT = general relativity theory and in ECT = the Einstein - Cartan theory the time dependence of global superenergetic quantities of a general closed system and the physical meaning of superenergy.

1. On the uniqueness of the global superenergetic quantities of a closed system in GRT

In the framework of GRT one considers many different energy-momentum complexes [2–6]. These complexes are globally equivalent for Schwarzschild's solution in the ALC in which the system is at rest. Locally, these complexes are not equivalent. In consequence, the total superenergy tensors corresponding to different energy-momentum complexes are different, e.g., the total superenergy tensor ${}_{\mathrm{E}}^{\mathrm{t}}S_{\mu .}^{\cdot \nu}(P; v^{\varrho})$ calculated from Einstein's energy-momentum complex, ${}_{\mathrm{E}}K_{\mu .}^{\cdot \nu}$, has the following analytic form

$$
\begin{aligned}
{}_{\mathrm{E}}^{\mathrm{t}}S_{\mu .}^{\cdot \nu}(P; v^{\varrho}) = \frac{2k}{9}(2v^{\alpha}v^{\beta}-g^{\alpha\beta})\big[&T^{\nu \cdots}_{\cdot \mu\alpha\beta}+\bar{T}^{\nu \cdots}_{\cdot \mu\alpha\beta}-\tfrac{1}{2}\,\delta^{\nu}_{\mu}K^{\gamma\delta\varrho\cdot}_{\cdots\beta}(K^{\cdots\cdot}_{\gamma\delta\varrho\alpha}+K^{\cdots\cdot}_{\gamma\varrho\delta\alpha}) \\
&+2\delta^{\nu}_{\mu}\bar{k}^{2}\,{}_{\mathrm{E}}T^{\cdot\ \gamma}_{(\alpha|\cdot}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{\gamma|\beta)}-3\bar{k}^{2}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{\mu(\alpha|}\,{}_{\mathrm{E}}T^{\nu\ \cdot}_{\cdot\ |\beta)}-2\bar{k}K^{(\gamma\cdot\nu)\cdot}_{\cdot\mu\cdot(\alpha}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{|\gamma|\beta)}\big] \\
&+(2v^{\alpha}v^{\beta}-g^{\alpha\beta})\big[\overset{*}{\nabla}_{(\alpha}\overset{*}{\nabla}_{\beta)}T_{\mu .}^{\cdot \nu}+\tfrac{1}{3}\,K^{\lambda\ \cdot\ \cdot\ \cdot}_{\cdot(\alpha|\mu|\beta)}T_{\lambda .}^{\cdot \nu}-\tfrac{1}{3}\,K^{\nu\ \cdot\lambda\cdot}_{\cdot(\alpha\cdot\beta)}T^{\cdot\ \cdot}_{\mu\lambda}\big],
\end{aligned}
\tag{1}
$$

while the total superenergy tensor ${}_{\mathrm{LL}}^{\mathrm{t}}S^{\mu\alpha}_{\cdot\ \cdot}(P; v^{\varrho})$ calculated from Landau-Lifshits' energy-momentum complex, ${}_{\mathrm{LL}}K^{\mu\nu}_{\cdot\ \cdot}$, possesses the form

$$
\begin{aligned}
{}_{\mathrm{LL}}^{\mathrm{t}}S^{\mu\alpha}_{\cdot\ \cdot}(P; v^{\varrho}) = k(2v^{\gamma}v^{\delta}-g^{\gamma\delta})\big[&\tfrac{4}{9}\,T^{\alpha\mu\cdot\cdot}_{\cdot\cdot\gamma\delta}+\tfrac{2}{3}\,K^{\alpha\lambda\sigma}_{\cdots(\gamma|}K^{\mu\cdot\ \cdot}_{\cdot\sigma\lambda|\delta)}-\tfrac{2}{9}\,g^{\mu\alpha}K^{\lambda\sigma\varrho\cdot}_{\cdots\gamma}K^{\cdots\cdot}_{\lambda\varrho\sigma\delta} \\
&+\tfrac{10}{9}\,\bar{k}^{2}g^{\mu\alpha}\,{}_{\mathrm{E}}T^{\cdot\ \lambda}_{(\gamma|\cdot}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{\lambda|\delta)}-2\bar{k}^{2}\,{}_{\mathrm{E}}T^{\mu\ \cdot}_{\cdot(\gamma|}\,{}_{\mathrm{E}}T^{\alpha\ \cdot}_{\cdot\ |\delta)}-\tfrac{10}{9}\,\bar{k}K^{\lambda\mu\alpha\ \cdot}_{\cdots(\gamma|}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{\lambda|\delta)}+\tfrac{10}{9}\,\bar{k}\,{}_{\mathrm{E}}T^{\cdot\ \cdot}_{\lambda(\lambda|}K^{\alpha\lambda\mu\ \cdot}_{\cdots|\delta)}
\end{aligned}
$$

* Address: Zakład Fizyki, Wyższa Szkoła Pedagogiczna, Wielkopolska 15, 70-451 Szczecin, Poland.

$$+\tfrac{2}{9}\, K^{\mu\,\cdot\,\sigma\lambda}_{\cdot\,(\gamma|\,\cdot\,\cdot} K^{\alpha\,\cdot\,\cdot\,\cdot}_{\cdot\,|\delta)\sigma\lambda} + \tfrac{2}{9}\, K^{\mu\lambda\sigma\,\cdot}_{\cdot\,\cdot\,\cdot\,(\gamma|} K^{\alpha\,\cdot\,\cdot\,\cdot}_{\cdot\,|\delta)\sigma\lambda} + \tfrac{2}{9}\, K^{\mu\,\cdot\,\sigma\lambda}_{\cdot\,(\gamma|\,\cdot\,\cdot} K^{\alpha\,\cdot\,\cdot\,\cdot}_{\cdot\,\lambda\sigma|\delta)}]$$

$$+(2v^\gamma v^\delta - g^{\gamma\delta})\,[\overset{*}{\nabla}_{(\gamma}\overset{*}{\nabla}_{\delta)} T^{\mu\alpha}_{\cdot\,\cdot} - \tfrac{1}{3}\, K^{\mu\,\cdot\,\cdot\,\cdot}_{\cdot\,(\gamma|\lambda|\delta)} T^{\lambda\alpha}_{\cdot\,\cdot} - \tfrac{1}{3}\, K^{\alpha\,\cdot\,\cdot\,\cdot}_{\cdot\,(\gamma|\lambda|\delta)} T^{\lambda\mu}_{\cdot\,\cdot}]. \tag{2}$$

By direct calculation one can immediately see that the global superenergetic quantities calculated with the help of (1) are different from the global superenergetic quantities calculated with the help of[1]

$${}_{\mathrm{LL}}^{t}S_{\mu\cdot}^{\cdot\alpha}(P;v^\varrho) := g_{\mu\nu}\,{}_{\mathrm{LL}}^{t}S^{\nu\alpha}_{\cdot\,\cdot}(P;v^\varrho)$$

already in the case of the simplest closed system described by Schwarzschild's solution of Einstein equations.

This would suggest that the global superenergetic quantities of a closed system are dependent on which total superenergy tensor is used, i.e., this would suggest the non-uniqueness of the global superenergetic quantities of a given closed system.

Difficulties of the same kind, although to lesser degree, appear also in the framework of the energetic global quantities of a given, general closed system [7]: different energy-momentum complexes give different values of global energetic quantities of a general closed system or give the same values as Einstein's, canonical energy-momentum complex ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$ and Bergmann's single index complex, ${}_{\mathrm{B}}J^{\mu}$, which is based on ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$, only, at most, in very special class of coordinates [8,9].

It is natural to assume that:

1° a real closed system possesses the global energetic quantities P_μ, which have definite values and which have definite transformational properties[2],

2° a real closed system possesses the definite superenergy distribution (after global fixing, of the vector field $\vec{v} : \vec{v}\cdot\vec{v} = 1$) and, in consequence, possesses the global superenergetic quantities which are also definite numerically and have definite transformational properties[3].

We will accept these assumptions in the following. As a consequence we must assume that in GRT there exists only one energy-momentum complex $K_{\mu\cdot}^{\cdot\nu}$. The superenergy tensor obtained from it by means of the method described in [1] gives the localization of the superenergy. How can this fact be compatible with the variety of energy-momentum complexes, which are considered in GRT? In our opinion, there exists only one possibility: to give a precise definition of the notion "the energy-momentum complex" in GRT. Lack of such a definition allows one to consider in GRT many different energy-momentum

---

[1] In order to calculate the global superenergetic quantities of a closed system we must, at first, fix the vector field $\vec{v}$ which is present in (1) and in (2). We do that in a natural manner as described in [1]. We do not present the results of calculations because they are very tedious.

[2] Under a fixed and under a variable hypersurface of integration and, at least, under linear coordinate transformations.

[3] We think that only under such circumstances can the superenergy possess physical meaning. The Newtonian limit of the superenergy tensor explicitly implies that superenergy should possess physical meaning, i.e., it supports our assumptions. Moreover, assumptions of that kind are obviously satisfied in SRT. In this case the energy and the superenergy are localized by the metric energy-momentum tensor.

complexes and leads to the above mentioned nonuniqueness of the global quantities of a closed system (energetic and superenergetic).

We propose the following definition of the energy-momentum complex in GRT.

Definition 1

By the energy-momentum complex in GRT we will mean the mixed, two-index affine tensor density having weigth (+1), $K_{\mu.}^{\cdot\nu}$, which possesses the following properties:[4]

1°. $$K_{\mu.}^{\cdot\nu} = \sqrt{(-g)}\,(T_{\mu.}^{\cdot\nu}+t_{\mu.}^{\cdot\nu}),$$

where $T^{\nu}_{\cdot\mu}$ is the dynamical[5] energy-momentum tensor of matter (in the framework of GRT this tensor coincides with the metric energy-momentum tensor) and $t_{\mu.}^{\cdot\nu}$ is the energy-momentum pseudotensor of the gravitational field.

2°. $$K_{\mu.}^{\cdot\nu} = U_{\mu\,..\,,\sigma}^{\cdot[\nu\sigma]},$$

where $U_{\mu..}^{\cdot[\nu\sigma]}$ are the so-called superpotentials and equations 2° are the so-called superpotential representation of Einstein's equations

$$\sqrt{(-g)}\,G_{\mu.}^{\cdot\nu} = \bar{k}\sqrt{(-g)}\,T_{\mu}^{\cdot\nu}$$

without any curl added to them.[6]

The energy-momentum complex, $K_{\mu.}^{\cdot\nu}$, introduced by Definition 1 automatically satisfies the continuity equations

$$K_{\mu.,\nu}^{\cdot\nu} = 0,$$

and the pseudotensor $t_{\mu.}^{\cdot\nu}$ which is also introduced by Definition 1 is a quadratic function of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$.

It is simple to check that this pseudotensor coincides with Einstein's canonical energy-momentum pseudotensor ${}_{E}t_{\mu.}^{\cdot\nu}$. Thus Definition 1 leads to an energy-momentum complex in GRT which is identical with Einstein's energy-momentum complex ${}_{E}K_{\mu.}^{\cdot\nu}$. Owing to that, the global energetic quantities of a closed system calculated with the help of the energy-momentum complex introduced by Definition 1 have the following, remarkable properties:

---

[4] These properties were chosen in such a way that the following conditions would be satisfied: a) Einstein's canonical energy-momentum complex, ${}_{E}K_{\mu.}^{\cdot\nu}$, should, in particular, possess these properties; b) Einstein's-Klein Theorem should be true for the global energetic quantities of a closed system calculated with the help of $K_{\mu.}^{\cdot\nu}$.

[5] The dynamical energy-momentum tensor appears as the source in the field equations. We assume that this tensor is uniquely determined.

[6] The different, non-vanishing in vacuum curls added to Einstein's equations lead to different energy-momentum complexes which consequently lead to different numerical values and different transformational properties of the global energetic quantities of a closed system. Moreover, the addition of any curl to Einstein's equations destroys the tensor character of these equations.

1° For Schwarzschild's space-time and in ALC in which the whole system is at rest we have

$$P_0 := \frac{1}{c} \int\limits_{x^0=\mathrm{const}} K_{0.}^{\cdot 0} dxdydz = mc,$$

$$P_k := \frac{1}{c} \int\limits_{x^0=\mathrm{const}} K_{k.}^{\cdot 0} dxdydz = 0 \quad (k = 1, 2, 3).$$

2° For the energetic global quantities, P_μ, of a general closed system, where

$$P_\mu := \frac{1}{c} \int\limits_{x^0=\mathrm{const}} K_{\mu.}^{\cdot 0} dxdydz,$$

calculated with the help of $K_{\mu.}^{\cdot\nu}$ the Einstein's-Klein theorem [2] is true.

Property 2° permits one to interpret physically the quantities P_μ as the components of the energy-momentum free-vector of a closed system.

Summing up, we can say that with the help of a precise but very simple definition of the energy-momentum complex, we arrive at the only energy-momentum complex in GRT and consequently at the only total superenergy tensor[7]. As a result, the global energetic quantities and the global superenergetic quantities of a closed system in GRT will be uniquely determined.

2. *On the uniqueness of the global superenergetic quantities of a closed system in ECT*

In the framework of ECT we can also arrive, with the help of a proper definition of the energy-momentum complex, at the uniquely determined global energetic quantities and uniquely determined global superenergetic quantities of a closed system.

In our opinion, we should define the energy-momentum complex in ECT in the same way as in GRT changing only { } in Γ and $T_{\mu.}^{\cdot\nu}$ in ${}_{\mathrm{can}}T_{\mu.}^{\cdot\nu}$ [8]. Thus, we assume the following definition for the energy-momentum complex in ECT.

Definition 2

By the energy-momentum complex in ECT we will mean the mixed, two-index affine tensor density of weigth (+1), $\bar{K}_{\mu.}^{\cdot\nu}$, which possesses the following properties:

1°. $$\bar{K}_{\mu.}^{\cdot\nu} = \sqrt{(-g)}\,({}_{\mathrm{can}}T_{\mu.}^{\cdot\nu} + \bar{t}_{\mu.}^{\cdot\nu}),$$

where ${}_{\mathrm{can}}T_{\mu.}^{\cdot\nu}$ is the canonical energy-momentum tensor of matter [11] and $\bar{t}_{\mu.}^{\cdot\nu}$ denotes the energy-momentum pseudotensor of geometry.

2°. $$\bar{K}_{\mu.}^{\cdot\nu} = \bar{U}_{\mu\,..}^{\cdot[\nu\sigma]}{}_{,\sigma}$$

---

[7] By definition, obtained from the sum $T_{\mu.}^{\cdot\nu} + t_{\mu.}^{\cdot\nu}$ by means of averaging described in [1].

[8] In the framework of ECT the dynamical energy-momentum tensor of matter coincides with the canonical energy-momentum tensor ${}_{\mathrm{can}}T_{\mu.}^{\cdot\nu}$.

where $\overline{U_{\mu\cdot\cdot}^{\cdot[\nu\sigma]}}$ are the so-called superpotentials and 2° gives the superpotential representation of the dynamical part of ECT equations:

$$\sqrt{(-g)}\,{}^{U_4}G_{\mu\cdot}^{\cdot\nu} = \bar{k}\sqrt{(-g)}\,{}_{\text{can}}T_{\mu\cdot}^{\cdot\nu}$$

(without any curl added to them).

In vacuum, complex $\bar{K}_{\mu\cdot}^{\cdot\nu}$ reduces to the energy-momentum complex of GRT, i.e., to Einstein's energy-momentum complex ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$. Definition 2 leads us to an energy-momentum complex identical in form with the energy-momentum complex which has been introduced in paper [1] as "the energy-momentum complex of ECT in Trautman's formulation of this theory". We will denote this complex ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu}$. ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu} = \sqrt{(-g)}({}_{\text{can}}T_{\mu\cdot}^{\cdot\nu}+t_{\mu\cdot}^{\cdot\nu})$, where the analytic form of $t_{\mu\cdot}^{\cdot\nu}$ was given in paper [1]. The energy-momentum complex ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu}$ follows from the dynamical part of the ECT equations in the same way as the energy-momentum complex of GRT, ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$, follows from GRT equations. Moreover, we can obtain the superpotential form of the complex, ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu}$, from the superpotential form of the energy-momentum complex of GRT, ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$, by simply changing { } onto Γ.

Thus, we may call the energy-momentum complex ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu}$ „the generalized (on ECT geometry) Einstein's energy-momentum complex".

We must remark here that if we express ECT in terms of Riemannian geometry[9], i.e., if we bring ECT equations to the combined form in which they have the form of the Einstein equations with the combined energy-momentum tensor of matter ${}_{\text{com}}T_{\mu\cdot}^{\cdot\nu}$ as the source of the gravitational field { }, we will come, on the strength of Definition 1, to the energy-momentum complex which is formally the same as the complex ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$. The difference consists in the following: $T_{\mu\cdot}^{\cdot\nu} \to {}_{\text{com}}T_{\mu\cdot}^{\cdot\nu}$. Let us denote this complex ${}_{\mathrm{E}}\bar{K}_{\mu\cdot}^{\cdot\nu}$. Complex ${}_{\mathrm{E}}\bar{K}_{\mu\cdot}^{\cdot\nu}$ can be obtained from the pure metric superpotentials which possess formally the same analytic form as Freud's superpotentials. Then, complex ${}_{\mathrm{E}}\bar{K}_{\mu\cdot}^{\cdot\nu}$ is a pure metric. Therefore, in our opinion, we should not interpret complex ${}_{\mathrm{E}}\bar{K}_{\mu\cdot}^{\cdot\nu}$ as the energy-momentum complex of matter and geometry in ECT. The energy-momentum complex of matter and geometry in ECT ought to be constructed from Γ which describes Cartan's geometry of space-time and from ${}_{\text{can}}T_{\mu\cdot}^{\cdot\nu}$ which is the dynamical energy-momentum tensor of matter in ECT [12], i.e., as in Definition 2.

It seems to us that complex ${}_{\mathrm{E}}\bar{K}_{\mu\cdot}^{\cdot\nu}$ rather should be interpreted physically as the energy-momentum complex of the above mentioned modification of GRT in the framework of Riemannian geometry which is given by the so-called "combined formulation of ECT".

On the strength of Definition 2 the complex ${}_{\mathrm{T}}K_{\mu\cdot}^{\cdot\nu}$ is the energy-momentum complex of matter and geometry in the framework of ECT.

Consequently, the tensor ${}_{\mathrm{T}}^{\mathrm{t}}S_{\mu\cdot}^{\cdot\nu}(P;v^{\varrho})$ given in [1] is the total superenergy tensor in ECT and it gives the proper, global superenergetic quantities of a closed system.

---

[9] In our opinion this reformulation of ECT equations can be interpreted as a new theory of gravity which is some modification of Einstein's theory in terms of Riemannian geometry. This modification consists in the change of the energy-momentum tensor of matter. It is a modification of GRT which is different from original ECT from the geometric point of view.

3. On the dependence of the global superenergetic quantities of a closed system on time

Let us consider the problem if, in GRT, the global superenergetic quantities of a closed system can be independent of time, i.e., if, in ALC,

$$\frac{d}{dx^0} \int\limits_{x^0=\mathrm{const}} {}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}\, dxdydz = 0. \tag{6}$$

Using the theorems on the differentiation of integrals with a parameter we can rewrite the left hand side of (6) in the following form

$$\frac{d}{dx^0} \int\limits_{x^0=\mathrm{const}} {}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}\, d^3x = \int\limits_{x^0=\mathrm{const}} \frac{\partial}{\partial x^0} ({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}) d^3x. \tag{7}$$

It is seen from (7) that (6) can be satisfied if and only if one of the following conditions is fulfilled:

1°.
$$\frac{\partial}{\partial x^0} ({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}) = 0.$$

We can have a situation of this kind only in the case of a stationary closed system.

2°.
$$\frac{\partial}{\partial x^0} ({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}) = (-)\partial_k({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot k} \sqrt{-g}),$$

i.e., if the total superenergy flux would satisfy the continuity equations. (Obviously, the total superenergy flux does not satisfy 2°.)

3°. The functions $\frac{\partial}{\partial x^0} ({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0}\sqrt{(-g)})$ possess such special analytic form that the integrals

$$\int\limits_{x^0=\mathrm{const}} \frac{\partial}{\partial x^0} ({}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}) d^3x$$

are equal to zero. Obviously, for a general closed system the conditions 3° are not satisfied. Thus, for a general closed system the integrals

$$S_\mu := \int\limits_{x^0=\mathrm{const}} {}_{\mathrm{E}}^{\mathrm{t}}S_{\mu.}^{\cdot 0} \sqrt{(-g)}\, d^3x$$

should be dependent on time.

The variation of the global superenergetic quantities of a closed system in time should be fluctuational in type because the values of these quantities ought to be finite for all times.

Summing up, we can say that the variations of the geometry of a closed system are accompanied by the corresponding variations of the values of the global superenergetic quantities of this closed system. We remark here that the variations of geometry of a closed system do not affect the energetic global quantities. All the above considerations can be repeated in the framework of ECT.

4. On the physical interpretation of the superenergy

The superenergy tensors describe the differential effects in the field of an energy-momentum tensor or in the field of an energy-momentum pseudotensor because for $T_{\mu.}^{.\nu} = \text{const}$ or $t_{\mu.}^{.\nu} = \text{const}$ we get $S_{\mu.}^{.\nu} = 0$.

In the framework of SRT the global superenergetic quantities of a closed system are equal to zero[10]. This appears to point out that the superenergy is physically meaningful only in the framework of GRT and in the framework of ECT. In general, we can easily see the close correspondence between the canonical (or metric) superenergy supertensors ${}^{\phi}_{s}T_{\mu.\alpha\beta}^{.\nu..}$ and the coefficient $\frac{k^2}{2}$ which appears in the potential energy of small, linear oscillations. Namely, let us consider the potential energy $U(x^k)$ of a small oscillation in Cartesian coordinates with the origin at the minimum of the potential energy which we put equal to zero[11].

Then, we have

$$U(x^k) = U(0)+U_{,l}(0)x^l+\tfrac{1}{2}\,U_{,kl}(0)x^kx^l+\,\ldots \cong \tfrac{1}{2}\,U_{,kl}(0)x^kx^l. \tag{8}$$

From the construction method of the 4-index superenergy tensors, the so-called superenergy supertensors [1], it is easy to see the close correspondence

$$\tfrac{1}{2}\,{}^{\phi}_{s}T_{\mu.\alpha\beta}^{.\nu..} \leftrightarrow \tfrac{1}{2}\,U_{,kl}(0).$$

In the case of small, linear oscillations we get from that the above mentioned correspondence

$$\tfrac{1}{2}\,{}^{\phi}_{s}T_{\mu.\alpha\beta}^{.\nu..} \leftrightarrow \frac{k^2}{2}, \tag{9}$$

where $k^2 := U''(0)$. From the construction method of the 4-index superenergy tensors we can also see that $\frac{1}{2}{}^{\phi}_{s}T_{\mu.\alpha\beta}^{.\nu..}y^\alpha y^\beta$, where y^α denotes normal coordinates, is the component of the relative energy-momentum density distribution[12] which gives the non-vanishing contribution in the process of the averaging of the differences $T_{\mu.}^{.\nu}-\mathring{T}_{\mu.}^{.\nu}$ over the space-time "cubes", which we have described in [1].

Concerning the vacuum gravitational field superenergy, we can see that it is connected, in the Newtonian limit, with the tidal forces. This fact can be immediately proved by direct calculations of the Newtonian limits of the components of the total superenergy tensor ${}^{t}_{E}S_{\mu.}^{.\nu}(P;v^\varrho)$. If we do the calculations, we will see (Appendix) that the Newtonian limits of the components of the pure geometric part[13] of this tensor are quadratic functions of the partial derivatives of the second order of the Newtonian gravitational potential φ which describe the tidal effects in the Newtonian gravitational field.

---

[10] This fact immediately follows from energy-momentum conservation.

[11] In the case of the gravitational field { } the normal coordinate system NCS(*P*) of the connection { } is in the most close correspondence to such coordinates.

[12] Relative with respect to the origin, *P*, of NCS(*P*) and in a sufficiently small neighbourhood of the origin.

[13] This part is a quadratic function of the curvature tensor.

The connection between the vacuum gravitational field superenergy and the tidal forces can be seen better in the following way. Let us consider the Newtonian limits of the components of Einstein s pseudotensor ${}_{\mathrm{E}}t_{\mu.}^{\cdot\nu}$ of the energy-momentum of the gravitational field. We have (Appendix)

$$ {}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0} = \frac{(\nabla\varphi)^2}{8\pi G} = \frac{\varphi_{,k}\varphi_{,k}}{8\pi G}. \tag{10} $$

The Newtonian expression which closely corresponds to the gravitational superenergy density is

$$ \nabla^2({}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0}) = \frac{\delta^{kl}}{4\pi G}\left[\varphi_{,p}\varphi_{,pkl} + \varphi_{,pk}\varphi_{,pl}\right]. \tag{11} $$

In the following we consider a spherically symmetric mass M which produces the gravitational potential

$$ \varphi = (-)\frac{GM}{\ } \tag{12} $$

at points exterior to it. Here r denotes the distance from the centre of the mass. Outside the mass we have

$$ \nabla^2({}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0}) = \frac{3GM^2}{2\pi r^6}. \tag{13} $$

The last expression is closely connected with the scalar product $\vec{p}(r)\cdot\vec{p}(r)$, where $\vec{p}(r) := (-)\frac{da(r)}{dr}\frac{\vec{r}}{r}$ and $a(r)$ is defined by the formula $(r)\,\vec{a} = -\frac{GM}{r^2}\frac{\vec{r}}{r} = :a(r)\frac{\vec{r}}{r}\cdot\vec{a}(r)$ denotes here the strength of the gravitational field outside the mass at distance $r = |\vec{r}|$ from the geometric centre of the mass.

We can physically interpret

$$ \vec{p}(r) = (-)\frac{da(r)}{dr}\frac{\vec{r}}{r} = (-)\frac{2GM}{r^3}\frac{\vec{r}}{r} $$

as the strength of tidal forces[14] at distance r from the geometric centre of the mass.

If we calculate $\vec{p}(r)\cdot\vec{p}(r)$, then we will see that

$$ \nabla^2({}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0}) = \frac{3GM^2}{2\pi r^6} = \frac{3}{8G}\vec{p}(r)\cdot\vec{p}(r) = \frac{3}{8G}[\vec{p}(r)]^2. \tag{14} $$

Formula (14) gives us the above mentioned connection.

---

[14] In analogy to the gravitational force strength, $\vec{a}(r) := (-)\frac{d\varphi(r)}{dr}\frac{\vec{r}}{r} = a(r)\frac{\vec{r}}{r}$ we understand by the strength of the tidal force in the gravitational field the expression $\vec{p}(r) = (-)\frac{da(r)}{dr}\frac{\vec{r}}{r} = \frac{d^2\varphi(r)}{dr^2}\frac{\vec{r}}{r}$.

This formula allows us to interpret the Newtonian superenergy density $\nabla^2({}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0})$ of the vacuum gravitational field as the "energy density" of tidal forces at distance, r, from the centre of the body.

The author wishes to thank Dr A. Staruszkiewicz for many useful, critical remarks.

APPENDIX

We give here the Newtonian limits of the components of the total superenergy tensor ${}^{t}_{\mathrm{E}}S_{\mu.}^{\cdot\nu}(P;v^{e})$ and the Newtonian limits of the components of the gravitational pseudotensor ${}_{\mathrm{E}}t_{\mu.}^{\cdot\nu}$

$$
\begin{aligned}
{}^{\mathrm{N}}_{\mathrm{E}}S_{0.}^{\cdot 0} &= \frac{2}{9\pi G}(\varphi_{,11}^2+\varphi_{,22}^2+\varphi_{,33}^2)+\frac{2}{9\pi G}(\varphi_{,12}^2+\varphi_{,13}^2+\varphi_{,23}^2)\\
&\quad+\frac{(\varphi_{,11}\varphi_{,22}+\varphi_{,11}\varphi_{,33}+\varphi_{,22}\varphi_{,33})}{6\pi G}-\tfrac{4}{3}\pi G\varrho^2+c^2\nabla^2\varrho,\\
{}^{\mathrm{N}}_{\mathrm{E}}S_{0.}^{\cdot k} &= {}^{\mathrm{N}}_{\mathrm{E}}S_{k.}^{\cdot 0} = 0,\\
\underset{(k=1,2,3)}{{}^{\mathrm{N}}_{\mathrm{E}}S_{k.}^{\cdot k}} &= \frac{1}{72\pi G}\Bigg[6\sum_{\substack{i=1\\ i\neq k}}^{3}\varphi_{,ii}^2-2\varphi_{,kk}^2+4\sum_{\substack{i=1\\ i\neq k}}^{3}\varphi_{,ki}^2+12\prod_{\substack{i=1\\ i\neq k}}^{3}\varphi_{,ii}-4\pi G\varrho\Big(\varphi_{,kk}+\sum_{i=1}^{3}\varphi_{,ii}\Big)\\
&\quad-48\pi^2\varrho^2G^2\Bigg],\\
\underset{\substack{(i,k=1,2,3)\\ i\neq k}}{{}^{\mathrm{N}}_{\mathrm{E}}S_{i.}^{\cdot k}} &= \frac{1}{18\pi G}\Bigg[\varphi_{,ik}\Big((-)2\sum_{l=1}^{3}\varphi_{,ll}-\underset{\substack{p\neq i\\ p\neq k}}{\varphi_{,pp}}\Big)+\underset{p\neq i,}{\varphi_{,ip}}\ \underset{p\neq k}{\varphi_{,kp}}-\pi G\varrho\varphi_{,ik}\Bigg].
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
{}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot 0} &= \frac{(\nabla\varphi)^2}{8\pi G},\\
{}^{\mathrm{N}}_{\mathrm{E}}t_{0.}^{\cdot k} &= {}^{\mathrm{N}}_{\mathrm{E}}t_{k.}^{\cdot 0} = 0,\\
{}^{\mathrm{N}}_{\mathrm{E}}t_{i.}^{\cdot k} &= \delta_i^{\,k}\frac{(\nabla\varphi)^2}{8\pi G}-\frac{\varphi_{,i}\varphi_{,k}}{4\pi G}\quad (i,k=1,2,3).
\end{aligned}
\tag{A2}
$$

φ which appears in (A1) and in (A2) denotes the Newtonian gravitational potential satisfying Poisson's equation

$$\Delta\varphi(\vec{r}) = 4\pi G\varrho(\vec{r}).$$

REFERENCES

[1] J. Garecki, *Acta Phys. Pol.* **B8**, 159 (1977); **B9**, 291 (1978).

[2] A. Trautman, *Conservation Laws in General Relativity*, in *Gravitation: an Introduction to Current Research*, edited by Louis Witten, John Wiley and Sons, Inc., New York-London 1962; § 7 in *Lectures on General Relativity*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey 1965.

[3] C. Møller, *Nordita Publications*, No. 190 (1966).
[4] J. N. Goldberg, *Phys. Rev.* **111**, 315 (1958).
[5] C. Cattaneo, *Conservation Laws*, in Proceedings of the International Conference on Relativistic Theories of Gravitation, Vol. II, London 1965.
[6] F. H. J. Cornish, *Proc. R. Soc.* A. **282**, 358 (1964); **282**, 372 (1964).
[7] J. Garecki, *Doctoral Dissertation*, Cracow 1973.
[8] C. Møller, *The Theory of Relativity*, Clarendon Press, Oxford 1972.
[9] A. Komar, *Phys. Rev.* **127**, 1411 (1962).
[10] J. L. Synge, *Relativity: The General Theory*, North-Holland Publishing Company, Amsterdam 1960.
[11] F. W. Hehl, *Gen. Relativ. Gravitation* **4**, 333 (1973); **5**, 491 (1974).
[12] F. W. Hehl, *Rep. Math. Phys.* **9**, 55 (1976).
[13] A. Trautman, *On the Structure of the Einstein-Cartan Equations*, Istituto Nazionale di Alta Matematica, Symposia Mathematica **12**, 139 (1973).