REMARKS CONCERNING THE GEOMETRY OF NULL STRINGS

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The geometry of totally null complex 2-surfaces (null strings) in a complex space-time is considered. Some theorems concerning the relationship between algebraic types of the energy-momentum tensor and the existence of null strings in the complex space-time are given.

1. Introduction

A totally null complex 2-surface in a complex space-time is called a null string. The existence of a congruence of null strings in the complex space-time simplifies an analysis of complex Einstein equations and in many cases enables one to obtain solutions of these equations [1–7]. Moreover, null strings are very interesting objects from a geometrical point of view. The geometry of null strings was done recently by Boyer and Plebański [8, 9].

The purpose of this paper is to prove some theorems "connecting" the existence of flat null strings in a given complex space-time with an algebraic type of an energy-momentum tensor [10]. These theorems, we hope, will play a role in the study of complex space-time with the "matter".

2. Induced connection on a null string

Let (M_4^c, ds^2) be a complex space-time [10]. A null string M_2^c of (M_4^c, ds^2) is a 2-dimensional complex imbedded submanifold of $M_4^c (M_2^c \subset M_4^c)$ so that for each point $p \in M_2^c$ and for each vector X tangent to M_2^c at p

$$ds^2(X, X) = 0. \tag{2.1}$$

Note: we consider objects of types (p, 0) ([11] Vol. II). Let $(U, \{x^\mu\})$, $\mu = 1, 2, 3, 4$ be a local chart and let (e_1, e_2, e_3, e_4) be a local null tetrad (see e.g., [1, 10]) such that (e_2, e_4) are tangent to M_2^c at each $p \in M_2^c \cap U$. Now if ∇ is the connection on M_4^c then ([11] Vol. I)

$$\nabla_{e_{\tilde{a}}} e_{\tilde{b}} = \Gamma^{\tilde{c}}{}_{\tilde{b}\tilde{a}} e_{\tilde{c}}; \quad \tilde{a}, \tilde{b}, \tilde{c} = 2,4 \tag{2.2}$$

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on $M_2^c \cap U$, where $\Gamma^c{}_{\tilde{b}\tilde{a}}$ are connection coefficients. The formula (2.2) follows from the fact that $\Gamma^3{}_{\tilde{a}\tilde{b}} = \Gamma^1{}_{\tilde{a}\tilde{b}} = 0$ $(\tilde{a}, \tilde{b} = 2,4)$ on $M_2^c \cap U$, [12]. From (2.2) one may conclude that there exists the induced connection $\tilde{\nabla}$ on M_2^c locally defined by formula ([11] Vol. II)

$$\tilde{\nabla}_X Y := \nabla_X Y \tag{2.3}$$

for any vector fields X, Y on $M_2^c \cap U$ tangent to M_2^c. Consequently, from (2.2) and (2.3) we find that $\tilde{\nabla}$ is locally defined by the following connection coefficients of ∇ (or the Ricci coefficients):

$$\begin{gathered}\Gamma^4{}_{22} = \Gamma_{322}, \quad \Gamma^2{}_{22} = \Gamma_{122}, \quad \Gamma^4{}_{24} = \Gamma_{324}, \quad \Gamma^2{}_{24} = \Gamma_{124}, \\ \Gamma^4{}_{42} = \Gamma_{342}, \quad \Gamma^2{}_{42} = \Gamma_{142}, \quad \Gamma^4{}_{44} = \Gamma_{344}, \quad \Gamma^2{}_{44} = \Gamma_{144}.\end{gathered} \tag{2.4}$$

The connection $\tilde{\nabla}$ is symmetric. It defines the curvature tensor $\tilde{R}$ on M_2^c.

Locally ([11] Vol. I)

$$\tilde{R}(X, Y)Z := \tilde{\nabla}_X(\tilde{\nabla}_Y Z) - \tilde{\nabla}_Y(\tilde{\nabla}_X Z) - \tilde{\nabla}_{[X,Y]} Z, \tag{2.5}$$

where X, Y, Z are arbitrary vector fields on $M_2^c \cap U$ tangent to M_2^c. One can show easily that $\tilde{R}$ is locally determined by the following tetrad components of the curvature tensor R on M_4^c

$$\begin{gathered}R^4{}_{424} = \tfrac{1}{2}(C_{42} - C^{(4)}), \quad R^4{}_{224} = \tfrac{1}{2} C_{22}, \quad R^2{}_{424} = -\tfrac{1}{2} C_{44}, \\ R^2{}_{224} = -\tfrac{1}{2}(C_{42} + C^{(4)}),\end{gathered} \tag{2.6}$$

where $C_{\tilde{a}\tilde{b}}$ $(\tilde{a}, \tilde{b} = 2,4)$ are null tetrad components of the traceless Ricci tensor $C_{\mu\nu}$

$$C_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{4} \mathscr{R} g_{\mu\nu}, \tag{2.7a}$$

$$R_{\mu\nu} := R^\alpha{}_{\mu\nu\alpha}; \quad \mathscr{R} := R^\mu{}_\mu, \tag{2.7b}$$

and $C^{(4)}$ is one of the null tetrad components of the conformal curvature tensor on M_4^c (see e.g., [1, 10]). When $C_{44} = C_{42} = C_{22} = 0$ on $M_2^c \cap U$, then $C^{(4)} = 0$ on $M_2^c \cap U$. These are generalized Goldberg–Sachs theorems [12]. Therefore, we have:

Proposition 1. The connection $\tilde{\nabla}$ on M_2^c is flat if and only if

$$C_{\mu\nu} X^\mu Y^\nu = 0 \tag{2.8}$$

for each point $p \in M_2^c$ and for arbitrary vectors X, Y tangent to M_2^c at p. □

The results of this section in terms of fibre bundle and spinor formalisms were given by Boyer and Plebański [8, 9].

3. *Theorems concerning null strings with flat connections*

The notations used here are defined in [10].

Theorem 1. If the connection $\tilde{\nabla}$ on a null string M_2^c, is flat, then for each point $p \in M_2^c$

(*i*) tensor $C_{\mu\nu}$ is one of the types $[2N_1 - 2N]_2$, $[2N_1 - 2N]_{(1-2)}$, $[2N_1 - 2N]_4$, $[4N]_1$, ${}^{(3)}[4N]_2$, ${}^{(2)}[4N]_2$, $[4N]_3$, $[4N]_4$, (*ii*) if $C_{\mu\nu}$ is of types $[4N]_3$ or $[4N]_4$, then the null eigenvector of $C_{\mu\nu}$ is tangent to M_2^c at p

(*iii*) if $C_{\mu\nu}$ is of the type $^{(2)}[4N]_2$, then at least one eigenvector of $C_{\mu\nu}$ is tangent to M_2^c at p,

(*iv*) if $C_{\mu\nu}$ is of the types $[2N_1-2N]_2$ or $[2N_1-2N]_{(1-2)}$ or $[2N_1-2N]_4$, then two null eigenvectors of $C_{\mu\nu}$ are tangent to M_2^c at p,

(*v*) if $C_{\mu\nu}$ is of types $[4N]_1$ or $^{(3)}[4N]_2$, then every vector ($\neq 0$) tangent to M_2^c at p is the null eigenvector of $C_{\mu\nu}$,

(*vi*) every vector ($\neq 0$) tangent to M_2^c at p is a multiple generalized Debever–Penrose vector.

Proof: If $\tilde{\nabla}$ is flat, then for each point $p \in M_2^c$ and for each null tetrad (e_1, e_2, e_3, e_4) at p such that (e_2, e_4) are tangent to M_2^c we have:

$$C_{44} = C_{42} = C_{22} = 0 \tag{3.1}$$

(see Proposition 1).
Then from the eigenvalue equation

$$C_{ab}X^b = \lambda X_a \tag{3.2}$$

and from (3.1) one finds that

$$\lambda = \pm\sqrt{(C_{12})^2 + C_{41} \cdot C_{32}}\,. \tag{3.3}$$

Hence, for each point $p \in M_2^c$, $C_{\mu\nu}$ possess one quadruple or two double eigenvalues and then $C_{\mu\nu}$ is one of types (at p) [10]: $[2N_1-2N]_2$, $[2N_1-2N]_{(1-2)}$, $[2N_1-2N]_4$, $[4N]_1$, $^{(3)}[4N]_2$, $^{(2)}[4N]_2$, $[4N]_3$, $[4N]_4$. Thus (*i*) is proved.

Now, for each point $p \in M_2^c$ at least one null eigenvector of $C_{\mu\nu}$ is tangent to M_2^c. Assume that it is not true. Then one can select the null tetrad (e_1, e_2, e_3, e_4) at some $p \in M_2^c$ in such a manner that (e_2, e_4) are tangent to M_2^c and that e_3 is the null eigenvector of $C_{\mu\nu}$ at p. Hence,

$$C_{33} = C_{32} = C_{31} = 0. \tag{3.4}$$

From (3.1) and (3.4) it follows that:

$$C_{22} = C_{24} = C_{23} = 0. \tag{3.5}$$

This means that e_2 is the null eigenvector of $C_{\mu\nu}$ at p. Therefore, for each point $p \in M_2^c$ at least one null eigenvector of $C_{\mu\nu}$ is tangent to M_2^c and hence (*ii*) and (*iii*) hold. We assert that if $C_{\mu\nu}$ possess at least three null eigenvectors at $p \in M_2^c$ then two of them are tangent to M_2^c. Let (e_1, e_2, e_3, e_4) be any null tetrad at $p \in M_2^c$ so that (e_2, e_4) are tangent to M_2^c and (e_2, e_3) are null eigenvectors of $C_{\mu\nu}$ at p. Let E_1 be the third null eigenvector of $C_{\mu\nu}$ at p. One can select the null tetrad (e_1, e_2, e_3, e_4) so that

$$ds^2(E_1, e_3) = 0 \quad \text{and} \quad ds^2(E_1, e_2) = 1, \tag{3.6a}$$

or

$$ds^2(E_1, e_3) = 1 \quad \text{and} \quad ds^2(E_1, e_2) = 0. \tag{3.6b}$$

Suppose (3.6a). Then,

$$E_1 = e_1 + ze_3, \tag{3.7}$$

where z is the complex number. Define the null tetrad (e'_1, e'_2, e'_3, e'_4) at p by the formulae

$$e'_1 := E_1; \quad e'_2 := e_2; \quad e'_3 := e_3; \quad e'_4 := e_4 - ze_2. \tag{3.8}$$

One easily finds that

$$C_{\mu\nu}e'^{\mu}_4 e'^{\nu}_4 = C_{\mu\nu}e'^{\mu}_4 e'^{\nu}_2 = C_{\mu\nu}e'^{\mu}_4 e'^{\nu}_1 = 0. \tag{3.9}$$

Consequently, the null vector e'_4 tangent to M_2^c at p is the null eigenvector of $C_{\mu\nu}$ at p. Now assume (3.6b). This implies

$$E_1 = e_4 + z'e_2, \tag{3.10}$$

and E_1 is tangent to M_2^c at p.

Thus, we have proved that if $C_{\mu\nu}$ possess at least three null eigenvectors at $p \in M_2^c$, then two of them are tangent to M_2^c. Hence, one concludes that (*iv*) for types $[2N_1-2N]_2$, $[2N_1-2N]_{(1-2)}$ and (*v*) hold. Notice that for types $[4N]_1$, $^{(3)}[4N]_2$ all eigenvalues of $C_{\mu\nu}$ vanish.

Finally, let $C_{\mu\nu}$ be of type $[2N_1-2N]_4$ at the point $p \in M_2^c$. Suppose only one null eigenvector of $C_{\mu\nu}$ at p is tangent to M_2^c. Choose the null tetrad (e_1, e_2, e_3, e_4) at p so that (e_2, e_3) are null eigenvectors of $C_{\mu\nu}$ at p, (e_2, e_4) are tangent to M_2^c and [10]

$$C_{\mu\nu} = N(e_{4\mu}e_{3\nu} + e_{3\mu}e_{4\nu} - e_{1\mu}e_{2\nu} - e_{2\mu}e_{1\nu}) + 2Nz(e_{3\mu}e_{2\nu} + e_{2\mu}e_{3\nu}) + e_{2\mu}e_{2\nu} + e_{3\mu}e_{3\nu}, \tag{3.11}$$

where $(-N, N)$ are eigenvalues of $C_{\mu\nu}$ at p, z is some complex number (in [10] $z = 0$ and $e_3 \to E_4$, $e_4 \to E_3$, $e_1 \to E_1$, $e_2 \to E_2$). From (3.11) one finds that

$$C_{44} := C_{\mu\nu}e_4^{\mu}e_4^{\nu} = 1 \neq 0, \tag{3.12}$$

but this formula contradicts (3.1).

Consequently, two eigenvectors of $C_{\mu\nu}$ are tangent to M_2^c at p. Finally (*i*$-$*v*) have been proved. Now $C^{(5)} = 0$ and $C^{(4)} = 0$ on M_2^c. Hence, (*vi*) holds and the proof of Theorem 1 is completed. □

An intermediate consequence of Theorem 1 is:

Corollary 1. If $\tilde{\nabla}$ is flat, then for each point $p \in M_2^c$ at least one eigenvector of $C_{\mu\nu}$ is tangent to M_2^c and every eigenvector of $C_{\mu\nu}$ tangent to M_2^c is a multiple generalized Debever–Penrose vector. □

Now we prove the theorem which is in some sense reciprocal to Theorem 1.

Theorem 2. Let M_2^c be a null string and let for each point $p \in M_2^c$ $C_{\mu\nu}$ be one of the types: $[2N_1-2N]_2$, $[4N]_3$, $[4N]_1$. If for each point $p \in M_2^c$ at least one null eigenvector of $C_{\mu\nu}$ is tangent to M_2^c, then the connection $\tilde{\nabla}$ on M_2^c is flat.

Proof: Let $C_{\mu\nu}$ be of type $[2N_1-2N]_2$ at some point $p \in M_2^c$. Hence, there exists the null tetrad (e_1, e_2, e_3, e_4) at p so that [10]

$$C_{\mu\nu} = N(e_{4\mu}e_{3\nu}+e_{3\mu}e_{4\nu}-e_{2\mu}e_{1\nu}-e_{1\mu}e_{2\nu}) \tag{3.13}$$

and e_4 is tangent to M_2^c.

Then we conclude that e_1 or e_2 is tangent to M_2^c and from (3.13) it follows that

$$C_{\mu\nu}X^\mu Y^\nu = 0 \tag{3.14}$$

for arbitrary vectors X, Y tangent to M_2^c at p. Now assume that $C_{\mu\nu}$ is of type $[4N]_3$ at p. So there exists the null tetrad (e_1, e_2, e_3, e_4) at p such that [10]

$$C_{\mu\nu} = \frac{\sqrt{2}}{2}\, i[e_{4\mu}(e_{1\nu}-e_{2\nu})+(e_{1\mu}-e_{2\mu})e_{4\nu}] \tag{3.15}$$

and e_4 is tangent to M_2^c. Hence e_1 or e_2 is tangent to M_2^c and using (3.15) one finds

$$C_{\mu\nu}X^\mu Y^\nu = 0 \tag{3.16}$$

for arbitrary vectors X, Y tangent to M_2^c at p.

If $C_{\mu\nu}$ is of type $[4N]_1$ at p then obviously

$$C_{\mu\nu}X^\mu Y^\nu = 0 \tag{3.17}$$

for every vectors X, Y tangent to M_2^c at p. Therefore using results of Proposition 1 one easily deduces that $\tilde{\nabla}$ is flat. □

It is well known that the energy-momentum tensor of the complex electromagnetic field (linear or non-linear) belongs to one of the types [10]: $[2N_1-2N]_2$ (general field), $^{(3)}[4N]_2$ (null field), $^{(2)}[4N]_2$ (one-sidedly null field). Consequently, using the results of our theorems one finds: 1° If (M_4^c, ds^2) is the complex space-time with the electromagnetic field and M_2^c is a flat null string ($\equiv \tilde{\nabla} = 0$) of M_4^c then for each point $p \in M_2^c$ one (at least) null eigenvector of the energy-momentum tensor is tangent to M_2^c; for types $[2N_1-2N]_2$, $^{(3)}[4N]_2$ *two* null eigenvectors are tangent to M_2^c. Moreover, for each point $p \in M_2^c$ each null eigenvector of the energy-momentum tensor tangent to M_2^c is a generalized Debever–Penrose vector; 2° If (M_4^c, ds^2) is the complex space-time with the general electromagnetic field and M_2^c is a null string of M_4^c such that for each point $p \in M_2^c$ one of null eigenvectors of the energy-momentum tensor is tangent to M_2^c then M_2^c is flat.

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REFERENCES

[1] J. F. Plebański, *J. Math. Phys.* **16**, 2395 (1975).
[2] J. D. Finley, III, J. F. Plebański, *J. Math. Phys.* **17**, 585 (1976).
[3] J. F. Plebański, I. Robinson, *Phys. Rev. Lett.* **37**, 493 (1976).
[4] C. P. Boyer, J. F. Plebański, *J. Math. Phys.* **18**, 1022 (1977).
[5] A. Garcia, J. F. Plebański, I. Robinson, *Gen. Relativ. Gravitation* **8**, 841 (1977).
[6] A. Garcia, J. F. Plebański, *Nouvo Cimento* **40B**, 224 (1977).

[7] J. D. Finley, III, J. F. Plebański, *J. Math. Phys.* **18**, 1662 (1977).
[8] C. P. Boyer, J. F. Plebański, *Rep. Math. Phys.* **14**, 111 (1978).
[9] C. P. Boyer, J. F. Plebański, *Complex General Relativity, $\mathscr{H}$ and $\mathscr{H}\mathscr{H}$ Spaces — A Survey*, Com. Tec. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas. Universidad Nacional Autonoma de Mexico, Mexico 20, D. F. **9**, No 174 (1978).
[10] M. Przanowski, J. F. Plebański, *Acta Phys. Pol.* **B10**, 485 (part I) (1979).
[11] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New York — London, Vol. I, 1963, Vol. II, 1969.
[12] J. F. Plebański, S. Hacyan, *J. Math. Phys.* **16**, 2403 (1975).