THE PERTURBED SINE-GORDON EQUATION

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We discuss some solvable modifications (perturbations) of the sine-Gordon equation which may be modelled by circular pendula in an outer field. The existence and shape, stability and mass formulae for solitons are illustrated by a double-sine-Gordon. A new class of so-called Jacobi–Gordon equations is introduced.

1. Introduction

The Klein-Gordon equation

$$\partial_x^2 \varphi - \partial_t^2 \varphi = \Phi(\varphi) \tag{1}$$

with $\Phi(\varphi) = \Phi_{KG}(\varphi) = \varphi$ provides a relativistic description of the scalar field $\varphi = \varphi(x, t)$. For $\|\varphi\| \ll 1$, Eq (1) may be interpreted as the first order approximation to its "more realistic" nonlinear form with $\Phi(\varphi) \neq \Phi_{KG}(\varphi)$. In the special sine-Gordon (SG) case $\Phi(\varphi) = \Phi_{SG}(\varphi) = \sin\varphi$, great interest in Eq. (1) was inspired by the existence of soliton solutions [1] as possible candidates for the description of elementary particles [2].

The SG equation is very exceptional from both the formal and physical point of view [1]: E. g., introduction of any small perturbation to Φ_{SG} admits just the approximate (numerical) treatment [3, 4] while other known solvable equations differ very significantly from the SG case [1, 5]. In the present paper, we suggest employing the analytic means for the investigation of some properties of special („Jacobi-Gordon") perturbed SG equations using the simplifying ansatz $\Phi(\varphi) = f(\varphi) f'(\varphi)$ and the closed shape, mass and stability formulae.

In Section 2 we start with the pendulum interpretation of SGE [6]. This admits the periodic perturbation to be added to Φ_{SG} preserving the physical (and possibly soliton) interpretation. The corresponding modification of the mechanical model is obtained just by a small change in the external forces.

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The double-sine equation $\left(\text{DSE, Eq. (1) with } \Phi(\varphi) = \Phi_{DS}(\varphi) = \sin\varphi + \frac{\lambda}{2}\sin\frac{\varphi}{2}\right)$ is investigated in Section 3 as the well known example. The numerical studies [3, 4] of the two (SG) soliton interaction are shown to reflect just the quasi-two-particle structure of the single soliton of DSE. The "non-linear" behaviour of the perturbation is stressed.

Finally, in Section 4, the class of the Jacobi–Gordon equations is introduced: Different doubly periodic f's are considered and the simplest closed formulae are tabulated complementing the recent soliton list given by Hu [5].

2. *Pendulum-Gordon equations*

The mechanical model of (1) with $\Phi = \Phi_{SG}$ has been described by Scott [6]. It is composed of the chain (along the x-coordinate) of pendula (angle $\varphi = \varphi(x, t)$) for which (1) represents an equation of motion. The kinetic term $\partial_t^2\varphi$ and the elastic connection term $\partial_x^2\varphi$ are defined by the geometry of the chain while the form of the potential energy term ($\sin\varphi$) depends on an outer field (the gravitational one).

Any nonlinear Klein-Gordon equation (1) with periodic $\Phi(\varphi) = \Phi(\varphi+2\pi)$ may be modelled by the same chain of pendula in an ad hoc constructed outer field corresponding to the functional form of nonlinearity Φ. The pendulum model works as an analogon computer, and, via analogy with the sine-Gordon case, it shows how the solitons appear as a simple consequence of the energy conservation in the model and what they look like.

The pure topological arguments [7] may be complemented by some simple formulae. The Lagrangian density

$$\mathscr{L} = \frac{2m^4}{\lambda}(\partial_t\varphi)^2 - \mathscr{V}(\varphi), \qquad \mathscr{V}(\varphi) = \frac{2m^4}{\lambda}[(\partial_x\varphi)^2 + f^2(\varphi) - C] \tag{2}$$

defines the one-dimensional Lorentz invariant field theory, the choice of the constant C helps us to put the vacuum potential energy density $\mathscr{V}$ (vac) equal to zero. Eq. (1) is the field equation corresponding to the Lagrangian density (2) when $\Phi(\varphi) = f(\varphi)\partial_\varphi f(\varphi) = ff'$. This expression is to be inserted into the right-hand side of (1).

Static soliton solutions $\varphi(x, t) = U(x)$ are most easily obtained by integrating (1) once and differentiating the ansatz

$$T[U(x)] = A\exp\gamma x. \tag{3}$$

We get the explicit form of the unknown function T in (3)

$$T(\varphi) = \exp\gamma\int_{\varphi_0}^{\varphi} d\tau[f^2(\tau) - C]^{-1/2}, \tag{4}$$

which becomes $\tan(\varphi/4)$ in the SG case ($f_{SG}(\varphi) = 2\sin\varphi/2$).

The general mass and stability formulae may also be obtained in close analogy with the sine-Gordon case. Using the potential energy functional

$$V[U] = \int_{-\infty}^{\infty} \mathscr{V}[U(x)]\frac{dx}{m}$$

corresponding to Lagrangian density (2), it is usual to define the classical mass of the soliton U as the difference

$$M = V[U] - V[\text{vac}] = 8m^3\lambda^{-1}\mu,$$

$$2\mu = \int_{-\infty}^{\infty} f^2[U(x)]dx = \int_{U(-\infty)}^{U(\infty)} f(U)dU \tag{5}$$

(for $C = 0$). When using the "scaling" $f(\varphi) \to \alpha f(\beta\varphi)$ in (2), and hence $\Phi(\varphi) \to \alpha^2\beta\Phi(\beta\varphi) = \Phi_S(\varphi)$ in (1) (e.g. to normalize the general period of $\Phi_S(\varphi)$ to 2π), the soliton mass is multiplied by the factor α/β.

The small linearized perturbations superimposed on the soliton $U(x)$ are governed by the Schrödinger-like equation [8]

$$-\partial_x^2\chi_n(x) + W(x)\chi_n(x) = \omega_n\chi_n(x),$$

$$W(x) = f(U)f''(U) + f'^2(U), \qquad U = U(x). \tag{6}$$

For most solvable cases, this is reducible to the hypergeometric equation [9]. Nevertheless, the zero frequency mode is known for all f's, $\chi_0(x) = f(U)$, $\omega_0 = 0$. Provided it is sufficiently smooth and has no zeros for finite x, the oscillation theorem [10] for (6) implies that χ_0 is an eigenstate of homogeneous equation (6) with minimal energy. This eigenstate corresponds to the translation invariance of (1), $\omega_n > 0$ for $n > 0$ and solution $U(x)$ is therefore stable. This will be satisfied for any input function $f(U)$ which has no "superfluous" vacua (minima of f^2) i.a. for all f's in the following sections.

3. *Double-sine-Gordon equation*

The specific choice of an auxiliary function

$$f_{DS}(\varphi) = ib(a\beta)^{-1/2} + i(a/\beta)^{1/2}\cos\beta\varphi \tag{7}$$

leads to DSE since

$$\Phi = f_{DS}f'_{DS} = b\sin\beta\varphi + \frac{a}{2}\sin 2\beta\varphi = \frac{a}{2}\,\Phi_{DS}\left(2\beta\varphi, \frac{2b}{a}\right).$$

The perturbation calculations by Newell [3] are based on the assumption $b \to 0$($a = 2$, $\beta = \frac{1}{2}$). Applying the considerations of Section 1, we may interpret his "synchronization of solitons" as follows:

In the pendulum model the outer field which defines Φ in (8) is similar to the homogeneous gravitational one in the limiting case $a \to 0$($b = \beta = 1$) only. Because of the discontinuity in the value of the period of Φ in the vicinity of $b = 0$, two entirely different pendulum models must be used for $b = 0$ and $b \neq 0$, respectively, and the perturbation of the sine-Gordon case $b = 0$ must be treated with due care.

Insertion of (7) into (4) yields the one-soliton solution for all positive values of a, b, β. After simple rearrangement, we get the shape formula

$$U(x) = \frac{2}{\beta} \arcsin z^{-1/2}, \qquad z = \operatorname{ch}^2 \gamma x - n^2 \operatorname{sh}^2 \gamma x,$$

$$n^2 = a/(a+b), \qquad \gamma = [(a+b)/\beta]^{1/2}, \tag{9}$$

and see again that the smooth perturbation of the sine-Gordon soliton occurs for $n^2 \ll 1$ only. "Non-perturbative" phenomena emerge even for $a > b > 0$, i.e. $n^2 > 1/2$. Although soliton (9) is still stable ($\chi_0(x) \neq 0$, $|x| < \infty$), it becomes localized at two symmetric points $x = \pm \frac{1}{\gamma} \operatorname{arch} n/n'$, $n'^2 = 1-n^2$ that move apart as $n \to 1$. The stability potential $W_{DS}(x) = a+b-2(b+4a)/z+8a/z^2$ to be inserted into (6) exhibits also two-dip structure. For a non-vacuum field $\varphi(x) = (2k+1)\pi/\beta$, the potential energy density $\mathscr{V}$ starts to have a local minimum $(f^2(\beta\varphi)-C)/\beta = 4b/\beta$ approaching zero as $n \to 1$. Therefore, the change in the period by a jump $2\pi \to \pi$ at $n = 1$ allows us to interpret the distant halves of soliton (9), for $n \lesssim 1$, as a system of two weakly bound SG solitons.

This interpretation is consistent with the "synchronized" motion due to the possible addition of small linear perturbations. We may further support it by giving the mass formula

$$M = \frac{M_0}{2\beta}\left(\frac{b}{2n} \ln \frac{1+n}{1-n} + a + b\right),$$

$$M = M_0(1+2n^2/3+O(n^4)),$$

$$M = 2M_0(1-b \ln b/4+O(b)) \tag{10}$$

for soliton (9). The sine-Gordon one-soliton mass $M_0 = 8M^3/\lambda$ is obtained for $a \sim 0$ ($b = \beta = 1$) while for $b \sim 0$ ($a = 2$, $\beta = \frac{1}{2}$) we get the masses of two solitons minus the nonpolynomial (!) binding energy correction.

The idea of coupling solitons by shifting the minima (and local minima) in the potential energy density $\mathscr{V}$ or by adding small "interaction" terms with greater period to $\Phi = ff'$ leads to interesting model constructions: bound states of the soliton-soliton type may reflect the clusterization of solitons (e.g. quarks and nucleons in nuclei with $ff' = \sin\varphi + \varepsilon_1 \sin\varphi/3 + \varepsilon_1\varepsilon_2 \sin\varphi/12 + \ldots$, $\varepsilon_1, \varepsilon_2 \ll 1$ etc.) in a way similar to the optical coherence phenomena [4].

4. *Jacobi–Gordon equations*

Perturbation of the sine-Gordon equation by higher harmonics is a problem to be solved on the computer. The Jacobi doubly-periodic functions are another and more natural means of smoothly deforming the sine term ($\sin\varphi = \operatorname{sn}(\varphi, k) + O(k^2)$). The advantage of using Jacobi functions lies in the existence of useful functional relations, relatively simple analyticity properties and symmetries [11].

To obtain a large variety of pendulum-Gordon equations with exact soliton solutions, we use directly (4), (5) and employ the Jacobi functions in place of f. Performing the scaling $\varphi \rightarrow \varphi/2 = u$ we eliminate the redundant numerical factors ($f_{SG}(u) = \sin u$) and obtain the list of the fundamental solvable perturbed SG equations given in Table I, where the following abbreviations are used: $s = \text{sn}(u, k)$, $c = \text{cn}(u, k)$, $d = \text{dn}(u, k)$, $S = \text{sn}(u/2, \text{k})$, $C = \text{cn}(u/2, k)$, $D = \text{dn}(u/2, k)$, $k^2 = 1 - k'^2$. The one-soliton solutions $U =$

TABLE I

The alternatives of the Jacobi-Gordon equation and corresponding one-soliton solutions and their masses

Equation		Soliton			Mass
m	$\Phi_m(u)$	$f(u)$	γ	$T_m(u)$	μ_m
1	sc/d^3	s/d	± 1	$s/c+1) = SD/C$	$2\psi/\sin 2\psi$
2	scd	s	± 1	$s/(c+d) = S/(CD)$	$\varrho/\text{th}\,\varrho$
3	$-scd$	c	$\pm k'$	$(k's+d)/c$	$\psi/\sin\psi$
4	$-k'^2 sc/d^3$	c/d	± 1	$(1+s)/c$	$\varrho/\text{th}\,\varrho$
5	$(3-2/d^2)sc/d^3$	s/d^2	± 1	$s/((c+d)(d-kc)^k)$	$1/\cos^2\psi$

$U_m(x)$ for different right-hand sides $\Phi = \Phi_m(U)$ of (1) are defined by $T = T_m(U)$ and (3) with $A = 1$ in an implicit way. The masses are given by (5) in terms of the hypergeometric functions μ_m of the parameter $k^2 = \text{th}^2\varrho = \sin^2\psi$. Using definition [11] of the elliptic integral of the first kind $F(r, k)$ we get the explicit forms

$$U_1(x) = F(2 \arctan e^x, k),$$

$$U_2(x) = 2(1+k)^{-1} F(\arctan e^x, 2k^{1/2}(1+k)^{-1}),$$

$$U_3(x) = F(\arccos \text{th}\, k'x, k) + F(\pi/2, k),$$

$$U_4(x) = F(\arcsin \text{th}\, x, k), \ldots$$

of one soliton solutions. Table I may be continued arbitrarily by halving arguments, inserting dn's etc. but starting with the fifth row, the inverse T^{-1} becomes a complicated function.

Let us consider the first row ($m = 1$) and specify the Jacobi-Gordon (JG) equation as

$$\partial_x^2 u(x, t) - \partial_t^2 u(x, t) = a^2 \Phi_{JG}[u(x, t)],$$

$$\Phi_{JG}(u) = \text{sn}(u, k)\,\text{cn}(u, k)/\text{dn}^3(u, k) \tag{11}$$

($a^2 = 1$). It "interpolates" between the sine-Gordon ($k = 0$) and another standard (sinh--Gordon, $k = 1$) equation. For $u \ll 1$, very close approximation of the polynomial $\lambda\varphi^6$ and $\lambda\varphi^4$ field theories by (11) may be achieved by the special choice $k_{(6)}^2 = 1/2$, $k_{(4)}^2 = (1 \pm 3/\sqrt{17})/2$ of the value of modulus which corresponds to the elimination of the $O(u^3)$

and $O(u^5)$ terms, respectively, in the Taylor expansion of $\Phi_{JG}(u)$. We may also find the connection between (11) and DSE since

$$\Phi_{JG}(u) = \sin\varphi - \frac{k^2}{2}\sin 2\varphi + O(k^4).$$

Thus, the transition from (11) to (8) (for $b = 1$, $a = -k^2$ and $\beta = 2$) and to the SG equation is smooth for small k. Moreover, the stability potential

$$W_{JG}(x) = 1-2(1-2k^2)/z - 3k^2(1-k^2)/z^2, \quad z = \mathrm{ch}^2\, x - k^2$$

to be used in (6) has the structure of $W_{DS}(x)$. It seems that both JG and DS cases have the same level of mathematical complexity.

The different forms of $\Phi(u)$ given in Table I are not independent. By analytic continuation and using the identities valid for Jacobi functions we get

$$\begin{aligned} \Phi_{JG}(u) &= \mathrm{sn}\,(u,k)\,\mathrm{cn}\,(u,k)/\mathrm{dn}^3\,(u,k) \\ &= -k'^{-2}\,\mathrm{sn}\,(u_1,k)\,\mathrm{cn}\,(u_1,k)\,\mathrm{dn}\,(u_1,k) \\ &= +k'^{-1}\,\mathrm{sn}\,(u_2,k_2)\,\mathrm{cn}\,(u_2,k_2)\,\mathrm{dn}\,(u_2,k_2), \end{aligned}$$

$$u_1 = u - F(\pi/2,k), \quad u_2 = k'u, \quad k_2 = ik/k', \quad k'^2 = 1-k^2,$$

which renders it possible to identify the first four items in Table I by means of simple re-interpretation of the corresponding pendulum model. Yet, they are physically distinguishable for $a^2 = 1$ in (11) due to the different outer fields needed to define different Φ's in the pendulum model. E.g. after π-normalization of the period we get increasing or decreasing k-dependence of soliton mass for $m = 1$ or 3 in Table I, respectively.

Since the non-elementary Jacobi functions are obtained even when solving the simple SG equation (Hu [5]), it is interesting to note that now the class of necessary functions is closed. Really, the ansatz for the one-dimensional static field $u = u(x)$

$$\mathrm{sn}\,(u,k) = \beta\,\mathrm{pq}\,(x,l), \tag{12}$$

where pq represents any Jacobi function (sn, cn or dn) and $x \in (-\infty, \infty)$, leads to the same type of equation (11) satisfied by $u(x)$ provided that we specify a^2, β and pq according to Table II. This may be verified by direct insertion.

TABLE II

The static "soliton-lattice" solutions Eq. (12), $u(x) = F(\arcsin\beta\,\mathrm{pq}(x,l), k)$, to Jacobi–Gordon equation ($k'^2 = 1-k^2$, $l'^2 = 1-l^2$)

pq (x,l)	β	a^2	m
cn (x,l)	1	$1-k'^2l'^2$	1
cn (x,l)	i l/l'	$-1+l^2k'^2$	
dn (x,l)	1	$1-k^2l'^2$	1
dn (x,l)	$1/l'$	k'^2-l^2	
sn (x,l)	1	$-l^2+k^2$	4
sn (x,l)	l	$-1+k^2l^2$	4

The general static solution given by (12) is not stable. Let us again consider the first row of Table II only. The solution (12) with $l \neq 1$ may be visualized as an infinite number of coils of the elastic connection of pendula. For $k = 0$ it becomes identical with the SG soliton lattice described by Hu [5]. For any $k \geqslant 0$, the finite energy per coil may be defined by the integral along one period ($C = 0$)

$$M_l = \frac{2m^3}{\lambda} \int_{x_0}^{x} [(\partial_x u)^2 + f^2(u)] dx$$

and evaluated in a closed form. In our case we have

$$M_l = \tfrac{1}{2} M_0 \left[\frac{a\pi}{kk'} \Lambda_0(\psi, l) - l'^2 F\left(\frac{\pi}{2}, k\right) \right], \qquad M_0 = 8m^3/\lambda, \qquad \sin\psi = k/a, \qquad (13)$$

where Λ_0 denotes Heumann's lambda function [11].

Because of the repulsion of the coils ($M_l > M_0$), the boundary conditions must be fixed by some additional external force in the mechanical model. Removal of this force either entails the time dependence of the solution or corresponds to the limit $l \to 1$, $x_0 \to -\infty$, $x_1 \to \infty$. Only one static coil (= one soliton) pertains and the energy (13) decreases to the one soliton mass $M = M_0 \cdot 2\psi/\sin 2\psi$. Further, the sine-Gordon limit ($l = 1$, $k \to 0$) of the mass M is equal to M_0 so that the transition to $k \neq 0$ is continuous and the free parameter k enables us to vary the nonlinearity in a smooth though nontrivial way.

Similar conclusions hold for other items in the Tables (the fifth row in Table I interpolates between SG case $k = 0$ and DSE for $k = 1$, $a = -1/2$, $b = 1$, $\varphi =$ pure imaginary, it provides similar approximations of the $\lambda\varphi^6$ and $\lambda\varphi^4$ theories with $k_{(6)}^2 = 1/5$ and $k_{(4)}^2 = (3+\sqrt{113})/26$, respectively, etc.). Unfortunately, it is not yet clear whether also the breather and/or the N-soliton solutions will be found in a closed analytic form in the near future. Nevertheless, the $N = 1$ and $N = \infty$ soliton solutions given here might also be of considerable help in numerical analysis of $N > 1$ cases.

5. *Concluding remarks*

Soliton is a new phenomenon in the field theory being an essentially non-perturbative solution of (nonlinear) field equations. The most popular sine-Gordon case underlying many speculations is exceptional in its simplicity (from many points of view). We have discussed here, on the classical level, some examples of the influence of the finite periodic perturbation on the sine-Gordon equation. They may help elucidate the role of perturbation in the nonlinear (non-polynomial) context. Many questions remain open, and not only formal ones (possibility to generalize the Bäcklund or inverse scattering transformations) but especially those concerning the quantization (through WKB and equation (6) related to the "next to solvable" class of Fuchs differential equations with four singularities [12]) and non-perturbative renormalizability.

APPENDIX

Reduced form of the JG equation

We perform the change of independent variable $u(x,t) \to v(x,t) = \mathrm{sn}(u/2,k)\,\mathrm{dn}(u/2,k)/\mathrm{cn}(u/2,k)$ and express $\partial_x^2 u$, $\partial_t^2 u$, $\Phi_{JG}(u)$ in terms of $\partial_x^2 v$, $\partial_t^2 v$, $(\partial_x v)^2$, $(\partial_t v)^2$, v. JG equation (11) with $a^2 = 1$ then becomes

$$k^2\psi(v) - k'^2\psi(iv) = 0,$$
$$\psi(v) = v(1-v^2)\left[(1-v^2)(\partial_x^2 v - \partial_t^2 v) + 2v((\partial_x v)^2 - (\partial_t v)^2) - v(1+v^2)\right]. \tag{A1}$$

We note that this is the common reduced form of the JG, SG and sinh-Gordon equations for $k^2 \neq 0, 1$, $k^2 = 0$ and $k^2 = 1$, respectively. Unfortunately, we have not found the multisoliton solutions for $k^2 \neq 0, 1$. E.g. the method of Osborne and Stuart [13] (separable ansatz for v) gives only the one-soliton solution again.

It is worth mentioning that the mixing of solutions of SG and sinh-Gordon equation in (Al) resembles the Bäcklund transformation formulas obtained for the 2+1 and 3+1 dimensional versions of the SG equation [14].

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