A NON-COMPACT GAUGE GROUP FOR THE DIRAC EQUATION

By D. Darby and Th. W. Ruijgrok

Instituut voor Theoretische Fysica, Rijksuniversiteit Utrecht*

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It is shown that infinitesimal transformations of a Dirac spinor, in which small amounts of negative energy states are mixed with positive energy states, together with infinitesimal phase transformations, form the generators of an sl(2,R)-algebra. The global invariance of the Dirac equation obtained in this way is extended to a local invariance by introducing, in addition to the electromagnetic potential, another complex potential, which carries negative energy and is doubly charged. The field equations for the Dirac field coupled to the new gauge fields are derived and a number of special solutions are given.

1. *Introduction*

The multiplications of a Dirac spinor by a phase factor form a one parameter continuous group of transformations, which leave the Dirac equation invariant. The only discrete invariance transformation which is not related to a change in the space and time coordinates is the charge conjugation C:

$$\psi(x) \to \psi'(x) = \gamma^2\psi^*(x) \equiv \psi_c(x). \tag{1.1}$$

It changes a state of positive energy into a state of negative energy, which follows from the fact that $\bar{\psi}_c\psi_c = -\bar{\psi}\psi$. Since the difference in sign of the energy cannot be declared to be physically irrelevant, it seems impossible to enlarge the U(1) group of the phase transformations by adding the discrete transformation (1.1).

However, it is possible to mix infinitesimal phase transformations with infinitesimal C-transformations

$$\psi \to \psi' = \psi + i\alpha\psi + \beta\psi_c + \dots \ (\alpha \text{ real}, \beta \text{ complex}) \tag{1.2}$$

such that $\bar{\psi}\psi$ is invariant.

In this paper we want to investigate the consequences of the assumption that for two states which are connected by the transformation (1.2) all observable effects are the same.

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* Address: Instituut voor Theoretische Fysica, Princetonplein 5, Postbus 80.006, 3508 TA Utrecht, The Netherlands.

We will first show that the transformations (1.2) form an invariance algebra sl (2, R) for the Dirac equation. We will then turn this global gauge invariance into a local one and introduce gauge fields in the usual way: the electromagnetic fields connected with the phase transformations and a new complex field, related to the infinitesimal transformations of charge conjugation. We stress that no probability interpretation of the theory is given and that the whole discussion is for classical fields. Transformation (1.2) is the infinitesimal form of a transformation discussed by Galindo [1]. Because it does not appear to be well known and because our method of deriving it can be readily generalized, we discuss this transformation in some detail.

In the remaining part of this section we want to derive the global invariance group of the Dirac equation $(\gamma^\mu\partial_\mu+m)\psi(x) = 0$, but in a slightly different way from that indicated above. We use the convention in which

$$\gamma^k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} (k = 1, 2, 3) \qquad \gamma^4 = i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3)$$

with the metric $-g^{00} = g^{11} = g^{22} = g^{33} = +1$.

By introducing the 8-component real spinor

$$X(x) = \begin{pmatrix} \mathrm{Re}\ \psi(x) \\ \mathrm{Im}\ \psi(x) \end{pmatrix}$$

the Dirac equation can be written as

$$(\Gamma^\mu\partial_\mu+m)X(x) = 0 \tag{1.3}$$

with the real 8×8 matrices Γ^μ defined by

$$\Gamma^0 = \begin{pmatrix} 0 & i\gamma^0 \\ -i\gamma^0 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & i\gamma^1 \\ -i\gamma^1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & i\gamma^3 \\ -i\gamma^3 & 0 \end{pmatrix}.$$

These satisfy the usual anti-commutation relations $\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu = 2g^{\mu\nu}$. The Lorentz scalar $\bar\psi\psi$, which we will also keep invariant under our new transformations, can be written as

$$\bar\psi\psi = \bar{X}X \equiv XGX,$$

where

$$G = \begin{pmatrix} \gamma^4 & 0 \\ 0 & \gamma^4 \end{pmatrix}.$$

Consider now all real and linear infinitesimal transformations

$$X_i(x) \to X_i'(x) = (\delta_{ij}+Y_{ij}+ \ldots)X_j(x), \tag{1.4}$$

which do not change the value of $\bar{X}X$ and are such that $X'(x)$ again satisfies Eq. (1.3). The latter condition requires that Y commute with all Γ^μ: $[Y, \Gamma^\mu] = 0$. We find for the most general infinitesimal transformation Y the form

$$Y = \theta^a L_a, \tag{1.5}$$

in which θ^a are the infinitesimal real parameters and

$$L_1 = \begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \gamma^2 \\ \gamma^2 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are 8×8 real matrices. They indeed commute with all Γ^μ: $[\mathrm{L}_a, \Gamma^\mu] = 0$.
The matrices $\frac{1}{2}L_a$ form an algebra

$$[\tfrac{1}{2} L_a, \tfrac{1}{2} L_b] = c^c_{.ab} \cdot \tfrac{1}{2} L_c$$

with the structure constants

$$c^1_{23} = -c^1_{32} = -1, \quad c^2_{31} = -c^2_{13} = -1, \quad c^3_{12} = -c^3_{21} = +1, \quad \text{all others zero.}$$

A metric in the "charge space" spanned by $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$ is defined by

$$K_{ab} = \tfrac{1}{2} c^c_{.ad} c^d_{.bc} \quad \text{or} \quad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and *this metric tensor will be used for raising and lowering the group indices $a, b, c, \ldots$* . In view of the gauge fields to be introduced later we will call a vector $\vec{\theta}$ of the electromagnetic type, of the charged type or a null vector, when its length, defined by

$$\theta^2 = \theta_a \theta^a = K^{ab} \theta_a \theta_b = \theta_1^2 + \theta_2^2 - \theta_3^2,$$

is negative, positive or zero respectively. The multiplication rules for the matrices L_i are

$$L_a L_b = K_{ab} + c^c_{.ab} L_c, \tag{1.6}$$

from which follows in particular that

$$L_a L_b + L_b L_a = 0 \quad \text{for} \quad a \neq b, \quad L_1^2 = L_2^2 = 1 \quad \text{and} \quad L_3^2 = -1.$$

L_3 is a compact generator, L_1 and L_2 are non-compact. This is related to the way in which the original ψ-field changes under an infinitesimal transformation. For this we find

$$\psi(x) \to \psi'(x) = \psi(x) - i\theta^3 \psi(x) + (\theta' + i\theta^2) \psi_c(x) + \ldots$$

with $\psi_c(x) = \gamma^2 \psi^*(x)$. We see that L_3 is connected with the change of phase and L_1 and L_2 with the charge conjugation as mentioned in the beginning of this section.

Finite transformations which leave the Dirac equation invariant are given by[1]

$$X'(x) = gX(x), \tag{1.7}$$

[1] We prefer this form, since for non-compact groups two exponentiations are generally necessary to map the algebra onto the group [2].

with

$$g = g^0 1 + g^a L_a \tag{1.8}$$

and

$$(g^0)^2 - g^a g_a = 1. \tag{1.9}$$

Using Eq. (1.6) it can be shown that the matrices g are the elements of a group with the inverse given by

$$g^{-1} = g^0 1 - g^a L_a.$$

In fact they form an eight-dimensional reducible representation of the non-compact group SL(2, R). The formulas (with T indicating the transposed)

$$GL_a^{\mathrm{T}} = -L_a G \quad \text{and} \quad G(\Gamma^\mu)^{\mathrm{T}} = -\Gamma^\mu G \tag{1.10}$$

are easily proved. With the first of these it then follows that $\bar{X}$ transforms as

$$\bar{X}(x) \to \bar{X}'(x) = \bar{X}(x) g^{-1}, \tag{1.11}$$

from which, combined with equation (1.7), it is seen that $\bar{X}X$ is indeed invariant, also under finite transformations. For the ψ-field these become

$$\psi'(x) = (g^0 - ig^3)\psi(x) + (g^1 + ig^2)\psi_c(x).$$

We see that $\psi'(x)$ can never be equal to $\psi_c(x)$ and the operation of charge conjugation is not a gauge transformation. Since $|g^0 - ig^3|^2 = |g^1 + ig^2|^2 + 1$ we also see that an increase in the amplitude of negative energy components of the wave function is accompanied by an equal increase in amplitude of the positive energy components. The energy of the state is invariant; the gauge invariant energy-momentum tensor is given in the next section.

The Dirac equation in the real form Eq. (1.3) can be obtained from the Lagrangian

$$L = -\bar{X}(\Gamma^\mu \partial_\mu + m)X, \tag{1.12}$$

or in terms of the ψ-field

$$L = -\tfrac{1}{2}\,\bar{\psi}(\gamma^\mu \vec{\partial}_\mu + m)\psi + \tfrac{1}{2}\,\bar{\psi}(\gamma^\mu \overleftarrow{\partial}_\mu - m)\psi.$$

Because of the new gauge group new conserved currents can be derived from this Lagrangian. We find

$$s_a^\mu = \bar{X} L_a \Gamma^\mu X, \qquad \partial_\mu s_a^\mu = 0. \tag{1.13}$$

In terms of the ψ-fields they can be written as

$$j^\mu \equiv s_3^\mu = -i\bar{\psi}\gamma^\mu\psi,$$

which is the usual current connected with phase changes, and a new current connected with charge conjugation and given by

$$s^\mu \equiv \frac{1}{\sqrt{2}}(s_1^\mu + is_2^\mu) = -\frac{1}{\sqrt{2}}\,\bar{\psi}_c\gamma^\mu\psi$$

and its complex conjugate

$$s^{\mu *} \equiv \frac{1}{\sqrt{2}}(s_1^{\mu} - i s_2^{\mu}) = \frac{1}{\sqrt{2}} \bar{\psi}\gamma^{\mu}\psi_c.$$

A Lorentz and gauge invariant quantity constructed with these currents is

$$s^{\mu a} s_{\mu a} = -j^{\mu} j_{\mu} + 2 s^{\mu} s_{\mu}^{*}.$$

Without spoiling the new gauge group we can therefore obtain a self-interacting Dirac field by adding to the Lagrangian (1.12) a current-current term

$$L_v = g_{\mu\nu} G_v [(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma^{\nu}\psi) - (\bar{\psi}_c\gamma^{\mu}\psi)(\bar{\psi}\gamma^{\nu}\psi_c)].$$

Another invariant interaction term, leading to the Thirring model, is

$$L_s = G_s(\bar{\psi}\psi)^2.$$

More generally we could extend any field theory and require invariance under continuous transformations mixing particle and anti-particle states as in Eq. (1.2). We will, however, not pursue these ideas at the moment, but rather study the interactions introduced by requiring that the global SL (2, R)-invariance found so far also holds locally. This will be done in the remaining sections.

We end this section by remarking that Pauli [3] and Gürsey [4] have considered a rather similar global transformation for quantized fields. In order to satisfy the anti-commutation relations an extra factor γ^5 had to be introduced [3]. It also was necessary to consider two fields for the case where the particles were not massless [4]. In contrast we are discussing one classical Dirac spinor with a mass.

2. *Local gauge invariance*

Local gauge invariance can be obtained in the standard way [5] by introducing twelve real gauge potentials $Z_{\mu}^{a}(x)$ and replacing $\partial_{\mu}X(x)$ in the Lagrangian by the covariant derivative

$$D_{\mu}X(x) = (\partial_{\mu} + Z_{\mu}(x))X$$

in which $Z_{\mu}(x) = eZ_{\mu}^{a}(x)L_a$ is an 8×8 matrix. Gauge transformations of the X-field are again given by the formulas (1.7–9), but now with $g^0(x)$ and $g^a(x)$ functions of x. For the Lagrangian to be invariant the gauge potentials should transform as

$$Z'_{\mu}(x) = gZ_{\mu}g^{-1} + g\partial_{\mu}g^{-1}.$$

The fields $F_{\mu\nu}^{a}(x)$ and their matrix forms are defined by

$$F_{\mu\nu} = eF_{\mu\nu}^{a}L_a$$

and

$$F_{\mu\nu} = \partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu} + [Z_{\mu}, Z_{\nu}]. \tag{2.1}$$

Under gauge transformations the latter behaves as

$$F'_{\mu\nu}(x) = g(x)F_{\mu\nu}(x)g^{-1}(x).$$

Adding an invariant term for the free gauge fields, the Lagrangian becomes

$$L = -\bar{X}(\Gamma^{\mu}D_{\mu}+m)X+\tfrac{1}{4}\,F^{a}_{\mu\nu}F^{\mu\nu}_{a}.$$

The field equations derived from this Lagrangian read:

$$(\Gamma^{\mu}D_{\mu}+m)X(x) = 0 \tag{2.2}$$

and

$$\partial_{\nu}F^{\nu\mu}_{a}(x) = -es^{\mu}_{a}(x) \tag{2.3}$$

with

$$s^{\mu}_{a}(x) = \bar{X}\Gamma^{\mu}L_{a}X+2c^{bc}_{a}F^{\mu\nu}_{b}Z_{\nu c}, \tag{2.4}$$

which are the new conserved currents, taking the place of Eq. (1.13). For future use we have also calculated the symmetric energy-momentum tensor and found

$$T^{\mu\nu} = -F^{\mu\varrho}_{a}F^{\nu a}_{\varrho}+\tfrac{1}{4}\,g^{\mu\nu}F^{a}_{\lambda\sigma}F^{\lambda\sigma}_{a}+\tfrac{1}{2}\,\bar{X}(g^{\mu\sigma}\Gamma^{\nu}+g^{\nu\sigma}\Gamma^{\mu})D_{\sigma}X. \tag{2.5}$$

It is easily checked that this is invariant under local gauge transformations. It is also conserved, i.e., $\partial_{\mu}T^{\mu\nu}(x) = 0$ and its trace is

$$g_{\mu\nu}T^{\mu\nu}(x) = -m\bar{X}X = -m\bar{\psi}\psi. \tag{2.6}$$

In order to display the electric charge of the new gauge potentials, we consider a special infinitesimal gauge transformation for which only $\theta^3 = \theta$ is nonvanishing and $\theta^1 = \theta^2 = 0$. The fields $A_{\mu} = Z^3_{\mu}$ and $W_{\mu} = \frac{1}{\sqrt{2}}(Z^1_{\mu}+iZ^2_{\mu})$ and the electron field ψ transform as follows

$$A'_{\mu}(x) = A_{\mu}(x)-\frac{1}{e}\,\partial_{\mu}\theta, \quad W'_{\mu}(x) = W_{\mu}(x)-2i\theta(x)W_{\mu}(x), \quad \psi'(x) = \psi(x)-i\theta(x)\psi(x).$$

This shows that $A_{\mu}(x)$ transforms like the electromagnetic field, whereas $W_{\mu}(x)$ transforms like a field carrying a charge which is twice as large as that of the ψ-field. Another uncommon feature of the W_{μ}-field shows up when the energy-momentum tensor for a pure gauge field is written in terms of

$$F_{\mu\nu} = F^{3}_{\mu\nu} = \partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+2ie(W_{\mu}W^{*}_{\nu}-W^{*}_{\mu}W_{\nu})$$

and

$$G_{\mu\nu} = \frac{1}{\sqrt{2}}(F^{1}_{\mu\nu}+iF^{2}_{\mu\nu}) = \partial_{\mu}W_{\nu}-\partial_{\nu}W_{\mu}+2ie(W_{\mu}A_{\nu}-W_{\nu}A_{\mu}).$$

We find

$$T^{\mu\nu} = T^{\mu\nu}_{F}+T^{\mu\nu}_{G},$$

with

$$T_F^{\mu\nu} = F^{\mu\varrho}F^{\nu}{}_{\varrho} - \tfrac{1}{4}\, g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \tag{2.7}$$

and

$$T_G^{\mu\nu} = -G^{\mu\varrho}G^{\nu *}{}_{\varrho} - G^{\mu\varrho *}G^{\nu}{}_{\varrho} + \tfrac{1}{2}\, g^{\mu\nu}G^{*}_{\lambda\sigma}G^{\lambda\sigma}. \tag{2.8}$$

In the same way as for the Maxwell field it can be shown that the energy density $T_F^{00}(x) \geqslant 0$. It is also true however, that the W_μ-field carries a negative energy $T_G^{00}(x) \leqslant 0$. This is, of course, connected with the fact that the W_μ-field mediates between positive and negative energy states. For this reason it must not be excluded that upon second quantization, when the positron acquires positive energy, also the W-boson will be of positive energy. It will keep its double charge, however.

Returning to the classical case we should like to remark that, because of the existence of negative energy states, there is probably no stable solution of the Dirac-Maxwell equations, describing an electron bound to its own electromagnetic field. The motto being: "If radiation can occur, it will occur". In the present case, however, the existence of a self binding $e^+ - W^{--}$ state should not be excluded a priori, because the radiation argument applies for the negative as well as for the positive energy states. Some aspects of this possibility will be discussed in the next section, where a number of exact special solutions of the field equations will be given.

3. *Special solutions of the field equations*

A. As a first example we take the equations (2.3) and (2.4) for the gauge fields and consider the currents

$$\varrho_a^\mu(\vec{x}) = \bar{X}\Gamma^\mu L_a X \tag{3.1}$$

as given external sources, independent of time. We now try to find a static solution for the gauge fields by making the Ansatz

$$Z_0^a(x) = \phi^a(\vec{x}) \quad \text{and} \quad Z_i^a(x) = 0 \quad (i = x, y, z).$$

The equations then become

$$\Delta\phi_a(\vec{x}) = e\varrho_a^0(\vec{x}) \quad \text{and} \quad 2\varepsilon_{abc}\phi^b\partial_i\phi^c = \varrho_a^i(\vec{x}),$$

which shows that the current densities $\varrho_a^i(\vec{x})$ are completely determined by the charge densities $\varrho_a^0(\vec{x})$. The contribution of the gauge fields to the total energy $\int T^{00}(\vec{x})d\vec{x}$ can be calculated from Eqs. (2.7) and (2.8) and gives

$$\varepsilon = +\frac{e^2}{8\pi}\int \frac{\varrho_a^0(\vec{x})\varrho_0^a(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}d\vec{x}'. \tag{3.2}$$

(Notice the position of all indices; summation over a is implied). A special case occurs for $\varrho_a^i(\vec{x}) \equiv 0$. This happens if and only if $\varrho_a^0(\vec{x}) = \alpha_a \varrho(\vec{x})$, where the α_a are the components of an arbitrary real gauge vector. In this case the energy becomes

$$\varepsilon = \frac{e^2}{8\pi} f(\vec{\alpha}) I \tag{3.3}$$

with

$$f(\vec{\alpha}) = -\alpha^a \alpha_a \quad \text{and} \quad I = \int \frac{\varrho(x)\varrho(x')}{|\vec{x}-\vec{x}'|} d\vec{x} d\vec{x}' > 0.$$

The force between the charges in two different regions can be calculated from Eq. (3.2) or (3.3). In the latter case it is seen that this force is repulsive if $f(\vec{\alpha})$ is positive and attractive if it is negative. In the repulsive case it is possible to find a global gauge transformation which makes $\alpha_1 = \alpha_2 = 0$ and the field has become purely electromagnetic.

An interesting situation arises if $\varrho_a^0(\vec{x})$ is a null vector for all $\vec{x}$, i.e.,

$$\varrho_a^0(\vec{x})\varrho^{0a}(\vec{x}) = 0,$$

which happens for instance if

$$\varrho_1^0(\vec{x}) = \cos\chi(\vec{x})\varrho_3^0(\vec{x}) \quad \text{and} \quad \varrho_2^0(\vec{x}) = \sin\chi(\vec{x})\varrho_3^0(\vec{x}).$$

Since we want to keep $Z_i^a(x) = 0$ and have a static solution, only global gauge transformations are allowed and with such a transformation it will in general be impossible to make $\varrho_2^0(\vec{x})$ vanish for all $\vec{x}$ and $\varrho_1^0(\vec{x})$ equal to $\varrho_3^0(\vec{x})$ everywhere. The total electric charge may thus be non zero, whereas the total W-charge

$$Q_1 + iQ_2 = \int [\varrho_1^0(\vec{x}) + i\varrho_2^0(\vec{x})] d\vec{x} = \int e^{i\chi(\vec{x})} \varrho_3^0(\vec{x}) d\vec{x}$$

may vanish if $\chi(\vec{x})$ behaves appropriately, so that asymptotically there is only an electromagnetic field. In any case the force of a volume element $d\vec{x}'$ on a volume element $d\vec{x}$ becomes

$$\frac{\varrho_3^0(\vec{x})\varrho_3^0(\vec{x}')}{|\vec{x}-\vec{x}'|^2} \cdot [1-\cos(\chi(\vec{x})-\chi(\vec{x}'))] \cdot \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x} d\vec{x}'.$$

For $\vec{x}$ and $\vec{x}'$ close together it is seen that this force is not singular anymore. The integral of $\vec{x}'$ over a small sphere around $\vec{x}$ is equal to zero. This may be called asymptotic freedom. It means that in principle a charge distribution can be stable, which is impossible for the electromagnetic and probably also for the Yang–Mills case, because for these theories no null-vectors exist in the charge space.

It should be emphasized that this conclusion is a consequence of the assumption that $\varrho_a^0(\vec{x})$ is a null-vector. Whether this is really the case can only be established by solving the equation for $X(x)$ and calculating $\varrho_a^0(x)$ using Eq. (3.1). So far we have not succeeded in solving this problem, except for a special case to be discussed in Section 3E.

B. If the fields inside a particle are not singular one may perhaps get some idea of the magnitude and type of these fields by looking for solutions of the field equations which are constant throughout all space and time. For the Dirac–Maxwell equations only the trivial solution $\psi = 0$ of this type exists. In the present case, however, a nonvanishing ψ of the form $\psi = (\psi_1, 0, 0, 0)$ can be found. A complete solution is given by

$$\psi_1^2 = \frac{8m^2}{27e^2}, \qquad -Z_y^1 = Z_x^2 = Z_t^3 = \frac{m}{3e},$$

as can be verified by substitution into the equations (2.2)–(2.4). For the energy–momentum tensor as given by Eq. (2.5) we find

$$T^{\mu\nu} = \begin{bmatrix} 3x & 0 & 0 & 0 \\ 0 & -3x & 0 & 0 \\ 0 & 0 & -3x & 0 \\ 0 & 0 & 0 & -3x \end{bmatrix},$$

with $x = 2m^4/81e^2$. The trace of T_ν^μ is equal to $-12x = -m\bar{\psi}\psi$, which is in agreement with Eq. (2.6). The total energy density is

$$T^{00} = 3x = \frac{2m^4}{27e^2}, \tag{3.4}$$

of which a part $T_{\mathrm{F}}^{00} = x$ can be assigned to the electromagnetic field, Eq. (2.7) and a part $T_{\mathrm{G}}^{00} = -2x$ is residing in the other gauge fields, Eq. (2.8). From Eq. (3.4) we can calculate that the total energy contained in a cube with a side equal to the Compton wavelength $1/m$ of the particle is equal to 0.8 m, when $e^2/4\pi = 1/137$. If a localized solution with finite energy exists and has a similar energy density, we obtain an equation for the electric charge by identifying this energy with the input mass of the particle $\int T^{00}(\vec{x})d\vec{x} = m$. For dimensional reason the charge obtained from this equation cannot depend on m and e^2 is a truly universal constant, if not of nature, at least of this theory.

C. In this section we will consider pure gauge fields, $\psi = 0$, and look for solutions which are constant in space and time. From Eq. (2.1) follows that the fields can be written as

$$F_{\mu\nu}^a = 2ec^a_{.bc}Z_\mu^b Z_\nu^c, \tag{3.5}$$

which should be substituted into the expression Eq. (2.4) for s_a^μ. The field equations (2.3) are simply $s_a^\mu = 0$, which becomes

$$M_\nu^\mu Z_a^\nu = \kappa Z_a^\mu, \tag{3.6}$$

where we have introduced $M_\nu^\mu = Z_a^\mu Z_\nu^a$ and $\kappa = M_\mu^\mu$. Multiplying Eq. (3.6) by $Z^{\lambda a}$ and summing over a gives

$$M_\nu^\mu M^{\nu\lambda} = \kappa M^{\mu\lambda}. \tag{3.7}$$

Before solving Eq. (3.6) we first substitute the fields of Eq. (3.5) into the expression (2.5) for the energy-momentum tensor to obtain

$$T^{\mu\nu} = -4e^2[M^\mu_\varrho M^{\varrho\nu} - \kappa M^{\mu\nu}] + e^2 g^{\mu\nu}[M^\mu_\varrho M^\varrho_\mu - \kappa^2].$$

From Eq. (3.7) we see immediately that $T^{\mu\nu} = 0$. In the case of a compact gauge group this would imply that the fields $F^a_{\mu\nu}$ are also zero. This is not so in the present case. Indeed it can be shown from Eq. (3.6) that all Z^a_μ for which the corresponding $F^a_{\mu\nu}$ do not necessarily vanish, must satisfy the relations

$$Z^3_\mu = \sqrt{\frac{a}{a+b}}\, Z^1_\mu + \sqrt{\frac{b}{a+b}}\, Z^2_\mu \tag{3.8}$$

and

$$Z^1_\mu Z^{\mu 2} = \sqrt{ab}\,, \tag{3.9}$$

where a and b must have the same sign and are defined by

$$a = Z^1_\mu Z^{1\mu} \quad \text{and} \quad b = Z^2_\mu Z^{\mu 2}. \tag{3.10}$$

Using these relations the fields become

$$F^3_{\mu\nu} = 2e(Z^1_\mu Z^2_\nu - Z^2_\mu Z^1_\nu), \quad F^1_{\mu\nu} = \sqrt{\frac{a}{a+b}}\, F^3_{\mu\nu}, \quad F^2_{\mu\nu} = \sqrt{\frac{b}{a+b}}\, F^3_{\mu\nu},$$

from which follows immediately that

$$(F^1_{\mu\nu})^2 + (F^2_{\mu\nu})^2 - (F^3_{\mu\nu})^2 = 0.$$

So for each μ, ν-pair $F^a_{\mu\nu}$ is a null-vector in the charge space, which cannot be made to vanish by a gauge transformation and therefore is a non-trivial solution of the field equations. We still have to show that the Eqs. (3.8–10) allow solutions which give a non-vanishing $F^a_{\mu\nu}$. This can be done by writing $Z^1_\mu = [V_1, \vec{R}_1]$ and $Z^2_\mu = [V_2, \vec{R}_2]$. Substitution into Eq. (3.9) and using Eq. (3.10) gives

$$\alpha^2 - 2v_1 v_2 \alpha + v_1^2 + v_2^2 - 1 = 0, \tag{3.11}$$

where

$$\alpha = \frac{\vec{R}_1 \cdot \vec{R}_2}{R_1 R_2}, \quad v_1 = \frac{V_1}{R_1} \quad \text{and} \quad v_2 = \frac{V_2}{R_2}.$$

Eq. (3.11) has a solution with $-1 \leqslant \alpha \leqslant 1$ if and only if $-1 \leqslant v_1 \leqslant 1$ and $-1 \leqslant v_2 \leqslant 1$. From Eq. (3.10) we then find that $R_1^2 = \dfrac{a}{1-v_1^2}$ and $R_2^2 = \dfrac{b}{1-v_2^2}$, which means that a and b must both be positive, i.e., Z^1_μ and Z^2_μ are both space-like. For arbitrary a, b, v_1 and v_2 satisfying the above conditions the vectors Z^1_μ and Z^2_μ and hence Z^3_μ can be constructed, such that Eqs. (3.8–10) hold. An example with $F^a_{\mu\nu} \neq 0$ is then easily given. Two solutions will be called equivalent if one can be obtained from the other by a gauge transformation.

This happens only if the quantity $[Z_\mu^1]^2+[Z_\mu^2]^2-[Z_\mu^3]^2$ is the same for the two solutions and for each μ.

Now, in a classical field theory any state for which $T^{\mu\nu}$ vanishes, cannot be distinguished from the vacuum. The above construction of a large class of non-equivalent solutions with $T^{\mu\nu}=0$ therefore means that there are many different vacuum states. This is very similar to the different vacua known to exist for the Yang–Mills theory. The difference is, however, that here we work in Minkowski space and not in Euclidean space and with a non-compact group instead of with SU(2). It seems likely that there exist finite energy solutions which interpolate between different vacua. So far, however, we have not been able to construct them explicitly.

D. For the Yang–Mills equations Coleman [6] has found non-abelian plane wave solutions. Also in our case they exist and are given by

$$Z_1^a = Z_2^a = 0 \quad \text{and} \quad Z_0^a = Z_3^a = \tfrac{1}{2}\,xf^a+\tfrac{1}{2}\,yg^a+\tfrac{1}{2}\,h^a,$$

where $f^a(u)$, $g^a(u)$ and $h^a(u)$ are arbitrary functions of the variable $u=z+t$. The only non-vanishing components of the fields and of the energy-momentum tensor are $F_{01}^a=F_{31}^a=-\frac{1}{2}f^a$ and $F_{02}^a=F_{32}^a=-\frac{1}{2}g^a$, and $T^{00}=T^{33}=-T^{03}=-T^{30}=-\frac{1}{4}f^af_a-\frac{1}{4}g^ag_a$, which are all functions of $u=z+t$. The functions $f^a(u)$ and $g^a(u)$ can be chosen in such a way that $T^{00}(u)$ is positive for some u and negative for other u-values. Such a choice therefore gives on observable difference with an electromagnetic plane wave, whereas for the Yang–Mills case this difference did not exist.

We obtain again a vacuum state by choosing $f^a(u)$ and $g^a(u)$ to be null-vectors in the charge space. Then also the fields $F_{\mu\nu}^a(u)$ are null-vectors and this shows that the class of constant vacuum fields, found in the preceding subsection, was not a complete characterization of all possible non-equivalent vacua.

E. In the preceding subsections we have seen that null-vectors in the charge space can give rise to interesting solutions of the field equations. Another example will now be constructed for the case where the X field does not vanish identically.

Let $X_0(x)$ be a solution of the free field equation

$$(\Gamma^\mu\partial_\mu+m)X_0(x)=0$$

and define

$$X(x)=\alpha^aL_aX_0(x) \tag{3.12}$$

with α^a a constant null-vector, i.e., $\alpha^a\alpha_a=0$. Eq. (3.12) is not a gauge transformation of $X_0(x)$. Using Eq. (1.6) it is easy to see that

$$\alpha^aL_aX(x)=0. \tag{3.13}$$

For the gauge potentials we try a solution of the form

$$Z_\mu^a(x)=\alpha^aA_\mu(x). \tag{3.14}$$

The fields calculated from Eq. (2.1) then become

$$F_{\mu\nu}^a=\alpha^aF_{\mu\nu},$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The second term in the right hand side of Eq. (2.4) for the current is zero. With the help of $\bar{X}_0 \Gamma^\mu X_0 = 0$, which can be proved with Eq. (1.10), we then obtain for the current of Eq. (2.4)

$$s_a^\mu(x) = -\alpha_a \bar{X}_0 \Gamma^\mu \alpha^b L_b X_0 \equiv \alpha_a s^\mu(x).$$

By substitution of Eq. (3.14) into Eq. (2.2) and using Eq. (3.13) it follows immediately that $X(x)$ satisfies the field equation (2.2). The remaining Eq. (2.3) for the gauge fields becomes

$$\partial_\nu F^{\nu\mu}(x) = -es^\mu(x),$$

which is now decoupled from the Dirac equation and which, for a given $X_0(x)$, i.e., for a given $s^\mu(x)$, can always be solved.

Since $F^a_{\mu\nu}$ is a null-field its contribution to the energy-momentum tensor is zero. It can even be shown, in the same way as

$$\bar{X}X = -\alpha^a \alpha^b \bar{X}_0 L_a L_b X_0 = -\alpha^a \alpha_a \bar{X}_0 X_0 = 0,$$

that the total energy-momentum tensor Eq. (2.5) vanishes: $T^{\mu\nu}(x) = 0$. We therefore have found a new vacuum solution for which the Dirac field does not vanish. Quantummechanically one may speak of vacuum fluctuations. These fluctuations radiate a gauge field, which however carries neither energy nor momentum. There are again many inequivalent vacuum states of this type. Two functions $X_0^{(1)}$ and $X_0^{(2)}$, for instance, which are connected by a Lorentz-transformation, cannot give two vacuum states $X^{(1)}$ and $X^{(2)}$, which are related by a gauge transformation. Also here there are probably finite energy solutions mediating between different vacua and possibly describing stable single particle states. However, an attempt to construct these states has not been made.

F. For the source free Yang–Mills equations Wu and Yang [7] have found a time--independent solution in which the indices for space-time are mixed with those for the isospin. Since, however, the metric of our gauge group differs from the SU(2) metric, their solution cannot just be copied to get a solution of our equations. Nevertheless it is possible to construct a similar solution. This is done by looking for a z-independent solution of the following form

$$Z_x^2 = -Z_y^1 = tg(u), \quad Z_x^3 = -Z_t^1 = -yg(u), \quad Z_y^3 = -Z_t^2 = xg(u),$$

all other components equal to zero, where $u = +\sqrt{t^2 - x^2 - y^2}$ and $g(u)$ is to be determined from the field equations. For a fixed t we thus consider the fields inside a cylinder, which is centered around the z-axis and has a radius equal to t. With the above potentials the currents s_a^μ and the fields $F_a^{\mu\nu}$ can be calculated and we find the following

$$s_1^0 = -s_3^1 = -4yB(u), \quad s_2^0 = -s_3^2 = 4xB(u), \quad s_2^1 = -s_1^2 = 4tB(u)$$

and

$$\begin{aligned} F_a^{01} &= -yC(u)(x, y, +t) - D(u)(0, 1, 0), \\ F_a^{02} &= xC(u)(x, y, +t) + D(u)(1, 0, 0), \\ F_a^{12} &= tC(u)(x, y, +t) + D(u)(0, 0, -1), \end{aligned} \tag{3.15}$$

all other components equal zero. We have used the abbreviations

$$B(u) = g^2+eu^2g^3, \quad C(u) = \frac{1}{u}\frac{dg}{du} - 2eg^2, \quad D(u) = 2g+u\frac{dg}{du}.$$

The field equations (2.3) reduce to the single equation

$$C(u)-4eB(u)+\frac{1}{u}\frac{d}{du}D(u) = 0,$$

which, with the substitution $f(u) = 2eug(u)$, becomes

$$f''+\frac{2}{u}f'-(1+uf)\left(\frac{2f}{u^2}+\frac{f^2}{u}\right) = 0.$$

This is identical to Eq. [6] of Wu and Yang [7], who discuss the solutions in some detail. One rather trivial solution, characterized by $D(u) = 0$, is

$$f(u) = -\frac{1}{u},$$

for which

$$B(u) = \frac{1}{8e^2u^4} \quad \text{and} \quad C(u) = \frac{1}{2eu^4}.$$

For this, as well as for the other solutions found by Wu and Yang, the fields become singular near the front of the expanding cylindrical wave.

For the solution with $D(u) = 0$ the relation

$$(F_1^{\mu\nu})^2+(F_2^{\mu\nu})^2-(F_3^{\mu\nu})^2 \leqslant 0$$

is satisfied in each space-time point and for all μ and ν. In this case it is seen from Eq. (3.15) that all $F^{\mu\nu}$ have the same direction in the charge space. There exists therefore a gauge transformation such that all $F_1^{\mu\nu}$ and all $F_2^{\mu\nu}$ become zero in each space-time point. This transformation is given by

$$g^0(x) = \sqrt{\frac{t+u}{2u}}, \quad g^1(x) = \frac{y}{\sqrt{2u(t+u)}}, \quad g^2(x) = \frac{-x}{\sqrt{2u(t+u)}}, \quad g^3(x) = 0.$$

The resulting field is purely electromagnetic and has the following non-vanishing components

$$E_x = \frac{+y}{2eu^3}, \quad E_y = \frac{-x}{2eu^3}, \quad B_z = \frac{-t}{2eu^3}.$$

The energy-momentum tensor can be written in terms of the functions $C(u)$ and $D(u)$. In particular we find for the energy density

$$T^{00}(x) = \tfrac{1}{2}(2t^2-u^2)(u^2C^2-2CD)-\tfrac{1}{2}D^2$$

or

$$T^{00}(x) = \frac{2t^2-u^2}{8e^2u^2}\left(\frac{\phi^2-1}{u^2}\right)^2 - \frac{t^2}{4e^2u^4}(\phi')^2 \tag{3.16}$$

with $\Phi(u) = 1+uf(u)$. From Eq. (3.16) it is seen that $T^{00}(x)$ is certainly positive if

$$\left(\frac{\phi^2-1}{u}\right)^2 \geqslant 2(\phi')^2. \tag{3.17}$$

We have not been able to prove this for all $u < t$. It seems, however, from the tabulation of ϕ in the paper by Wu and Yang, that Eq. (3.17) is satisfied, at least for points not too close to the wavefront. But even for points where the energy density is positive the field is certainly not of the purely electromagnetic type. This is shown directly by the form (3.15), but can also be proved from the following observation. It is known [8] that, when the energy-momentum tensor is constructed from a single skew symmetric field, like the Maxwell field, the 4×4 matrix $T^{\mu\lambda}T^{\nu}_{\lambda}$ is a multiple of the metric tensor $g^{\mu\nu}$. A direct calculation of this matrix shows that for $D(u) \neq 0$ there are non-vanishing off-diagonal elements, which proves that the above mentioned property of $T^{\mu\nu}T^{\nu}_{\lambda}$ in general does not hold for non-abelian gauge fields and in particular that the field of Eq. (3.15) cannot be purely electromagnetic.

4. *Final remarks and conclusions*

In the first section of this paper we have observed that the Dirac equation is invariant under mixing of states with positive energy with certain amounts of negative energy states. Therefore, the differentiation between an electron and a positron is, to a certain extent, an arbitrary choice. We have shown that this invariance is true, not only for the free Dirac equation, but also when certain interactions are introduced. For the Thirring model the invariance group remains the same, whereas for other interactions it may change. In trying to make this invariance hold also locally we can say, quoting the paper by Yang and Mills, mutatis mutandis, that "... we wish to explore the possibility of requiring all interactions to be invariant under arbitrary but equal changes in the positive and negative energy densities at all space-time points, so that a change in these densities from one point to another becomes physically meaningless (the weak interactions being neglected)".

In the present paper this exploration was limited to the Dirac equation. We showed that, after the introduction of gauge fields in the usual way, a mathematically consistent classical field theory resulted, which is invariant under transformations forming the non--compact group SL(2, R). At first sight it seemed that the physical consistency was lost, because two of the three gauge fields carry negative energy. This, however, could turn into an advantage, because it may invalidate the heuristic argument why a classical Dirac field cannot be stable. This argument is a variation of one of Murphy's rules and says: "if radiation can occur, it will occur". With the presence of the negative energy gauge fields, however, an electron may make transitions to states of higher and lower energy with equal ease. Indeed we showed by explicit construction that many non-trivial vacuum states exist, which at most radiate a null-field. We would like to classify all these vacua by finding a topologically invariant quantity, which could differentiate between them in the same way as this is done by Pontryagin's winding number for the Yang–Mills equations in Euclidean space. However, the source of the topological current, defined by

$$\partial_\mu J^\mu = 8e^2 \tilde{F}_a^{\mu\nu} F^a_{\mu\nu},$$

turns out to be zero for all our vacuum solutions and can therefore not serve as characteristic quantity. We have not been able to find another classification and therefore cannot draw firm conclusions from the existence of inequivalent vacua about the reality of interpolating fields with finite energy. The search for fields is complicated by the fact that in a Minkowski space no self-dual fields exist. Self-binding gauge fields (glueballs) are not ruled out by Coleman's argument [9], because his proof is valid for a compact group, but not for the group SL (2, R).

We therefore consider the existence of the many gauge-inequivalent vacua, as found in Sections 3C and 3E, as an indication that stable field configurations with finite energy may exist.

We believe that our model is an example of a classical field theory, which is free of internal inconsistencies and may have interesting solutions, in addition to the ones we have given, relevant to the description of elementary particles and their interactions.

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